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THE EXISTENCE OF n–SHAPE THEORY FOR ARBITRARY COMPACTA

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Abstract. Shape theory is an extension of homotopy theory which uses the idea of homotopy in its conception. By comparison, the theory of n-shape, which heretofore only has been defined for metrizable compacta, has as its basic notion that of n-homotopy instead of homotopy. We shall demonstrate that the theory of n-shape extends to the class of all Hausdorff compacta.

1. Introduction

This work will deal with n-shape theory for compact Hausdorff spaces. Shape theory, which of course was originally introduced by K. Borsuk for metrizable compacta only ([Bo1], [Bo2]), can be defined ([MS1], [Mo]) for all topological spaces. On the other hand, the theory of n-shape, introduced by A. Chigogidze (see [Ch] for a good exposition) in order to provide the correct medium for complement theorems in universal Menger compacta, only applies to metrizable compacta. His results have been expanded in [A] and [AS]. An addition theorem was proved in [JR]. The theory of n-shape previously has not been extended, even to the category of compact Hausdorff spaces. In the sequel, we shall show how to make this extension.

To accomplish the goal just stated, we are going to employ a strategy which involves a type of "resolution" of our compactum X, i.e., a map f of a specially chosen n-dimensional compactum Z into X. The basis for this approach comes from something which was done in [MR2] (see section 9). Beginning with an approximate system of compact polyhedra with X as its limit, another approximate system of n-dimensional compacta was constructed and which mapped to the given approximate system, thus inducing a map f of its limit Z to X. In section 4 we will modify somewhat the approach given there in order to produce the map we want. We shall begin our procedure with an inverse system of compact polyhedra representing X, which as will be seen can be treated as a commutative approximate system.

Section 2 of this work will contain a description of shape and n-shape theories. The next section will provide the reader with the needed theory of approximate systems. Our main construction will appear in section 4, with a proof of our main theorem finally occurring in section 5. In this paper we shall use map to indicate continuous function, and compactum will mean compact Hausdorff space.

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2. Shape and n-shape

The theory of shape in general [MS1] is quite abstract and can be applied to many different settings. When one wants to define shape for say topological spaces only, a good reference to use is [Mo], although of course, all the information can be found in [MS1]. We shall review some of the ideas behind the shape theory of topological spaces. Having done that, it will be easier for the reader who is not familiar with the abstract approach to shape to see what is needed for shape theory in general, and in particular to see what we have to do for n-shape. The specific item to key on is Definition 1.2 of [Mo] or Theorem 1 of I.2.1 of [MS1]. First a little notation.

Let HOMOT denote the homotopy category of topological spaces and homotopy classes of maps and HPOL the full subcategory of HOMOT whose objects are all topological spaces having the homotopy type of a CW-complex. (We really only need compact polyhedra in this paper.)

2.1. Definition. We say that an inverse system $(X_a, [p_{a\,a'}], A)$ in HOMOT or HPOL is **associated** with or is an **expansion** of a topological space X if there are maps $p_a : X \to X_a$ for $a \in A$ such that the following conditions are satisfied.

(i) $[p_{a\,a'}][p_{a'}] = [p_a]$, if $a < a'$.

(ii) For any map $f : X \to Q$ with $Q \in \mathrm{Obj(HPOL)}$, there exist $a \in A$ and a map $f_a : X_a \to Q$ such that $[f] = [f_a][p_a]$.

(iii) For $a \in A$ and for maps f_a, $g_a : X_a \to Q$ with $Q \in \mathrm{Obj(HPOL)}$ such that $[f_a][p_a] = [g_a][p_a]$, there exists $a' \in A$ with $a \leqslant a'$ such that $[f_a][p_{a\,a'}] = [g_a][p_{a\,a'}]$.

Technically speaking, there should be an adjective, either HOMOT or HPOL in conjunction with the terms “associated” and “expansion”. The abstract theory of shape yields that there is a shape theory for HOMOT (and hence a shape theory for all topological spaces) if and only if every topological space has an HPOL-expansion, i.e., if and only if HPOL is a so-called “dense” subcategory of HOMOT. The work in [Mo] and elsewhere is designed to show that such expansions exist. Let us give a brief discussion of system maps which will be involved in this setting (cf., p. 256 of [Mo]).

2.2. Definition. Let $\mathbf{X} = (X_a, [p_{a\,a'}], A)$ and $\mathbf{Y} = (Y_b, [q_{b\,b'}], B)$ be inverse systems in the category HOMOT. A **system map** $\mathbf{f} = (f, f_b) : \mathbf{X} \to \mathbf{Y}$ consists of a function $f : B \to A$ and maps $f_b : X_{f(b)} \to Y_b$ for $b \in B$ such that if $b < b'$ in B, then there exists $a \in A$ with $f(b), f(b') < a$ and so that

$$[q_{b\,b'} f_{b'} p_{f(b')\,a}] = [f_b p_{f(b)\,a}].$$

The system map $\mathbf{g} = (g, g_b) : \mathbf{X} \to \mathbf{Y}$ is said to be **equivalent** to $\mathbf{f}$ if for each $b \in B$ there exists $a \in A$ such that $f(b), g(b) < a$ and

$$[f_b p_{f(b)\,a}] = [g_b p_{g(b)\,a}].$$

Composition of system maps is defined simply by composing the maps of the index sets to get a new index map, and then composing the maps in HOMOT indicated by the new index map (p.5 of [MS1]). The relation indicated in Definition 2.2 for system maps is an equivalence relation, and it turns out that composition of equivalence classes is well-defined by composing representatives. This composition of equivalence classes (but with the same set of objects) determines a category, referred to as the Pro-HOMOT category. One can now state a theorem about this.

2.3. THEOREM. *Let X and Y be topological spaces. Then X and Y have the same shape,* $\operatorname{sh} X = \operatorname{sh} Y$, *if and only if for each HPOL-expansion* $\mathbf{X} = (X_a, [p_{a\,a'}], A)$ *of X and HPOL-expansion* $\mathbf{Y} = (Y_b, [q_{b\,b'}], B)$ *of Y there exist system maps* $\mathbf{f} = (f, f_b) : \mathbf{X} \to \mathbf{Y}$ *and* $\mathbf{g} = (g, g_a) : \mathbf{Y} \to \mathbf{X}$ *such that the compositions* $\mathbf{gf}$ *and* $\mathbf{fg}$ *are equivalent to the respective identity system maps* $1_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ *and* $1_{\mathbf{Y}} : \mathbf{Y} \to \mathbf{Y}$.

We note that these identity system maps are, as expected, those which are induced by the identity on everything in sight.

To have an abstract theory of shape, one just replaces the categories HOMOT and HPOL by any pair $\mathscr{H}$ and $\mathscr{P}$ of categories, and simply requires that $\mathscr{P}$ be dense in $\mathscr{H}$. Thus we demand that each object X in $\mathscr{H}$ have a $\mathscr{P}$-expansion. Then Theorem 2.3 with $\mathscr{P}$ in place of HPOL becomes true for the shape theory defined in this particular sense–it is a shape theory for the objects in the category $\mathscr{H}$ (relative to $\mathscr{P}$). So, in order to define our theory of n-shape for compact Hausdorff spaces, we are obliged to tell what our categories $\mathscr{H}$ and $\mathscr{P}$ will be and then to prove that the latter is dense in the former.

First let us recall a definition.

2.4. *Definition.* Let $f, g : X \to Y$ be maps of compacta and $n \in \mathbb{N} \cup \{0\}$. Then we say that f is **n–homotopic** to g if for each compactum B with $\dim B \leqslant n$ and map $h : B \to X$, it is true that $fh \simeq gh$. One readily determines that the relation of being n-homotopic is an equivalence relation. We write $f \overset{n}{\simeq} g$ to indicate that f is n-homotopic to g and use $[f]_n$ to designate the equivalence class of f under this relation. Herein we shall use $\mathscr{H} = n$-HOMOT to denote the category whose objects are all compacta and whose morphisms are the equivalence classes of n-homotopic maps and where composition is (well-defined) by composing representatives of the given classes. The full subcategory $\mathscr{P} = n$-HPOL is determined by using only compact polyhedra as objects. These are going to be our $\mathscr{H}$ and $\mathscr{P}$ for the theory of n-shape.

Here is an obvious fact.

2.5. LEMMA. *Let $f, g : X \to Y$ be maps of compacta such that $f \simeq g$. Then $f \overset{n}{\simeq} g$.*

Our task, according to the above discussion, is simply to show that n-HPOL is a dense subcategory of n-HOMOT. Having done that, our definition of n-shape (n-sh) will be as in the above cited references or, more simply, by analogy with Theorem 2.3.

3. Approximate Systems

Approximate (inverse) systems $\mathbf{X}=(X_a, \epsilon_a, p_{a\,a'}, A)$ over directed indexing sets $(A, \leqslant)$ will be taken as in [MR1]. Thus the meshes $\epsilon_a > 0$ are numerical (in contradistinction with the open cover meshes of [MW]). Since we shall make heavy use of the approximate system concept, let us review the definition now.

3.1. Definition. An approximate (inverse) system of metric compacta $\mathbf{X} = (X_a, \epsilon_a, p_{a\,a'}, A)$ consists of the following: A directed ordered set $(A, \leqslant)$ with no maximal element; for each $a \in A$, a compact metric space X_a with metric $d = d_a$ and a real number $\epsilon_a > 0$; for each pair $a \leqslant a'$ from A, a map $p_{a\,a'} : X_{a'} \to X_a$, satisfying the following conditions:

(A1) $d(p_{a_1\,a_2}p_{a_2\,a_3}, p_{a_1\,a_3}) \leqslant \epsilon_{a_1}$, $a_1 \leqslant a_2 \leqslant a_3$; $p_{a\,a} = id$.

(A2) $(\forall a \in A)(\forall \eta > 0)(\exists a' \geqslant a)(\forall a_2 \geqslant a_1 \geqslant a')\, d(p_{a\,a_1}p_{a_1\,a_2}, p_{a\,a_2}) \leqslant \eta$.

(A3) $(\forall a \in A)(\forall \eta > 0)(\exists a' \geqslant a)(\forall a'' \geqslant a')(\forall x, x' \in X_{a''})\,(d(x,x') \leqslant \epsilon_{a''}) \Rightarrow (d(p_{a\,a''}(x), p_{a\,a''}(x')) \leqslant \eta)$.

We refer to the numbers ϵ_a as the meshes of $\mathbf{X}$. We say that $\mathbf{X}$ is cofinite if A is cofinite, i.e., every element $a \in A$ has only finitely many predecessors.

For such a system there is always a limit lying in $\prod_{a \in A} X_a$ (called the canonical limit in [MW]), which we define as follows.

3.2. Definition. A point $x = (x_a) \in \prod X_a$ belongs to $X = \lim \mathbf{X}$ provided for every $a \in A$,

$$x_a = \lim_{a_1}(p_{a\,a_1}(x_{a_1})).$$

The natural projection $p_a : X \to X_a$ is nothing but the restriction of the coordinate projection of $\prod X_a$ to X_a.

It is known, for example, that if each $X_a \neq \emptyset$, then X is a nonempty, compact Hausdorff space. As is well known ([MS1]), each compactum X can be written as the limit of an ordinary inverse system (that is, one in which the diagrams of maps in (A1) commute) of compact polyhedra X_a with a cofinite indexing set A. If $\dim X = 0$, then we may assume that $\dim X_a = 0$ for all $a \in A$. Applying Remark 2 and Proposition 1 of [MR1], we may conclude the following fact.

3.3. PROPOSITION. *Let X be a nonempty compact Hausdorff space. Then there exists an approximate system $\mathbf{X} = (X_a, \epsilon_a, p_{a\,a'}, A)$ of nonempty compact polyhedra X_a, such that $X = \lim \mathbf{X}$, A is cofinite, and $\mathbf{X}$ is commutative in the sense that*

$p_{a a'} \circ p_{a' a''} = p_{a a''}$ *whenever* $a \leqslant a' \leqslant a''$. *If* $\dim X = 0$, *then we may assume that* $\dim X_a = 0$ *for all* $a \in A$.

Here are two important attributes of approximate systems of compacta, the so-called "resolution" properties.

3.4. PROPOSITION. *Let* $\mathbf{X} = (X_a, \epsilon_a, p_{a a'}, A)$ *be an approximate system of compact metric spaces* X_a *and* $X = \lim \mathbf{X}$. *Then the following statements are true:*

(E1) *For every map* $h : X \to P$ *into a compact polyhedron* P, *there exist an* $a \in A$ *and a map* $f : X_a \to P$ *such that* $h \simeq f p_a$.

(E2) *If* P *is a compact polyhedron,* $a \in A$, *and* $f, f' : X_a \to P$ *are maps such that* $f p_a \simeq f' p_a$, *then there exists an* $a' \geqslant a$ *such that* $f p_{a a'} \simeq f' p_{a a'}$.

The validity of these statements is demonstrated in the proof of Theorem 7 of [MS2]; their truth also can be obtained from Theorem 4.2 of [MW].

4. The Core of a Complex

In Sections 5–8 of [MR2] the authors produced, for a given compact Hausdorff space X with $\dim_{\mathbb{Z}} X \geqslant 1$, two approximate systems $\mathbf{X}$ and $\mathbf{Z}$, and a map f of $Z = \lim \mathbf{Z}$ onto $X = \lim \mathbf{X}$. This was done in such a way that $\mathbf{Z}$ was an approximate system of metrizable compacta Z_a with $\dim Z_a \leqslant \dim_{\mathbb{Z}} X$ for each index a. It was required there that $\dim_{\mathbb{Z}} X < \infty$, but we are not concerned with that here.

We want to perform such a construction just with the assumption that $\dim X \geqslant 0$. We will not determine any properties of the fibers of the map f, nor will it be necessary that f is surjective, but in spite of that and because of the changed preconditions, the theory as given in [MR2] unfortunately does not contain what we need. To avoid repeating everything that was done there, we will review the main ideas and indicate what changes could be made to produce the desired effects.

The notions of the "n-dimensional core" and the "stacked n-dimensional core" of a finite (simplicial) complex K were introduced in Section 5 of [MR2]. We shall describe these metrizable, compact, n-dimensional spaces which were respectively denoted, Z_K and Z_K^*.

We denote by K^k the k-th iterated barycentric subdivision of K. For any complex L, $L^{(n)}$ will designate its n-skeleton. Now for each $k \geqslant 0$, select a simplicial approximation $q_{k\,k+1} : K^{k+1} \to K^k$ of the identity map $1 : |K^{k+1}| \to |K^k|$ and let $q_{k\,k+j} = q_{k\,k+1} \cdots q_{k+j-1\,k+j} : K^{k+j} \to K^k$. Restricting these simplicial maps to the n-skeleta, treating them as maps of polyhedra, and noting that

$$q_{k\,k+1}(|(K^{k+1})^{(n)}|) \subset |(K^k)^{(n)}|,$$

we get an inverse sequence

$$\mathbf{K} = (|(K^k)^{(n)}|, q_{k\,k+1})$$

of compact n-dimensional polyhedra. By definition,

$$Z_K = \lim \mathbf{K}.$$

As is shown in [MR2], there is a special map $f_K : Z_K \to |K|$ which is surjective and which is the limit of the Cauchy sequence (q_k) consisting of the coordinate projections $q_k : Z_K \to |(K^k)^{(n)}| \subset |K|$.

The stacked n-dimensional core Z_K^* of K can be written (see p. 65 of [MR2] for details) as the disjoint union of Z_K and an open subspace which is the topological sum of the polyhedra of the n-skeleta of the sequence of k-th barycentric subdivisions of K. This is written

$$Z_K^* = \left(\bigoplus_{k \geqslant 0} |(K^k)^{(n)}| \right) \bigcup Z_K.$$

There is a map $f_K^* : Z_K^* \to |K|$ which equals f_K on Z_K and is the inclusion on each of the other summands (see items (21), (22) of Section 5 of [MR2]).

There are further (see (17), (18) of Section 5 of [MR2]) the retractions $q_k^* : Z_K^* \to |K^{*k}|$, where

$$K^{*k} = (K^0)^{(n)} \oplus \cdots \oplus (K^k)^{(n)},$$
$$q_k^* |\, |(K^{k+j})^{(n)}| = q_{k\,k+j}, \ j \geqslant 1,$$

and

$$q_k^* |\, Z_K = q_k.$$

We plan to mimic the construction of a certain approximate system $\mathbf{Z}$ as found in sections (6)-(8) of [MR2], but with a somewhat different set of hypotheses. We ask the reader to refer to that paper as we proceed with the following. Assume for the remainder of this section that X is a compact Hausdorff space with $\dim X \geqslant 0$, $\mathbf{X} = (X_a, \epsilon_a, p_{a\,a'}, A)$ is an approximate (commutative) system satisfying Proposition 3.3, and that $n \geqslant 0$ is an integer.

We may assume that $\dim X_a = 0$ for all a in case $\dim X = 0$, and that $p_{a_1\,a_2}(X_{a_2})$ is infinite for all $a_1 < a_2$ if $\dim X \geqslant 1$.

Each X_a is given a triangulation K_a satisfying (1) of Section 6 of [MR2], that is,

$$6\,\mathrm{mesh}(K_a) \leqslant \epsilon_a, \quad a \in A.$$

We shall define an order $\leqslant'$ on A. Consider any $a_1 < a_2$, integer $k \geqslant 0$, and the statement,

$$(1) \quad d(x, x') \leqslant \epsilon_{a''} \Rightarrow d(p_{a_1\,a''}(x), p_{a_1\,a''}(x')) \leqslant \mathrm{mesh}(K_{a_1}^k) \text{ for } a'' \geqslant a_2.$$

Put $a_1 <' a_2$ provided $a_1 < a_2$ and condition (1) holds for $k = 0$. Then define $a_1 \leqslant' a_2$ if either $a_1 <' a_2$ or $a_1 = a_2$. Next we have analogues of Lemmas 2–4 of [MR2].

4.1. LEMMA. *$(A, \leqslant')$ is a directed set with no maximal element. For any $a_1 \in A$ and integer $k \geqslant 0$, there exists $a_2 > a_1$ such that (1) is true.*

Proof. See the proof of Lemma 2 of [MR2]. The only adjustment needed is that in case $\dim X = 0$, then $\mathrm{mesh}\, K_{a_1}^k = 0$; so one should choose $\eta > 0$ to be a number with the property that $d(v, w) > \eta$ for any two vertices v, w of $K_{a_1}^k$.

4.2. LEMMA. *If* $\dim X \geqslant 1$ *and* $a_1 < a_2$, *then the set of integers* $k \geqslant 0$ *which satisfy (1) is finite.*

Proof. The proof of Lemma 3 of [MR2] shows that if this set of integers is infinite, then $p_{a_1 a_2}$ maps X_{a_2} to a finite set, contrary to our above assumption.

4.3. Definition. Whenever $a_1 <' a_2$, define $k(a_1, a_2)$ to be the maximal integer satisfying (1) if $\dim X \geqslant 1$ and to be equal to the number of predecessors of a_2 relative to $\leqslant'$ if $\dim X = 0$.

Lemma 4 of [MR2] reduces to the following.

4.4. LEMMA. *If* $a_1 <' a_2$, *then (1) holds for* $k = k(a_1, a_2)$ *and*

(2) $d(p_{a_1 a'} p_{a'}, p_{a_1}) \leqslant \operatorname{mesh}(K^k_{a_1})$ *for* $a' \geqslant a_2$,

(3) $k(a_1, a_2) \leqslant k(a_1, a_3)$ *whenever* $a_2 \leqslant a_3$.

Furthermore, for $a_1 \in A$ *and any integer* $k \geqslant 0$, *there is an* $a_2 \in A$ *such that* $a_1 <' a_2$ *and*

(4) $k \leqslant k(a_1, a_2)$.

Proof. See the proof of Lemma 4 of [MR2] (changing (7)–(9) to (8)–(10) therein).

For the reader trying to compare this work with that in [MR2], the conditions (2)–(4) on page 67 are replaced by our single condition (1). Condition (2) of [MR2] is automatically true by commutativity. The last one, (4), is not relevant to this exposition.

We do not need Lemma 5 of [MR2], but rather just the following. Whenever $a_1 <' a_2$, then select a map

$$g_{a_1 a_2} : |K^{(n)}_{a_2}| \to |(K^k_{a_1})^{(n)}|$$

where $k = k(a_1, a_2)$, in such a manner that

$$d(g_{a_1 a_2}, p_{a_1 a_2} |\, |K^{(n)}_{a_2}|) \leqslant 2 \operatorname{mesh}(K^k_{a_1}).$$

This can be done, for example, by a simplicial approximation, so we state,

4.5. LEMMA. *The maps* $g_{a_1 a_2} : |K^{(n)}_{a_2}| \to |(K^k_{a_1})^{(n)}| \hookrightarrow |K_{a_1}|$ *and* $p_{a_1 a_2} |\, |K^{(n)}_{a_2}| : |K^{(n)}_{a_2}| \to |K_{a_1}|$ *are homotopic.*

Item (17) of Lemma 6 of [MR2] is replaced by the preceding. For the next, compare with (18)–(20) after Lemma 6 of [MR2].

4.6. Definition. Let a and $a_1 <' a_2$ in A; we put

(a) $Z^*_a = Z^*_{K_a}$,

(b) $r_{a_1 a_2} = g_{a_1 a_2} q^*_{0\, a_2} : Z^*_{a_2} \to Z^*_{a_1}$.

Here $q^*_{0\, a_2} : Z^*_{a_2} \to |K^{(n)}_{a_2}|$ is the map $q^*_0 : Z^*_K \to |K^{(n)}|$ where $K = K_{a_2}$. Note that

(c) $r_{a_1 a_2}(Z^*_{a_2}) \subset |(K^k_{a_1})^{(n)}|$ where $k = k(a_1, a_2)$.

Observing that the maps $q^*_{0a_2}$ indicated in (b) land in $|K^{(n)}_{a_2}|$, one sees that $r_{a_1 a_2}$ is well-defined. One can check that all the steps in the proof of Lemma 7 of [MR2] (see section 7) hold true as long as one always replaces the superscript $(n+1)$ by (n). (In case $\dim X = 0$, then (15) on page 71 of [MR2] is true because always $\operatorname{mesh} K^j_a = 0$). Let us state the result analogous to Lemma 7 of [MR2].

4.7. PROPOSITION. $\mathbf{Z} = (Z^*_a, \epsilon_a, r_{a a'}, (A, \leqslant'))$ *is an approximate system of nonempty metric compacta* Z^*_a *with* $\dim Z^*_a \leqslant n$. *The limit* $Z = \lim \mathbf{Z}$ *is a nonempty compact Hausdorff space with* $\dim Z \leqslant n$ *and* $\operatorname{wt} Z \leqslant \mathit{card}(A) \leqslant \operatorname{wt} X$. *For each* $a \in A$, *there is a fixed triangulation* K_a *of* X_a, *and* Z^*_a *is the stacked n-dimensional core of* K_a.

We shall denote the projections $r_a : Z \to Z^*_a$.

Proceeding now to Section 8 of [MR2], note that we define

$$f^*_a = f^*_{K_a} : Z^*_a \to X_a$$

just as done there. Then Lemma 10 of [MR2] holds true for the current situation, and we state this result as follows.

4.8. PROPOSITION. *There is a map* $f : Z \to X$ *such that for all* $a \in A$,

$$f^*_a r_a = p_a f.$$

Later we are going to need information about maps of the form $q^*_{0a} : Z^*_a \to |K^{(n)}_a|$. Specifically,

4.9. LEMMA. *For each* $a \in A$, *let* $j : |K^{(n)}_a| \hookrightarrow Z^*_a$ *denote the inclusion map. Then* $q^*_{0a} j : |K^{(n)}_a| \to |K^{(n)}_a|$ *is the identity.*

Proof. This is an obvious feature of the construction given on pages 65–66 of [MR2].

Regarding maps f^*_a, the formula (21) on p. 66 of [MR2] shows that,

4.10. LEMMA. *For each* $a \in A$, $f^*_a \mid |(K^k_a)^{(n)}|$ *is inclusion into* $|K_a|$.

5. n-shape for Arbitrary Compacta

5.1. THEOREM. *For each* $n \geqslant 0$, *the category n-HPOL is a dense subcategory of n-HOMOT. Indeed, for each compact Hausdorff space X and approximate system* $\mathbf{X} = (X_a, \epsilon_a, p_{a a'}, A)$ *of compact polyhedra* X_a *such that* $\mathbf{X}$ *satisfies the conditions of Proposition 3.3, the system* $(X_a, [p_{a a'}]_n, A)$ *is an n-HPOL-expansion of X.*

Proof. First, $(X_a, [p_{a a'}]_n, A)$ is an inverse system in the category n-HPOL because of commutativity in $\mathbf{X}$. Now we have to show that (i)-(iii) of Definition 2.1 are true for the current situation. Item (i) follows by the same argument we just used. Item (ii) is obtained from (E1) in Proposition 3.4 and Lemma 2.5.

Getting (iii) is different. We must show that for $a \in A$, compact polyhedron Q, and for maps $f_a, g_a : X_a \to Q$ having the property that $f_a p_a \overset{n}{\simeq} g_a p_a$, there exists $a' \in A$, $a \leqslant a'$, such that $f_a p_{a a'} \overset{n}{\simeq} g_a p_{a a'}$.

Let $\mathbf{Z}$, its limit $Z = \lim \mathbf{Z}$ and $f : Z \to X$ be as in Section 4 (Propositions 4.7, 4.8) employing the current value of n and the approximate system $\mathbf{X}$ of the hypothesis, which does satisfy the conditions of Proposition 3.3. We know therefore that $\dim Z \leqslant n$. This shows that the maps $f_a p_a f$ and $g_a p_a f$ are homotopic. Using Proposition 4.8, we get,

$$f_a f_a^* r_a \simeq g_a f_a^* r_a.$$

An application of (E2) in Proposition 3.4 and the definition of $<'$ detects an $a' \geqslant a$ such that

$$f_a f_a^* r_{a a'} \simeq g_a f_a^* r_{a a'}.$$

To show that this choice of a' works for (iii), let B be a compact space with $\dim B \leqslant n$ and $h : B \to X_{a'}$ be a map. Our proof will be complete if we can show that $f_a p_{a a'} h \simeq g_a p_{a a'} h$.

By definition, the map $r_{a a'}$ is $g_{a a'} q^*_{0a'} : Z^*_{a'} \to Z^*_a$. So,

$$f_a f_a^* g_{a a'} q^*_{0a'} \simeq g_a f_a^* g_{a a'} q^*_{0a'}.$$

There is no loss of generality in assuming that $h(B) \subset |K^{(n)}_{a'}| \subset Z_{a'} \subset Z^*_{a'}$. Because of this and Lemma 4.9, $g_{a a'} q^*_{0a'} h = g_{a a'} h : B \to Z^*_a$. So we may write,

$$f_a f_a^* g_{a a'} h \simeq g_a f_a^* g_{a a'} h.$$

Using Lemma 4.10, this simplifies to,

$$f_a g_{a a'} h \simeq g_a g_{a a'} h.$$

Finally, an application of Lemma 4.5 gives us the needed result, that

$$f_a p_{a a'} h \simeq g_a p_{a a'} h.$$

Our proof is complete.

5.2. COROLLARY. *For each $n \geqslant 0$, there is a theory of n-shape for the category of all compact Hausdorff spaces.*

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