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BANACH ALGEBRAS WITH COMMUTING IDEMPOTENTS, NO IDENTITY

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1.

In the recent paper [5] we showed that certain algebras, with identity, can be represented as algebras of continuous functions on a Boolean space (a totally disconnected compact space). Now we shall show that the same is essentially true even if we do not assume existence of an identity.

More specifically, we shall characterize the space $C_p(S)$ of continuous complex-valued functions on a Boolean space S, each vanishing on certain point $p \in S$, specified in advance. We shall formulate the main results in such a way that the result of [5] is an easy consequence, following as a special case. Our proofs will use a different approach (we shall utilize Gelfand theory rather than that of Stone).

2.

We shall use terminology and notation from [5] and [2]. A Boolean space is a totally disconnected compact space. A clopen set is a set which is both open and closed, $C(S)$ will denote the space of all continuous complex valued functions on a space S. An idempotent is a member e of a Banach algebra such that $e^2 = e$. The set of all idempotent members will be denoted by I, A_0 will denote the space of all finite linear combinations $\sum_{i=1}^{n} \lambda_i e_i$ of members $e_1, ..., e_n$ of I.

3.

Let S be a Boolean space, $p \in S$ and let $C_p(S)$ denote the space of all members of $C(S)$ vanishing at p.

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THEOREM 1. *The class A_0 of all finite linear combinations $\sum_{i=1}^{n} \lambda_i \chi_{E_i}$ of characteristic functions of clopen subsets $E_1, ..., E_n$ of $S \sim \{p\}$ is dense in $C_p(S)$ for any $p \in S$ (here χ_{E_i} denotes the characteristic function of E_i : $\chi_{E_i}(t) = 1$ if $t \in E_i$ and $\chi_{E_i}(t) = 0$ otherwise).*

Proof. One can use essentially the proof of Theorem 1 in [5].

Now let A be a Banach algebra such that the set $A_0 = \{\sum_{i=1}^{n} \lambda_i e_i : e_i \in \mathrm{I}$ and $\lambda_1, ..., \lambda_n$ are complex numbers$\}$ is dense in A and such that the members of I commute. Assume further that $\| x \| \leqslant \max_i |\lambda_i|$ for each member x of A_0 of the form $x = \sum_{i=1}^{n} \lambda_i e_i$, where $e_1, ..., e_n$ are disjoint in the sense that $e_i e_j = 0$ if $i \neq j$.

THEOREM 2. *For each algebra A with the above three properties there exists a Boolean space S and $p \in S$ such that A is isomorphic and isometric to $C_p(S)$ (the norm in $C_p(S)$ is the supnorm: $\| x \| = \sup_t |x(t)|$ for each $x \in C_p(S)$). If A has an identity, then p is an isolated point of S (the set $\{p\}$ is both open and closed).*

Proof. First note that the first two assumed conditions imply that A is commutative. So, we can invoke Gelfand theory. Let S be the set of all multiplicative members of A^*. Then

$$S = \{F_M : M \in \mathfrak{M}\} \cup \{\theta\},$$

where $\mathfrak{M}$ is the set of all maximal regular ideals of A and θ is the member of A^* that maps everything into 0 : $\theta(x) = 0$ for all $x \in A$. (We use the notation of [2] here: F_M is the multiplicative linear functional on A whose kernel is $M \in \mathfrak{M}$). We equip S with the topology τ which is the restriction of the weak* topology of A^*. Since S is a closed subset of the norm closed unit ball of A^*, we conclude that S is compact in τ. Consider the Gelfand map $x \to x^\wedge(\ )$ of A into $C(S)$, where $x^\wedge(F_M) = F_M(x)$ if a member of S is of this form for some $M \in \mathfrak{M}$ and $x^\wedge(\theta) = 0$. This Gelfand map does not have to be one-to-one but it maps members e of I (idempotents) into characteristic functions χ_E of some clopen subset E of S : the fact that $e^\wedge(s)^2 = e^\wedge(s)$ implies that $e^\wedge(s)$ is either 0 or 1, so we can take $E = \{s \in S : e^\wedge(s) = 1\}$. It follows from the continuity of $e^\wedge(\ )$ that both E and its complement are closed subsets of S. Also it is not difficult to see that if $e_1, ..., e_n$ are disjoint in the sense that $e_i e_j = 0$ if $i \neq j$, then the corresponding sets $E_1, ..., E_n$ are disjoint in set-theoretic sense. This means that if a member $x = \sum_{i=1}^{n} \lambda_i e_i$ in A_0 is such that $e_1, ..., e_n$ are disjoint, then $\| x^\wedge \|_\infty = \max |\lambda_i|$ (the norm $\| \ \|_\infty$ is defined in [2, 6A]). But $\| x^\wedge \|_\infty \leqslant \| x \|$ [2,23B]. Hence the last assumption about the set A_0 implies that $\| x^\wedge \|_\infty = \| x \|$, and so A is semi-simple. Thus, the Gelfand map $x \to x^\wedge$ is one-to-one and isometric. One can show, using Theorem 1 above, that it maps A onto $C_p(S)$, where $p = \theta$.

It remains to show that S is totally disconnected. Let M_1, M_2 be two different members of $\mathfrak{M}$, take $x \in M_1 \sim M_2$. Then $F_{M_1}(x) = 0$ and $x = \mu u + m$ for some $m \in M_2$, some complex number μ and a relative identify u of M_2. Also $F_{M_2}(x) = \mu$, and we may assume that $\mu = 1$. Then x is also a relative identiy of M_2. It follows from [2,22D] that the set $\{y \in A : \| y - x \| < 1\}$ does not intersect M_2. Take $x_0 \in A_0$ such that $\| x_0 - x \| < \frac{1}{2}$; then $x_0 = \sum_{i=1}^{n} \lambda_i e_i$ for some $e_1, ..., e_n \in \mathrm{I}$ and complex numbers $\lambda_1, ..., \lambda_n$. It follows from Proposition 2 in [5] that we may assume $e_1, ..., e_n$ to be disjoint. Then the corresponding sets $E_1, ..., E_n$ are also disjoint. There exists exactly one set E_i such that $M_1 \in E_i$. This set E_i is clopen and it is not difficult to show that $M_2 \notin E_i$: "$M_2 \in E_i$" would imply that $F_{M_2}(x_0) = F_{M_1}(x_0)$ from which it would follow that $|F_{M_1}(x_0) - 1| = |F_{M_2}(x_0) - F_{M_2}(x)| \leqslant \| x_0 - x \| < \frac{1}{2}$, which is in contradiction to the fact that $|F_{M_1}(x_0)| < \frac{1}{2}$ (indeed, $|F_{M_1}(x_0)| = |F_{M_1}(x_0) - F_{M_1}(x)| \leqslant \| x_0 - x \| < \frac{1}{2}$).

This shows that any two members s_1 and s_2 of S of the form $s_1 = F_{M_1}, s_2 = F_{M_2}$ can be separated by a clopen set E_i.

The case when $s_1 = \theta$ and $s_2 = F_M$ for some $M \in \mathfrak{M}$ is even simpler. Let u be a relative identity modulo M and take $x_0 \in A_0$ such that $\| x_0 - u \| < \frac{1}{2}$. Then, as above, we find a clopen subset E_i of S such that $s_2 \in E_i$. Obviously, $\theta \notin E_i$.

The last statement of the Theorem is also easy to establish, it follows from the fact that $\mathbf{1}^\wedge(F_M) = 1$, $M \in \mathfrak{M}$, and $\mathbf{1}^\wedge(\theta) = 0$ (here $\mathbf{1}$ denotes the identity of A).

Note that Theorem 2 and 3 in [5] are simple consequence of this theorem: all we have to do here to take $S = \mathfrak{M}$.

THEOREM 3. *For each algebra A above (the same as in the statement of Theorem 2) there exits a locally compact totally disconnected space S such that A is isomorphic and isometric to the algebra $C_\infty(S)$ of all continuous complex valued functions $x(t)$ on S vanishing at ∞ (which means that for each $\epsilon > 0$ there exists a compact set C_ϵ such that $|x(t)| \leqslant \epsilon$ for each $t \in S \sim C_\epsilon$). The space S has the property that each point in S has a neighborhood which is both open and compact.*

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