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WHICH CONDITIONS FOR AN APPROXIMATE RESOLUTION ARE ESSENTIAL?

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Abstract. Among all conditions which characterize an approximate polyhedral resolution of a topological space, only two ((AS) and (B1)) are essential.

1. Introduction

The notion of a resolution of a topological space was introduced by S. Mardešić [2], [3]. The idea was to improve some lacks of the inverse limit theory in the noncompact case. A few years later S. Mardešić and L. R. Rubin [4] introduced the notion of an approximate (inverse) system. The intention was to relax the commutativity condition, which has been too rigid for some purposes even in the compact nonmetric case. Finally, S. Mardešić and T. Watanabe [6] put the both ideas together, introducing the notion of an approximate resolution of a space.

In any case, the basic idea (due to P. S. Aleksandrov, 1920's) did not change: to express a "bad" object (an arbitrary space) in terms of "nice" ones (polyhedra, ANR's, ...) on purpose of much easier studying some of its properties.

By a space we mean a topological space, and by a mapping, a continuous function. A covering $\mathcal{U}$ of a space X is an indexed family of its subsets whose union equals X. If $A \subseteq X$ and $\mathcal{U}$ is a covering of X, then $st(A, \mathcal{U})$ denotes the union of all $U \in \mathcal{U}$ meeting A. If $\mathcal{U}$ and $\mathcal{V}$ are coverings of X, then $st(\mathcal{U}, \mathcal{V})$ denotes the covering $\{st(U, \mathcal{V}) \mid U \in \mathcal{U}\}$; $st\mathcal{U}$ and $st^{n+1}\mathcal{U}$ are the abbreviations for $st(\mathcal{U}, \mathcal{U})$ and $st(st^n\mathcal{U}, \mathcal{U})$, $n \in N$, respectively. $\mathcal{U} \leqslant \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$. $Cov(X)$ denotes the collection of all normal (or numerable) coverings of a space X. Every normal covering $\mathcal{U}$ of a space X admits a normal covering $\mathcal{V}$ of X such that $st\mathcal{V} \leqslant \mathcal{U}$. If $f, g : Y \to X$ are $\mathcal{U}$-near mappings, i.e., for any $y \in Y$ there is a $U \in \mathcal{U}$ with $f(y)$, $g(y) \in U$, we write $(f, g) \leqslant \mathcal{U}$.

By a polyhedron we mean a triangulable space (CW-topology; [7]). If (K, h) is a triangulation of a polyhedron P, we formally identify P with the geometric realization $|K|$. A subspace $Q \subseteq P$ is a subpolyhedron of P, if there exist a triangulation K of P and a subcomplex $L \subseteq K$ such that $Q = |L|$.

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Recall now the basic definitions (comp. [6], [1]).

An *approximate (inverse) system* $\boldsymbol{X} = (X_a, p_{aa'}, A)$ consists of the following data: A preordered set $A = (A, \leqslant)$ which is directed and unbounded; for each $a \in A$, a space X_a; for each pair $a \leqslant a'$, a mapping $p_{aa'} : X_{a'} \to X_a$ ($p_{aa} = 1_{X_a}$). Furthermore, one condition is required:

(A2) $(\forall a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists a' \geqslant a)\ (\forall a_2 \geqslant a_1 \geqslant a')\quad (p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leqslant \mathcal{U}$.

If each X_a is a polyhedron, $a \in A$; we speak of an approximate polyhedral system. An approximate system $\boldsymbol{X}$ *admits meshes* if for each $a \in A$ there exists a $\mathcal{U}_a \in Cov(X_a)$ satisfying two additional conditions:

(A1) $(\forall a \leqslant a' \leqslant a'')\quad (p_{aa'}p_{a'a''}, p_{aa''}) \leqslant \mathcal{U}_a$;

(A3) $(\forall a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists a' \geqslant a)\ (\forall a'' \geqslant a')\quad \mathcal{U}_{a''} \leqslant p_{aa''}^{-1}\mathcal{U}$.

An approximate system $\boldsymbol{X}$ which admits meshes is *uniform* provided it satisfies the following condition:

(AU) $(\forall a \leqslant a')\quad \mathcal{U}_{a'} \leqslant p_{aa'}^{-1}\mathcal{U}_a$.

An *approximate map* $\boldsymbol{p}$ of a space X into an approximate system $\boldsymbol{X}$, $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$, is a collection of mappings $p_a : X \to X_a$, $a \in A$, such that the following condition holds:

(AS) $(\forall a \in A)\ (\forall \mathcal{U} \in Cov(X_a)\ (\exists a' \geqslant a)\ (\forall a'' \geqslant a')\quad (p_{aa''}p_{a''}, p_a) \leqslant \mathcal{U}$.

An *approximate resolution* of a space X is an approximate map $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$ which satisfies two following conditions:

(B1) $(\forall \mathcal{U} \in Cov(X))\ (\exists a \in A)\ (\exists \mathcal{V} \in Cov(X_a))\quad p_a^{-1}\mathcal{V} \leqslant \mathcal{U}$;

(B2) $(\forall a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists a' \geqslant a)\quad p_{aa'}(X_{a'}) \subseteq st(p_a(X), \mathcal{U})$.

If each X_a is a normal space, (B2) is equivalent to

(B3) $(\forall a \in A)\ (\forall$ open $U \subseteq X_a$ with $Cl(p_a(X)) \subseteq U)\ (\exists a' \geqslant a)\quad p_{aa'}(X_{a'}) \subseteq U$.

Note that the notion of an approximate resolution (which admits uniform meshes) includes conditions (A2), (AS) (B1) and (B2) (and (A1), (A3), (AU)). Although they are mutually independent, only two of them, (AS) and (B1), are essential (at least in the polyhedral case). More precisely, we are proving that, taking appropriate subpolyhedra and the restriction mappings, all the conditions can be obtained by means of (AS) and (B1).

2. Construction of the resolution

Let us state our main result:

THEOREM. *Let A be an unbounded and directed preordered set, and let* $\boldsymbol{X} = (X_a, p_{aa'}, A)$ *be a collection consisting of polyhedra* X_a, $a \in A$, *and of mappings* $p_{aa'} : X_{a'} \to X_a$, $a \leqslant a'$ ($p_{aa} = 1_{X_a}$). *Let* $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$ *be a collection of mappings* $p_a : X \to X_a$, $a \in A$, *satisfying conditions*

(AS) $(\forall a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists a' \geqslant a)\ (\forall a'' \geqslant a')\quad (p_{aa''}p_{a''}, p_a) \leqslant \mathcal{U}$;
(B1) $(\forall \mathcal{U} \in Cov(X))\ (\exists a \in A)\ (\exists \mathcal{V} \in Cov(X_a))\quad p_a^{-1}\mathcal{V} \leqslant \mathcal{U}$.

Then there exists an approximate polyhedral resolution $\boldsymbol{q} = (q_b) : X \to \boldsymbol{Y} = (Y_b, q_{bb'}, B)$ *such that*

(i) $B = (B, \leqslant)$ *is partially ordered and cofinite;*
(ii) $(\forall b \in B)\ (\exists a \in A)\quad Y_b \subseteq X_a$ *is a subpolyhedron;*
(iii) $(\forall b \leqslant b')\ (\exists a \leqslant a')\quad q_{bb'} : Y_{b'} \to Y_b$ *is the restriction mapping of* $p_{aa'}$;
(iv) $\boldsymbol{Y}$ *admits uniform meshes;*
(v) $(\forall b \in B)\ (\exists a \in A)\quad q_b = p_a : X \to Y_b \subseteq X_a$.

In addition, if $\boldsymbol{X}$ *is a (commutative) system and (AS) of* $\boldsymbol{p}$ *is strengthened up to commutativity, then (AS) of* $\boldsymbol{q}$ *turns into commutativity.*

In order to prove the theorem, i.e., to construct a desired resolution, we need two lemmata. The first one shows how to enlarge an indexing set to obtain a more convenient one.

LEMMA 1. *Let* $\boldsymbol{X} = (X_a, p_{aa'}, A)$ *be a collection of spaces* X_a, $a \in A$, *and mappings* $p_{aa'} : X_{a'} \to X_a$, $a \leqslant a'$, *over a preordered, directed and unbounded set* $A = (A, \leqslant)$. *Then there exists a collection of spaces and mappings* $\boldsymbol{X}' = (X'_\lambda, p'_{\lambda\lambda'}, \Lambda)$ *over* Λ *such that*

(i) $\Lambda = (\Lambda, \leqslant)$ *is partially ordered, directed, unbounded and cofinite;*
(ii) $(\forall \lambda \in \Lambda)\quad card(\Lambda_\lambda) = card(\Lambda) \geqslant card(A)$,
where $\Lambda_\lambda = \{\lambda' \in \Lambda \mid \lambda' \geqslant \lambda\}$;
(iii) $(\forall \lambda \in \Lambda)\quad card(\Lambda_\lambda) \geqslant cw(X'_\lambda)$,
where $cw(X'_\lambda)$ *is the covering weight of* X'_λ;
(iv) $(\forall \lambda \in \Lambda)\ (\exists a \in A)\quad X'_\lambda = X_a$;
(v) $(\forall \lambda \leqslant \lambda')\ (\exists a \leqslant a')\quad p'_{\lambda\lambda'} = p_{aa'}$.

Proof. For each $a \in A$ choose a cofinal subfamily $\mathcal{C}_a \subseteq Cov(X_a)$, and let $\mathcal{L} = \{(a, \mathcal{U}) \mid a \in A, \mathcal{U} \in \mathcal{C}_a\} = \cup_{a\in A}(\{a\} \times \mathcal{C}_a)$ (see [9], Sec. 2.). Apply now the well known "Mardešić trick" on $\mathcal{L}$, i.e., let $\Lambda = F(\mathcal{L})$ be the set of all finite subsets $\lambda \subseteq \mathcal{L}$ ordered by inclusion. Clearly, Λ satisfies conditions (i), (ii) and (iii) (see [8], [5]). Furthermore, there exists an increasing surjection $s : \Lambda \to A$ such that $s(\{(a, \mathcal{U})\}) = a$, $(a, \mathcal{U}) \in \mathcal{L}$. By putting $X'_\lambda = X_{s(\lambda)}$, $\lambda \in \Lambda$, and $p'_{\lambda\lambda'} = p_{s(\lambda)s(\lambda')}$ whenever $\lambda \leqslant \lambda'$, one establishes (iv) and (v). □

LEMMA 2. *Let* $\boldsymbol{X} = (X_a, p_{aa'}, A)$ *be a collection of polyhedra* X_a, $a \in A$, *and mappings* $p_{aa'} : X_{a'} \to X_a$, $a \leqslant a'$, *over a preordered, directed and unbounded set* $A = (A, \leqslant)$. *Let* $\{\mathcal{U}_a \in Cov(X_a) \mid a \in A\}$ *be a family of open coverings satisfying condition*

(A3) $(\forall a \in A)\ (\forall \mathcal{U} \in Cov(X_a))\ (\exists a' \geqslant a)\ (\forall a'' \geqslant a')\quad \mathcal{U}_{a''} < p_{aa''}^{-1}\mathcal{U}$.

Let X *be a space and let* $\boldsymbol{p} : X \to \boldsymbol{X}$ *be a family of mappings* $p_a : X \to X_a$, $a \in A$, *satisfying condition*

(AS) $(\forall a \in A)\ (\forall \mathscr{U} \in Cov(X_a))\ (\exists a' \geqslant a)\ (\forall a'' \geqslant a')\quad (p_{aa''}p_{a''}, p_a) < \mathscr{U}$.
Then there exist an approximate polyhedral system $\boldsymbol{X}' = (X'_a, p'_{aa'}, A')$ *and an approximate map* $\boldsymbol{p}' = (p'_a) : X \to \boldsymbol{X}'$ *satisfying*

(i) $A' = (A, \leqslant')$ *is a preordered, directed and unbounded set such that:*
$a \leqslant' a' \Rightarrow a \leqslant a'$ *and* $a \not\leqslant' a' \leqslant a'' \Rightarrow a \leqslant' a''$;
if $\leqslant$ *is a partial order, then so is* $\leqslant'$;
if $(A, \leqslant)$ *is cofinite, then so is* $(A, \leqslant')$;

(ii) X'_a *is a subpolyhedral neighbourhood of* $Cl(p_a(X))$ *in* X_a, $a \in A$;

(iii) $p'_{aa'} = p_{aa'} \mid X'_{a'} : X'_{a'} \to X'_a \subseteq X_a$, $a \leqslant' a'$;

(iv) $p'_a = p_a : X \to X'_a \subseteq X_a$, $a \in A$.

Proof. Let K_a be a triangulation of X_a such that $st\mathscr{S}_a < \mathscr{U}_a$ (see [7], p. 126; [10]), where $\mathscr{S}_a = \{st(v, K_a) \mid v \in K^0_a\} \in Cov(X_a)$, $a \in A$. Let $X'_a = |N_a|$ be the minimal K_a-neighbourhood of $Cl(p_a(X))$ in X_a. ($|N_a|$ is the union of all closed simplexes of $|K_a|$ meeting $Cl(p_a(X))$).
Define a new relation $\leqslant'$ on A by putting

$$a \leqslant' a' \overset{def.}{\Longleftrightarrow} \begin{cases} a = a' \\ \text{or} \\ a \leqslant a', a \neq a' \wedge & (\forall a'' \geqslant a')\, p_{aa''}(X'_{a''}) \subseteq X'_a \quad \wedge \\ & (p_{aa''}p_{a''}, p_a) \leqslant \mathscr{S}_a \quad \wedge \\ & \mathscr{S}_{a''} \leqslant p^{-1}_{aa''}(\mathscr{S}_a) \end{cases}$$

We first show that the relation $\leqslant'$ contains more than the diagonal of $A \times A$. Let $a \in A$. Since X'_a is a neighbourhood of $Cl(p_a(X))$ in the normal space X_a, there exists an open neighbourhood V of $Cl(p_a(X))$ such that $V \subseteq Cl(V) \subseteq Int(X'_a) \subseteq X'_a$. Take $\mathscr{W} = \{Int(X'_a), X_a \setminus Cl(V)\} \in Cov(X_a)$. By (A3) and (AS), for a and $\mathscr{W}$, there exists an $a'_1 \geqslant a$ such that, for every $a'' \geqslant a'_1$,

$$\mathscr{S}_{a''} \leqslant \mathscr{U}_{a''} \leqslant p^{-1}_{aa''}(\mathscr{W}) \tag{1}$$

and

$$(p_{aa''}p_{a''}, p_a) \leqslant \mathscr{W} \tag{2}$$

hold. In the same way, for a and $\mathscr{S}_a$, there exists an $a'_2 \geqslant a$ such that, for every $a'' \geqslant a'_2$,

$$\mathscr{S}_{a''} \leqslant \mathscr{U}_{a''} \leqslant p^{-1}_{aa''}(\mathscr{S}_a) \tag{3}$$

and

$$(p_{aa''}p_{a''}, p_a) \leqslant \mathscr{S}_a \tag{4}$$

hold. Take an $a' \geqslant a'_1, a'_2$, $a' \neq a'_1, a'_2$. In order to prove $a \leqslant' a'$, we need to verify

$$p_{aa''}(X'_{a''}) \subseteq X'_a, \tag{5}$$

whenever $a'' \geqslant a'$. Let $x'_{a''} \in X'_{a''}$. Then there exist an $S_{a''} \in \mathscr{S}_{a''}$ and a $U_{a''} \in \mathscr{U}_{a''}$ such that

$$x'_{a''} \in S_{a''} \subseteq U_{a''} \quad \text{and} \quad S_{a''} \cap Cl(p_{a''}(X)) \neq \emptyset.$$

Therefore, since $S_{a''}$ is open, $S_{a''} \cap p_{a''}(X) \neq \emptyset$. Let $x_{a''} \in S_{a''} \cap p_{a''}(X)$ and let $x \in X$ such that $p_{a''}(x) = x_{a''}$. By (2), $(p_{aa''}p_{a''}, p_a) \leqslant \mathscr{W}$ holds, and thus

$$p_{aa''}(x_{a''}) \in W, \tag{6}$$

for some $W \in \mathscr{W}$. Note that $W = Int(X'_a)$, since $x_a \equiv p_a(x) \in p_a(X) \subseteq Int(X'_a) \in \mathscr{W}$. Further, $\mathscr{U}_{a''} < p^{-1}_{aa''}(\mathscr{W})$ implies $U_{a''} \subseteq p^{-1}_{aa''} Int(X'_a)$. Hence, $p_{aa''}(U_{a''}) \subseteq Int(X'_a)$, and $p_{aa''}(x'_{a''}) \in Int(X'_a)$, which verifies (5).

The proof from the above also shows that $A' = (A, \leqslant')$ is unbounded. Now, one easily verifies all the properties of $\leqslant'$ stated in (i).

Let

$$p'_{aa'} = p_{aa'} \mid X'_{a'} : X'_{a'} \to X'_a \subseteq X_a, \ a \leqslant' a', \tag{7}$$

and

$$p'_a = p_a : X \to X'_a \subseteq X_a, \ a \in A. \tag{8}$$

Since $\boldsymbol{p}'$ inherits property (AS) of $\boldsymbol{p}$, it remains to prove that $\boldsymbol{X}' = (X'_a, p'_{aa'}, A')$ is an approximate system, i.e., to verify condition

(A2) $(\forall a \in A)\ (\forall \mathscr{U} \in Cov(X'_a))\ (\exists \bar{a} \in A)\ (\forall a'' \geqslant' a' \geqslant' \bar{a})\quad (p'_{aa'}p'_{a'a''}, p'_{aa''}) \leqslant \mathscr{U}$.

Let $a \in A'$ and $\mathscr{U} \in Cov(X'_a)$. Since $X'_a \subseteq X_a$ is a subpolyhedron, it is normally embedded, and there exists a covering $\mathscr{W} \in Cov(X_a)$ such that

$$\mathscr{W} \mid X'_a \leqslant \mathscr{U}. \tag{9}$$

Choose a $\mathscr{V} \in Cov(X_a)$ such that

$$st^2\mathscr{V} \leqslant \mathscr{W}. \tag{10}$$

By (A3) and (AS), for a and $\mathscr{V}$, there is an $\bar{a} \in A'$ such that

$$\mathscr{S}_{a'} \leqslant st\mathscr{S}_{a'} \leqslant \mathscr{U}_{a'} \leqslant p^{-1}_{aa'}(\mathscr{V}) \quad \text{and} \quad (p_{aa'}p_{a'}, p_a) \leqslant \mathscr{V}, \tag{11}$$

whenever $a' \geqslant \bar{a}$. Let us show that, for all $a'' \geqslant' a' \geqslant' \bar{a}$,

$$(p'_{aa'}p'_{a'a''}, p'_{aa''}) \leqslant \mathscr{U} \tag{12}$$

holds. Let $x'_{a''} \in X'_{a''}$. Then there exists an $S_{a''} \in \mathscr{S}_{a''}$ such that

$$x'_{a''} \in S_{a''} \quad \text{and} \quad S_{a''} \cap p_{a''}(X) \neq \emptyset.$$

Choose any $x_{a''} \in S_{a''} \cap p_{a''}(X)$ and $x \in X$ such that $x_{a''} = p_{a''}(x)$. By (11), $\mathscr{S}_{a''} \leqslant \mathscr{U}_{a''} \leqslant p^{-1}_{aa''}(\mathscr{V})$ holds ($a'' \geqslant' \bar{a}$), and there exists a $V_1 \in \mathscr{V}$ such that $x_{a''}$, $x'_{a''} \in p^{-1}_{aa''}(V_1)$, i.e.,

$$p_{aa''}(x'_{a''}),\ p_{aa''}(x_{a''}) = p_{aa''}p_{a''}(x) \in V_1. \tag{13}$$

Further, (11) implies $(p_{aa''}p_{a''}, p_a) \leqslant \mathscr{V}$, and there exists a $V_2 \in \mathscr{V}$ such that

$$p_{aa''}p_{a''}(x),\ p_a(x) \in V_2. \tag{14}$$

By definition of $\leqslant'$, $a' \leqslant' a''$ implies $\mathscr{S}_{a''} \leqslant p^{-1}_{a'a''}(\mathscr{S}_{a'})$. Thus there exists an $S_{a'} \in \mathscr{S}_{a'}$ such that

$$p_{a'a''}(x_{a''}),\ p_{a'a''}(x'_{a''}) \in S_{a'}.$$

By the same argument, $(p_{a'a''}p_{a''}, p_{a'}) \leqslant \mathscr{S}_{a'}$ holds, and there exists an $S'_{a'} \in \mathscr{S}_{a'}$ such that

$$p_{a'a''}p_{a''}(x) = p_{a'a''}(x_{a''}),\ p_{a'}(x) \in S'_{a'}.$$

Similarly, $a' \geqslant' \bar{a}$ implies $\mathscr{S}_{a'} \leqslant p^{-1}_{aa'}(\mathscr{V})$ (see (11)), and there are $V_3, V_4 \in \mathscr{V}$ such that $p_{a'a''}(x_{a''}), p_{a'a''}(x'_{a''}) \in p^{-1}_{aa'}(V_3)$ and $p_{a'a''}p_{a''}(x), p_{a'}(x) \in p^{-1}_{aa'}(V_4)$, i.e.,

$$p_{aa'}p_{a'a''}(x_{a''}),\ p_{aa'}p_{a'a''}(x'_{a''}) \in V_3 \tag{15}$$

and

$$p_{aa'}p_{a'a''}(x_{a''}),\ p_{aa'}p_{a'}(x) \in V_4. \tag{16}$$

Finally, $a' \geqslant' \bar{a} \geqslant' a$ implies $(p_{aa'}p_{a'}, p_a) \leqslant \mathscr{V}$, and there exists a $V \in \mathscr{V}$ such that

$$p_{aa'}p_{a'}(x),\ p_a(x) \in V. \tag{17}$$

Relations (13), (14), (15), (16) and (17) imply

$$p_{aa''}(x'_{a''}),\ p_{aa'}p_{a'a''}(x'_{a''}) \in st^2V.$$

Now, by (7), (9) and (10), we establish

$$(p'_{aa'}p'_{a'a''}, p'_{aa''}) = ((p_{aa'} \mid X'_{a'})(p_{a'a''} \mid X'_{a''}), (p_{aa''} \mid X'_{a''})) \leqslant \mathscr{W} \mid X'_a \leqslant \mathscr{U}.$$

Consequently, (12) holds true, i.e., condition (A2) for $\boldsymbol{X}'$ is verified, and the lemma is proved. □

Proof of the theorem. By Lemma 1, there is an $\boldsymbol{X}' = (X'_\lambda, p'_{\lambda\lambda'}, \Lambda)$, associated with $\boldsymbol{X}$, having corresponding properties (i)-(v). Note that Lemma 1 (iii) is the stability condition (C) (see [8], [5]), which (in the cofinite case) implies condition (A3) for $\boldsymbol{X}'$. Further, $\boldsymbol{p} : X \to \boldsymbol{X}$ yields the collection of mappings $\boldsymbol{p}' = (p'_\lambda = p_{s(\lambda)}) : X \to \boldsymbol{X}'$. Obviously, $\boldsymbol{p}'$ inherits conditions (AS) and (B1) of $\boldsymbol{p}$. Applying Lemma 2 on $\boldsymbol{X}'$ and $\boldsymbol{p}'$, one obtains an approximate polyhedral system $\boldsymbol{X}'' = (X''_\lambda, p''_{\lambda\lambda'}, \Lambda')$ and an approximate map $\boldsymbol{p}'' = (p''_\lambda) : X \to \boldsymbol{X}''$ having corresponding properties (i)-(iv). We will now prove that $\boldsymbol{p}'' : X \to \boldsymbol{X}''$ is an approximate resolution. It suffices to check conditions (B1) and (B3). Note that (B1) of $\boldsymbol{p}''$ is inherited of the same condition of $\boldsymbol{p}'$. In order to verify (B3) for $\boldsymbol{p}''$, let any $\lambda \in \Lambda'$ and any open set $U \supseteq Cl(p''_\lambda(X))$ in $X''_\lambda \subseteq X'_\lambda = X_{s(\lambda)}$ are given. Choose an open set $W \subseteq X'_\lambda$ such that $W \cap X''_\lambda = U$. Since every polyhedron is a normal space, there exists an open set $V \subseteq X'_\lambda$ such that $Cl(p''_\lambda(X)) \subseteq V \subseteq Cl(V) \subseteq W \cap Int(X''_\lambda)$. Let $\mathscr{W} = \{W_1 = W \cap Int(X''_\lambda), W_2 = X'_\lambda \setminus Cl(V)\}$, and take a $\mathscr{W}' \in Cov(X'_\lambda)$ with $st\mathscr{W}' \leqslant \mathscr{W}$. By (A3) of $\boldsymbol{X}'$ and (AS) of $\boldsymbol{p}'$, for λ and $\mathscr{W}'$, there is a $\lambda' \geqslant \lambda$ such that, for every $\lambda'' \geqslant \lambda'$,

$$\mathscr{U}_{\lambda''} \leqslant (p'_{\lambda\lambda''})^{-1}\mathscr{W}' \quad \text{and} \quad (p'_{\lambda\lambda''}p'_{\lambda''}, p'_\lambda) \leqslant \mathscr{W}'.$$

(Clearly, one may choose $\lambda' \geqslant' \lambda$, and then any $\lambda'' \geqslant' \lambda'$.) Let $x''_{\lambda''} \in X''_{\lambda''}$. Then there exists an $S_{\lambda''} \in \mathscr{S}_{\lambda''}$ such that $x''_{\lambda''} \in S_{\lambda''}$ and $S_{\lambda''} \cap p_{\lambda''}(X) \neq \emptyset$. Choose any $x_{\lambda''} \in S_{\lambda''} \cap p_{\lambda''}(X)$. Since $\mathscr{S}_{\lambda''} \leqslant \mathscr{U}_{\lambda''} \leqslant (p'_{\lambda\lambda''})^{-1}\mathscr{W}'$, there are an $U_{\lambda''} \in \mathscr{U}_{\lambda''}$ and a $W' \in \mathscr{W}'$ such that

$$x_{\lambda''},\ x''_{\lambda''} \in S_{\lambda''} \subseteq U_{\lambda''} \subseteq (p'_{\lambda\lambda''})^{-1}W', \quad \text{i.e.,}$$

$$p'_{\lambda\lambda''}(x_{\lambda''}),\ p'_{\lambda\lambda''}(x''_{\lambda''}) \in W'.$$

Take an $x \in X$ such that $p'_{\lambda''}(x) = x_{\lambda''}$. Then $(p'_{\lambda\lambda''}p'_{\lambda''}, p'_{\lambda}) \leqslant \mathscr{W}'$ implies

$$p'_{\lambda}(x),\ p'_{\lambda\lambda''}p'_{\lambda''}(x) = p'_{\lambda\lambda''}(x_{\lambda''}) \in W'',$$

for some $W'' \in \mathscr{W}'$. Obviously, $W' \cap W'' \neq \emptyset$ and

$$p'_{\lambda}(x),\ p'_{\lambda\lambda''}(x''_{\lambda''}) \in W' \cup W'' \subseteq st(W', \mathscr{W}') \subseteq W_1,$$

consequently, $p'_{\lambda\lambda''}(x''_{\lambda''}) \in W_1$. Therefore,

$$p''_{\lambda\lambda''}(X''_{\lambda''}) = p'_{\lambda\lambda''}(X''_{\lambda''}) \subseteq W_1 = W \cap Int(X''_{\lambda}) \subseteq W \cap X''_{\lambda} = U$$

which establishes condition (B3) for $\boldsymbol{p}''$.

Finally, modifying ordering $\leqslant'$ into $\leqslant^*$ (see [5], Remark 2.10, and [6], (1.6) Remark), one also obtains condition (A1) and the uniformity condition (AU) for $\boldsymbol{X}''$. To complete the proof, one only has to adapt the notations: $B \equiv (\Lambda')^*$, $Y_b \equiv X''_{\lambda}$, $q_{bb'} \equiv p''_{\lambda\lambda'}$, and $q_b \equiv p''_{\lambda}$. The additional statement in the commutative case is obviously true. □

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