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HYPERBOLAS, ORTHOLOGY, AND ANTIPEDAL TRIANGLES

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Abstract. We obtain several characterisations of the Kiepert, Jarabek, and Feuerbach hyperbolas of a triangle ABC using the antipedal triangles of a variable point P in the plane and the notion of orthologic triangles. Our arguments are algebraic and use complex numbers.

1. Introduction

Among conics that pass through the vertices A, B, C of a scalene triangle ABC and its orthocenter H – all of them are equilateral hyperbolas – the most interesting are Feuerbach, Kiepert, and Jarabek hyperbolas. These are hyperbolas that go through the incenter I, the centroid G, and the circumcenter O, respectively. They have been extensively studied in the past. The following are some more recent papers that consider them: [1], [2], [6], [7], [5], [12], [21], [20], and [24].

In this paper we shall present new characterisations of the Kiepert, Jarabek, and Feuerbach hyperbolas associated to a triangle ABC. We shall use the same method for all three hyperbolas. Our idea is to associate to every point P its antipedal triangle $P^aP^bP^c$ and to look for triangles XYZ having the property that P lies on a hyperbola if and only if the triangles $P^aP^bP^c$ and XYZ are orthologic.

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars from the vertices of ABC on the sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC, XYZ]$. It is well-known (see [8] or [17]) that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars from the vertices of XYZ on the sides BC, CA, and AB of ABC are concurrent at the point $[XYZ, ABC]$.

In this definition and throughout this paper all triangles are nondegenerate, that is, their vertices are not collinear. The last assumption implies that in our approach we must exclude some points P so that ours are characterisations of three named hyperbolas without a small number of their points.

For a triangle ABC, let $W(ABC)$ denote the complement in the plane of the union of the side lines BC, CA, AB and the circumcircle γ_0 of ABC. Recall that for a point P in $W(ABC)$, the *antipedal triangle* $P^aP^bP^c$ *of* P has intersections of

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perpendiculars from A, B, C on AP, BP, CP as vertices. Let $\mathscr{A}$ denote the function that associates to a point P its antipedal triangle $P^a P^b P^c$.

The domain of the function $\mathscr{A}$ is $W(ABC)$ because it follows immediately from Thales Theorem and incidence arguments that the points P^a, P^b, and P^c coincide if and only if P lies on the circumcircle γ_0.

Let γ be a curve in the plane. Let $\mathscr{F}$ be a function from a subset S of the plane that associates to each point P of S a triangle $\mathscr{F}(P)$. A triangle XYZ is called $(\mathscr{F}, \gamma)$-*generating* provided XYZ is orthologic to $\mathscr{F}(P)$ if and only if a point P is in the set $\gamma \cap S$.

Let γ_F, γ_J, and γ_K denote the Feuerbach, Jarabek, and Kiepert hyperbola of the triangle ABC, respectively. With the above definitions and notation we can formulate the results of this paper as contributions to the following problem.

PROBLEM. *For* $\gamma \in \{\gamma_F, \gamma_J, \gamma_K\}$, *find* $(\mathscr{A}, \gamma)$-*generating triangles.*

Observe that when we know that, for $\gamma \in \{\gamma_F, \gamma_J, \gamma_K\}$, a triangle XYZ is $(\mathscr{A}, \gamma)$-generating, then we have the following characterisation of γ:

The hyperbola γ *is the closure of all points* P *in* $W(ABC)$ *such that the triangles* $P^a P^b P^c$ *and* XYZ *are orthologic.*

The vertices of the triangles XYZ which we prove in this paper to provide solutions to the above problem are all endpoints of segments of controlled length perpendicular to sides of ABC. A more formal description uses the following notation.

For a triple $h = (s_1, s_2, s_3)$ of real numbers and for triangles ABC and XYZ, let $[ABC, XYZ, h]$ denote the triangle UVW such that UX, VY, WZ are perpendicular to BC, CA, AB and the directed distances $|UX|$, $|VY|$, $|WZ|$ are equal to s_1, s_2, s_3, respectively. When $s_1 = 0$, we put $U = X$, and we do similar assignments when s_2 and s_3 are zero. For $s_1 > 0$ the vector $\vec{XU}$ points towards outside of ABC while for $s_1 < 0$ it points towards inside.

For an expression ε in terms of side lengths a, b, and c of the triangle ABC and a real number h, let $\varepsilon[h]$ denote the triple $(h\,\varepsilon,\ h\,\varphi(\varepsilon),\ h\,\psi(\varepsilon))$. More precisely, the coordinates $\varepsilon[h]_1$, $\varepsilon[h]_2$, $\varepsilon[h]_3$ of $\varepsilon[h]$ are products with h of ε, the first cyclic permutation $\varphi(\varepsilon)$ of ε, and the second cyclic permutation $\psi(\varepsilon)$ of ε, respectively. For example, $a[h] = (h\,a,\ h\,b,\ h\,c)$ and if $w_a = \frac{b+c-a}{2}$, $w_b = \frac{c+a-b}{2}$, and $w_c = \frac{a+b-c}{2}$, then $w_a[h] = (h\,w_a,\ h\,w_b,\ h\,w_c)$.

With this notation at hand, we can describe our task in this paper as a search for expressions ε and points X, Y, and Z in the plane of the triangle ABC such that the triangles $[ABC, XYZ, \varepsilon[h]]$ are $(\mathscr{A}, \gamma)$-generating for γ either γ_F, γ_J, or γ_K.

Recall that the triangles $[ABC, XYZ, \varepsilon[h]]$ have already been used for characterisations of Kiepert and Feuerbach hyperbolas. Indeed, the original description of the Kiepert hyperbola is that it is the locus of centers of perspective of triangles ABC and $X_hY_hZ_h$, where $X_hY_hZ_h = [ABC, A_mB_mC_m, a[h]]$ and A_m, B_m, C_m are midpoints of sides (see [6]).

Another application of triangles $X_hY_hZ_h$ on vertices of similar isosceles triangles build on sides of ABC is a result in [13] which shows that triangles ABC and $X_hY_hZ_h$ are orthologic and the point $[ABC,\ X_hY_hZ_h]$ traces again the Kiepert hyperbola as h goes through the reals.

The Feuerbach hyperbola is the locus of centers of perspective of triangles ABC and $P_hQ_hR_h$, where $P_hQ_hR_h = [ABC,\ A_pB_pC_p,\ 1[h]]$ and A_p, B_p, C_p are projections of the incenter onto sides (see [11]).

Another application of triangles $P_hQ_hR_h$, whose vertices are intersections of circles concentric to the incircle with perpendiculars through incenter to sides, is a result which shows that triangles ABC and $P_hQ_hR_h$ are orthologic and the point $[ABC,\ P_hQ_hR_h]$ again traces the Feuerbach hyperbola as h goes through the reals.

2. Preliminaries on complex numbers

We shall use complex numbers because they lead to the simplest expressions. Hence, our proofs are entirely algebraic. Every book on the use of complex numbers in geometry from the references below gives excellent and adequate introductions to this technique of proof. In this section we give only the most basic notions and conventions.

A point P in the Gauss plane is represented by a complex number p. This number is called the *affix* of P and we write $\widetilde{P} = p$ or $P(p)$ to indicate this. The complex conjugate of p is denoted $\bar{p}$. However, we shall be avoiding this notation by using next letter (now letter q) for the complex conjugate and sometimes write $P(p,\ q)$ or $\widetilde{P} = (p,\ q)$ in order to describe affix of a point and to describe its complex conjugate. In order to avoid quotients, we shall use z^* for $1/z$.

In the sections on the Kiepert and Jarabek hyperbolas, we follow the standard assumption that the vertexes A, B, and C of the reference triangle are represented by numbers u, v, and w on the unit circle so that the circumcentre O of ABC is the origin. Hence, the affix of O is number 0 (zero) and complex conjugates of u, v, and w are $1/u$, $1/v$, and $1/w$ (or, in our notation, u^*, v^*, and w^*).

Most interesting points, lines, circles, curves,... associated with the triangle ABC are expressions that involve symmetric functions of u, v, and w that we denote as follows.

$$\sigma = u + v + w, \quad \tau = vw + uw + uv, \quad \mu = uvw,$$

$$\sigma_a = -u + v + w, \quad \sigma_b = u - v + w, \quad \sigma_c = u + v - w,$$

$$\tau_a = -vw + wu + uv, \quad \tau_b = vw - wu + uv, \quad \tau_c = vw + wu - uv,$$

$$\mu_a = vw, \quad \mu_b = wu, \quad \mu_c = uv, \quad \delta_a = v - w,$$

$$\delta_b = w - u, \quad \delta_c = u - v, \quad \zeta_a = v + w, \quad \zeta_b = w + u, \quad \zeta_c = u + v.$$

For each $k \geqslant 2$, σ_k, σ_{ka}, σ_{kb}, and σ_{kc} are derived from σ, σ_a, σ_b, and σ_c with the substitution $u = u^k$, $v = v^k$, $w = w^k$. In a similar fashion we can define analogous expressions using letters τ, μ, δ, and ζ. We shall use corresponding small Latin letters to denote analogous symmetric functions in a, b, and c (lengths of sides of ABC). For example, $m = a\,b\,c$, $s = a + b + c$, $t = b\,c + c\,a + a\,b$, $z_a = b + c$, and $s_{2a} = b^2 + c^2 - a^2$.

The expressions which appear in triangle geometry usually depend on sets that are of the form $\{a,\ b,\ c,\ \ldots,\ x,\ y,\ z\}$ (that is, union of triples of letters). Let φ and ψ stand for permutations $|\,b,\ c,\ a,\ \ldots,\ y,\ z,\ x\,|$ and $|\,c,\ a,\ b,\ \ldots,\ z,\ x,\ y\,|$.

Let $f = f(x,\ y,\ \ldots)$ be an expression that depends on a set $S = \{x,\ y,\ \ldots\}$ of variables and let $\varrho : S \to S$ be a permutation of S. Then f^{ϱ} is a short notation for $f(\varrho(x),\ \varrho(y),\ \ldots)$. For permutations $\varrho,\ \ldots,\ \xi$ of S we shall use $\mathbb{S}_{\varrho,\ldots,\xi}\, f$ and $\mathbb{P}_{\varrho,\ldots,\xi}\, f$ to shorten $f + f^{\varrho} + \cdots + f^{\xi}$ and $f f^{\varrho} \ldots f^{\xi}$. Finally, $\mathbb{S}f$ and $\mathbb{P}f$ replace $\mathbb{S}_{\varphi,\psi} f$ and $\mathbb{P}_{\varphi,\psi} f$.

For real numbers f, g, and h, let $\langle f,\ g,\ h\rangle$ be a notation for $-f\mu + \mathbb{S}u^2(gv + hw)$. For example, $\langle 6,\ 1,\ 1\rangle = u^2v + u^2w + v^2u + v^2w + w^2u + w^2v - 6uvw$. Let S be the area of ABC.

Since points, lines, conics, ... associated to a triangle often appear in triples in which two members are build from a third by appropriate permutation, we shall often give only one of them while the other two (*relatives*) are obtained from it by cyclic permutations.

Let us close these preliminaries with few words on analytic geometry that we shall use.

In triangle geometry lines play an important role so that we have special notation $[f,\ g,\ h]$ for the set of all points $P(p,\ q)$ that satisfy the equation $f\,p + g\,q + h = 0$. This set is a line iff there is a complex number $z \neq 0$ such that $z\,g$ is the complex conjugate of $z\,f$ and $z\,h$ is a real number.

Let $X(x,\ a)$, $Y(y,\ b)$, and $Z(z,\ c)$ be three points and let ℓ be a line $[f,\ g,\ h]$ in the plane. Then the line XY is $[a - b,\ y - x,\ b\,x - a\,y]$, the parallel to ℓ through X is $[f,\ g,\ -g\,a - f\,x]$ and the perpendicular to ℓ through X is $[f,\ -g,\ g\,a - f\,x]$. The conditions for points X, Y, and Z to be collinear and for lines $\ell_1 = [f,\ g,\ h]$, $\ell_2 = [k,\ m,\ n]$, and $\ell_3 = [r,\ s,\ t]$ to be concurrent are $\Delta = 0$ and $\Gamma = 0$, where

$$\Delta = \Delta(X,\ Y,\ Z) = \begin{vmatrix} x & a & 1 \\ y & b & 1 \\ z & c & 1 \end{vmatrix}, \qquad \text{and} \qquad \Gamma = \Gamma(\ell_1,\ \ell_2,\ \ell_3) = \begin{vmatrix} f & g & h \\ k & m & n \\ r & s & t \end{vmatrix}.$$

Moreover, the (oriented) area of the triangle XYZ is $\frac{I\Delta}{4}$, where $I = \sqrt{-1}$ is the imaginary unit.

There are some notable exceptions to the rule that complex numbers give simpler expressions than trilinear and barycentric coordinates. As a convenience to readers that are more familiar with these coordinates, we shall describe transformation formulas between all three systems and present these shorter forms in relevant cases.

The above formula for the area of a triangle implies that a point $P(p,\ q)$ has with respect to the base triangle ABC the absolute barycentric coordinates $(\alpha,\ \beta,\ \gamma)$, where $\alpha = \frac{-u\, n_a}{\delta_b\, \delta_c}$ and β and γ are relatives of α and $n_a = p + \mu_a\, q - \zeta_a$. It follows that a point P with barycentric coordinates $(\alpha,\ \beta,\ \gamma)$ has affix

$$p = \frac{u\,\alpha + v\,\beta + w\,\gamma}{\alpha + \beta + \gamma}.$$

In order to have connection between complex and trilinear coordinates it suffices to recall that absolute trilinear coordinates $(x,\ y,\ z)$ and absolute barycentric coordinates $(\alpha,\ \beta,\ \gamma)$ of the same point are related by formulas

$$\alpha = \frac{a\,x}{2\,S}, \qquad \beta = \frac{b\,y}{2\,S}, \qquad \gamma = \frac{c\,z}{2\,S},$$

where S is the area of ABC and the lengths of the sides are

$$a = |\delta_a| = \sqrt{2 - \zeta_{2a}\,\mu_a^*}, \quad b = |\delta_b| = \sqrt{2 - \zeta_{2b}\,\mu_b^*}, \quad c = |\delta_c| = \sqrt{2 - \zeta_{2c}\,\mu_c^*}.$$

3. Statements of results

Let $n_F = 6$ and $n_J = n_K = 5$. For $X = K,\ F,\ J$, let X_i, for $i = 1,\ \ldots,\ n_X$, denote the following expressions.

$$K_1 = a, \quad K_2 = a\,z_a^*, \quad K_3 = a^2\,z_a^*, \quad K_4 = a\,s_a\,z_a^*, \quad K_5 = a^3.$$

$$F_1 = 1, \quad F_2 = s_a^*, \quad F_3 = a\,s_a, \quad F_4 = a\,s_a^*, \quad F_5 = s_{2a}\,z_a\,s_a^*, \quad F_6 = z_a\,s_a^*.$$

$$J_1 = a^*, \quad J_2 = s_a\,z_a^*, \quad J_3 = a\,s_{2a}^*, \quad J_4 = z_a^*\,s_{2a}^*, \quad J_5 = a^*\,s_{2a}^*.$$

For $i = J_3, J_4, J_5$ we must assume in addition that ABC has no right angle.

THEOREM 1. *Let $h \neq 0$ be a real number. For any triangle PQR homothetic to the triangle ABC, for $X = K,\ J,\ F$, and for $i = 1,\ \ldots,\ n_X$, the triangle $[ABC,\ PQR,\ X_i[h]]$ is $(\mathscr{A},\ \gamma_X)$-generating.*

Remark. Since there can be at most two values of the parameter h for which the vertices of the "triangle" $[ABC,\ PQR,\ X_i[h]]$ are collinear, we must exclude these values in addition to the value $h = 0$. In the above statement this is implicit in the assumption that we consider only nondegenerate triangles.

In the above theorem the triangle PQR can be, for example, the triangle ABC, the complementary triangle $A_mB_mC_m$, the anticomplementary triangle $A_aB_aC_a$, the Euler triangle $A_fB_fC_f$, and the opposite triangle $A_sB_sC_s$, where A_m, B_m, C_m denote midpoints of sides of the triangle ABC, A_a, B_a, C_a intersections of parallels through vertices to sides, A_f, B_f, C_f midpoints of segments joining vertices with the orthocenter H, and A_s, B_s, C_s reflections of vertices at the circumcenter O.

THEOREM 2. *For any triangle PQR homothetic to the triangle ABC, for $X = K,\ F,\ J$, and for $i,\ j = 1,\ \ldots,\ n_X$, the triangle*

$$[ABC,\ [ABC,\ PQR,\ X_i[h]],\ X_j[k]]$$

is $(\mathscr{A},\ \gamma_X)$*-generating for all real numbers* h *and all real numbers* k *except the value* $-h\,a_{ij}$*, where* a_{ij} *is the* $(i,\ j)$ *entry of the matrix* M_X *with*

$$M_K = \begin{bmatrix} 1 & \frac{2z}{y} & \frac{2z}{v} & \frac{2z}{u} & 2x \\ \frac{y}{2z} & 1 & \frac{y}{v} & \frac{y}{u} & \frac{xy}{z} \\ \frac{v}{2z} & \frac{v}{y} & 1 & \frac{v}{u} & \frac{vx}{z} \\ \frac{u}{2z} & \frac{u}{y} & \frac{u}{v} & 1 & \frac{ux}{z} \\ \frac{1}{2x} & \frac{z}{xy} & \frac{z}{vx} & \frac{z}{ux} & 1 \end{bmatrix},$$

$$x = s_2^*, \quad y = s^2, \quad z = z_a\,z_b\,z_c, \quad u = 4\,s\,t - 4\,m - s^3, \quad v = s^3 - 2\,s\,t + 2\,m,$$

$$M_F = \begin{bmatrix} 1 & \frac{w}{v} & \frac{s}{x} & \frac{y}{x} & \frac{w}{z} & \frac{y}{u} \\ \frac{v}{w} & 1 & \frac{v}{xy} & \frac{v}{sx} & \frac{v}{z} & \frac{v}{su} \\ \frac{x}{s} & \frac{xy}{v} & 1 & \frac{y}{s} & \frac{xy}{z} & \frac{xy}{su} \\ \frac{x}{y} & \frac{sx}{v} & \frac{s}{y} & 1 & \frac{sx}{z} & \frac{x}{u} \\ \frac{z}{w} & \frac{z}{v} & \frac{z}{xy} & \frac{z}{sx} & 1 & \frac{z}{su} \\ \frac{u}{y} & \frac{su}{v} & \frac{su}{xy} & \frac{u}{x} & \frac{su}{z} & 1 \end{bmatrix},$$

$$x = 2\,m,\ y = s_a\,s_b\,s_c,\ z = s_{2a}\,s_{2b}\,s_{2c},\ u = 4\,s\,t - 6\,m - s^3,\ v = u + 2\,m,\ w = 16\,S^2,$$

$$M_J = \begin{bmatrix} 1 & \frac{z}{ms} & \frac{y}{4m^2} & \frac{yz}{msu} & \frac{y}{sx} \\ \frac{ms}{z} & 1 & \frac{sy}{4mz} & \frac{y}{u} & \frac{my}{xz} \\ \frac{4m^2}{y} & \frac{4mz}{sy} & 1 & \frac{4mz}{su} & \frac{4m^2}{sx} \\ \frac{msu}{yz} & \frac{u}{y} & \frac{su}{4mz} & 1 & \frac{mu}{xz} \\ \frac{sx}{y} & \frac{xz}{my} & \frac{sx}{4m^2} & \frac{xz}{mu} & 1 \end{bmatrix},$$

$$x = s_a\,s_b\,s_c, \quad y = s_{2a}\,s_{2b}\,s_{2c}, \quad z = z_a\,z_b\,z_c, \quad u = 4\,s\,t - 4\,m - s^3.$$

Remark. Observe that some important triangles related to the triangle ABC are of the form $[ABC,\ A_mB_mC_m,\ K_1[h]]$ for a suitable constant h. For example, the first Brocard triangle $A_bB_bC_b$ (for $h = 2\,S/s_2$), the Torricelli triangles $A_vB_vC_v$ and $A_uB_uC_u$ on vertices of equilateral triangles build on sides either towards outside or towards inside (for $h = \pm\sqrt{3}/2$), and Napoleon triangles $A_{vn}B_{vn}C_{vn}$ and $A_{un}B_{un}C_{un}$ on centers of these equilateral triangles (for $h = \pm\sqrt{3}/6$).

The orthic triangle $A_oB_oC_o$ and the three images triangle $A_rB_rC_r$ whose vertices are reflections of A, B, and C at opposite sides of ABC are of the form $[ABC,\ ABC,\ J_1[h]]$. Also, the tangential triangle $A_tB_tC_t$ (formed by tangents to the circumcircle at vertices of ABC) has the form $[ABC,\ A_mB_mC_m,\ J_3[h]]$.

Let v denote the expression $a\,(b^2 + c^2 - a^2)$.

THEOREM 3. *Let $k \neq 0$ and h be real numbers. For any triangle PQR homothetic to the triangle ABC, for $X = K, F, J$, and for $j = 1, \ldots, n_X$, the triangle*

$$[ABC, [ABC, PQR, v[h]], X_j[k]]$$

is $(\mathscr{A}, \gamma_X)$-generating.

THEOREM 4. *Let k and $h \neq 0$ be real numbers. For any triangle PQR homothetic to the triangle ABC, for $X = K, F, J$, and for $j = 1, \ldots, n_X$, the triangle*

$$[ABC, [ABC, PQR, X_j[h]], v[k]]$$

is $(\mathscr{A}, \gamma_X)$-generating.

Let $I_K = \{b, K, Kb, Km, u, ub, v, vb, un, vn\}$, $I_J = \{h, o, r, t, tr, w, Hh, Ho, O, Ot\}$, and $I_F = \{e, ep, er, k, kr, p, pp, Ie, Ik, Ip, Oi\}$. For each element i of these three sets we define a triangle $A_iB_iC_i$ by describing the vertex A_i. The vertices B_i and C_i have analogous descriptions. Let A_e be the centre of the A-excircle, A_{ep} the projection of A_e onto BC, the point A_{er} is the reflection of A_e at BC, the vertex A_k is the second intersection of the bisector of the angle A with the circumcircle, A_{kr} is the reflection of A_k at BC, the point A_p is the projection of the incentre I onto BC, the vertex A_{pp} is the projection of A_p onto AI, A_{Oi} is a projection onto BC of any point different from O on line IO joining the incentre with the circumcentre, A_{Ip} is a projection onto B_pC_p of any point different from central point X_{65} [10] on line IO, A_{Ie} and A_{Ik} are projections onto B_eC_e and B_kC_k of any point on IO different from the incentre I, the point A_b is the projection of the Grebe-Lemoine point K onto the perpendicular bisector of BC, the vertices A_K, A_{Kb}, and A_{Km} are the projection of any point different from the circumcentre O on the line KO onto BC, B_bC_b, and OA_m, vertices A_u and A_{un} are the vertex and the centre of the equilateral triangle build on BC towards inside, A_v and A_{vn} are the vertex and the centre of the equilateral triangle build on BC towards outside, A_{ub} and A_{vb} are projections of the Grebe-Lemoine points of $A_uB_uC_u$ and $A_vB_vC_v$ onto perpendicular bisectors of B_uC_u and B_vC_v, A_h is the second intersection of altitude line AH with the circumcircle, A_o is the projection of A onto BC, the point A_r is the reflection of A at BC, the intersection of tangents to the circumcircle at B and C is A_t, the reflection of A_t at BC is A_{tr}, A_w is the intersection of common tangents of the A-excircle with B-excircle and C-excircle, A_{Hh} and A_{Ho} are the projections onto B_hC_h and B_oC_o of any point X on the Euler line of ABC different from the orthocentre H, and A_O and A_{Ot} are the projections onto BC and B_tC_t of any point X on the Euler line of ABC different from the circumcentre O.

Some of the cases in the following theorem are clearly consequences of the previous theorem (for example, the first Brocard triangle $A_bB_bC_b$ has the form $[ABC, A_mB_mC_m, K_1[k]]$, for a suitable $k \neq 0$). Moreover, in some cases we must make additional assumptions about the triangle ABC. For example, for $i = b$, the triangle ABC can not be equilateral and for $i = t$ and $i = w$ it can not have right angle.

THEOREM 5. *For* $X = K, J, F$, *for* $i \in I_X$, *and for all real numbers* h, *the triangle* $[ABC, A_iB_iC_i, v[h]]$ *is* $(\mathscr{A}, \gamma_X)$*-generating.*

THEOREM 6. *For* $X = K, J, F$, *for any* $i \in I_X$, *and for all* $j, j' = 1, \dots, n_X$ *the triangles*

$$[ABC, [ABC, A_iB_iC_i, v[h]], X_j[k]],$$
$$[ABC, [ABC, A_iB_iC_i, X_j[h]], v[k]],$$
$$[ABC, [ABC, A_iB_iC_i, X_{j'}[h]], X_j[k]],$$

are $(\mathscr{A}, \gamma_X)$*-generating for all real values of constants* h *and* k *except exactly one value of either* h *or* k. *The matrices of exceptions are similar to the matrices* M_K, M_J, *and* M_F *from the Theorem 2.*

An important source of $(\mathscr{A}, \gamma_X)$-generating triangles is the following general result.

THEOREM 7. *Let* $Q \in W(ABC)$ *be a point different from the orthocentre* H. *The antipedal triangle* $Q^aQ^bQ^c$ *of* Q *with respect to* ABC *is orthologic with the triangle* $P^aP^bP^c$ *if and only if* P *lies on a conic through the points* A, B, C, H, *and* Q.

COROLLARY . *For* $X = K, J, F$, *the antipedal triangle* $Q^aQ^bQ^c$ *with respect to* ABC *of any point* Q *on the hyperbola* γ_X *outside the circumcircle* γ_0 *and different from the orthocentre* H *is* $(\mathscr{A}, \gamma_X)$*-generating.*

The next result can also be useful in search for $(\mathscr{A}, \gamma_X)$-generating triangles.

THEOREM 8. *For* $X = F, J, K$, *if* DEF *is a* $(\mathscr{A}, \gamma_X)$*-generating in* $W(ABC)$, *then triangles*

$$[ABC, DEF, v[h]] \quad \text{and} \quad [ABC, DEF, X_i[h]]$$

$(i = 1, \dots, n_X)$ *are also* $(\mathscr{A}, \gamma_X)$*-generating for all real numbers* h *with at most one exception.*

4. Preliminaries for proofs

Let us first determine the affixes of points P^a, P^b, and P^c. Since the affix of A is u and the affix of P is p, the line AP is $[1 - u\,q,\ u(p-u),\ u^2 q - p]$, where q is a complex conjugate of p. The lines BP and CP are relatives of AP. It follows that the perpendicular per_A at A to AP is $[1 - u\,q,\ u(u-p),\ u^2 q + p - 2\,u]$, while the perpendiculars per_B and per_C at B and C to BP and CP are its relatives. The intersection of per_B and per_C is a point P^a with affix $n_a^*[\mu_a\,(p\,q - 2) + p\,(\zeta_a - p)]$, where $n_a = p + \mu_a\,q - \zeta_a$. The affixes of P^b and P^c are relatives of the affix of P^a.

Recall that points P^a, P^b, P^c coincide if and only if the point P lies on the circumcircle γ_0 with the equation $p\,q - 1 = 0$.

THEOREM 9. *Triangles* XYZ *and* PQR *with affixes of vertices* x, y, z, p, q, *and* r *are orthologic if and only if* $(XYZ,\ PQR) = 0$, *where*

$$(XYZ,\ PQR) = \mathbb{S}\,[x\,(\bar{q} - \bar{r}) + \bar{x}\,(q - r)].$$

Proof. The line QR is $[\bar q - \bar r,\ r - q,\ q\,\bar r - \bar q\, r]$ so that the perpendicular per^X_{QR} through X onto QR is the line $[\bar q - \bar r,\ q - r,\ x\,(\bar r - \bar q) + \bar x\,(r - q)]$. The perpendiculars per^Y_{RP} and per^Z_{PQ} through Y and Z onto RP and PQ are relatives of per^X_{QR}. These three perpendiculars are concurrent if and only if $\Theta = 0$, where Θ denotes the determinant

$$\begin{vmatrix} \bar q - \bar r & q - r & x\,(\bar r - \bar q) + \bar x\,(r - q) \\ \bar r - \bar p & r - p & y\,(\bar p - \bar r) + \bar y\,(p - r) \\ \bar p - \bar q & p - q & z\,(\bar q - \bar p) + \bar z\,(q - p) \end{vmatrix}.$$

But, $\Theta = (XYZ,\ PQR)\,m$, where $m = p\,(\bar q - \bar r) + q\,(\bar r - \bar p) + r\,(\bar p - \bar q)$. Since $m = 0$ if and only if points P, Q, and R are collinear (and our assumptions exclude this possibility), we conclude that the triangles XYZ and PQR are orthologic if and only if $(XYZ,\ PQR) = 0$.

□

Let us observe that the condition for orthology of two triangles in both barycentric and trilinear coordinates is very complicated because it involves eighteen coordinates of vertices of these triangles. This is the main reason why we are making all calculations with complex numbers.

5. Proof of Theorem 1 for $X = K$ and $i = 1$

Since triangles ABC and PQR are homothetic, there is a point $T(x,\ y)$ and a real number $\xi \neq -1$ such that $\widetilde P = \eta\,(u + \xi\,x)$, $\widetilde Q = \varphi(\widetilde P)$, and $\widetilde R = \psi(\widetilde P)$, where $\eta = (\xi + 1)^*$.

Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC,\ PQR,\ K_1[h]]$. Then $\widetilde U = \widetilde P + Ih(v - w)$, where $I = \sqrt{-1}$. Also, $\widetilde V = \varphi(\widetilde U)$ and $\widetilde W = \psi(\widetilde U)$.

Since

$$\Delta(U,\ V,\ W) = \begin{vmatrix} \eta\,(u + \xi\,x) + I\,h\,\delta_a & u^*\,\eta\,(1 + \xi\,u\,y) + I\,h\,\delta_a\,\mu_a^* & 1 \\ \eta\,(v + \xi\,x) + I\,h\,\delta_b & v^*\,\eta\,(1 + \xi\,v\,y) + I\,h\,\delta_b\,\mu_b^* & 1 \\ \eta\,(w + \xi\,x) + I\,h\,\delta_c & w^*\,\eta\,(1 + \xi\,w\,y) + I\,h\,\delta_c\,\mu_c^* & 1 \end{vmatrix},$$

has up to a constant factor the form

$$\left(h - \frac{I\,\eta\,(\sigma\,\tau - 9\,\mu)}{3\,\mathbb{P}\,\delta_a}\right)^2 + \frac{4\,\eta^2\,(\tau^2 - 3\,\mu\,\sigma)\,(\sigma_2 - \tau)}{9\,\mathbb{P}\,\delta_a^2},$$

it follows that points U, V, and W will not be collinear if and only if h is different from

$$\frac{s_2 \pm 2\,\sqrt{s_4 - t_2}}{12\,S\,(\xi + 1)}.$$

Then $(P^a P^b P^c,\ UVW) = 4\,I\,h\,j_K\,j_0\,\mathbb{P}\,u^*\,n_a^*$, where $j_0 = 1 - p\,q$ and

$$j_K = (\tau^2 - 3\mu\sigma)p^2 - \mu^2(\sigma^2 - 3\tau)q^2 + (4\mu\sigma^2 - \sigma\,\tau^2 - 3\mu\,\tau)p$$
$$- \mu(4\tau^2 - \sigma^2\tau - 3\mu\,\sigma)q + \tau^3 - \mu\,\sigma^3.$$

Notice that $j_0 = 0$ is the equation of the circumcircle of ABC while $j_K = 0$ is the equation of the Kiepert hyperbola of ABC since the vertices $A(u,\ u^*)$, $B(v,\ v^*)$, and $C(w,\ w^*)$, the orthocenter $H(\sigma,\ \tau\,\mu^*)$, and the centroid $G(3^*\,\sigma,\ 3^*\,\tau\,\mu^*)$ satisfy it. This shows that UVW is $(\mathscr{A},\ \gamma_K)$-generating for all $h \neq 0$ except for at most two additional values of h found above when points U, V, and W are collinear.

The polynomial j_K is rather complicated. When we transfer it to barycentric coordinates it becomes significantly shorter so that the equation of the Kiepert hyperbola is $m_K = 0$, where $m_K = \mathbb{S}\,\delta_a\,(u^2 - \mu_a)\,y\,z$ or $m_K = \mathbb{S}\,d_{2a}\,y\,z$. The last form is simpler than the equation (5) in the excellent recent review article [6] on the Kiepert conics.

6. Proof of Theorem 1 for $X = J$ and $i = 1$

We first determine $\widetilde{P}$, $\widetilde{Q}$, and $\widetilde{R}$ as above. Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC,\ PQR,\ J_1[h]]$. Then $\widetilde{U} = \widetilde{P} + I\,h\,v\,w\,(v - w)^*$, $\widetilde{V} = \varphi(\widetilde{U})$, and $\widetilde{W} = \psi(\widetilde{U})$.

Notice that points U, V, and W will not be collinear if and only if h is different from

$$\frac{m\,(3\,m \pm \sqrt{3\,m_2 - \mathbb{S}\,a^4\,s_{2a}})}{4\,S\,(\xi + 1)\,s_2}.$$

This follows from the fact that

$$\Delta(U,\ V,\ W) = \begin{vmatrix} \eta\,(u + \xi\,x) + I\,h\,\delta_a^*\,\mu_a & u^*\,\eta\,(1 + \xi\,u\,y) + I\,h\,\delta_a^* & 1 \\ \eta\,(v + \xi\,x) + I\,h\,\delta_b^*\,\mu_b & v^*\,\eta\,(1 + \xi\,v\,y) + I\,h\,\delta_b^* & 1 \\ \eta\,(w + \xi\,x) + I\,h\,\delta_c^*\,\mu_c & w^*\,\eta\,(1 + \xi\,w\,y) + I\,h\,\delta_c^* & 1 \end{vmatrix},$$

has up to a constant factor the form

$$\left(h + \frac{3\,I\,\eta\,\langle 0,\ 1,\ -1\rangle}{\langle 6,\ 1,\ 1\rangle}\right)^2 + \eta^2\,\sigma\,\tau\,\mathbb{P}\,\delta_a^2.$$

Then $(P^a P^b P^c,\ UVW) = 2\,I\,h\,j_J\,j_0\,\mathbb{P}\,n_a^*$, where

$$j_J = \sigma\,p^2 - \mu\,\tau\,q^2 + (\tau - \sigma^2)\,p + (\tau^2 - \mu\,\sigma)q.$$

The equation of the Jarabek hyperbola of ABC is $j_J = 0$ since the vertices A, B, and C, the orthocenter H, and the circumcenter $O(0,\ 0)$ satisfy it.

The polynomial j_J which represents the Jarabek hyperbola is quite simple so that its forms in other coordinate systems do not lead to significant simplifications. For completeness, let us observe that in barycentric coordinates the equation of the Jarabek hyperbola is $m_J = 0$, where $m_J = \mathbb{S}\,u\,\delta_a^2\,\zeta_a\,(u^2 - \mu_a)\,y\,z$ or $m_J = \mathbb{S}\,a^2\,d_{2a}\,s_{2a}\,y\,z$.

7. Proof of Theorem 1 for $X = F$ and $i = 1$

In contrast with the previous two sections, in order to avoid square roots, here we shall assume that the vertices A, B, and C of the base triangle have affixes u^2, v^2, and w^2, with the same assumption about u, v, and w. Let ϱ denote a transformation which replaces variables u, v, and w with u^2, v^2, and w^2.

This time $\widetilde{P} = (\xi + 1)^* \, (u^2 + \xi\, x)$, $\widetilde{Q} = \varphi(\widetilde{P})$, and $\widetilde{R} = \psi(\widetilde{P})$. Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC,\ PQR,\ F_1[h]]$. Then $\widetilde{U} = \widetilde{P} + h\, v\, w$, $\widetilde{V} = \varphi(\widetilde{U})$, and $\widetilde{W} = \psi(\widetilde{U})$.

Since

$$\Delta(U,\ V,\ W) = \begin{vmatrix} \eta\,(u^2 + \xi\, x) + h\,\mu_a & \frac{\eta\,(1+\xi\, u^2 y)}{u^2} + h\,\mu_a^* & 1 \\ \eta\,(v^2 + \xi\, x) + h\,\mu_b & \frac{\eta\,(1+\xi\, v^2 y)}{v^2} + h\,\mu_b^* & 1 \\ \eta\,(w^2 + \xi\, x) + h\,\mu_c & \frac{\eta\,(1+\xi\, w^2 y)}{w^2} + h\,\mu_c^* & 1 \end{vmatrix},$$

has up to a constant factor the form $(h + \eta)^2 - \eta^2\, \mu^*\, \sigma\, \tau$, it follows that points U, V, and W will not be collinear if and only if h is different from

$$\frac{m \pm \sqrt{m\,(s^3 - 4\,m\,t + 9\,m)}}{4\,S(\xi + 1)}.$$

Then $(P^a P^b P^c,\ UVW) = 2\,h\, j_F\, j_0\, \mathbb{P}\, \delta_a\, u^*\, \varrho(n_a)^*$, where

$$j_F = \tau\, p^2 - \mu_3\, \sigma\, q^2 + (\mu\, \sigma + 2\,\mu\, \sigma^2 - \sigma^2\, \tau)\, p + \mu\, (\sigma\, \tau^2 - 2\,\sigma^2\, \mu - \mu\, \tau)\, q + \tau^3 - \sigma^3\, \mu.$$

Observe that $j_F = 0$ is the equation of the Feuerbach hyperbola of ABC since the vertices $A(u^2,\ 1/u^2)$, $B(v^2,\ 1/v^2)$, and $C(w^2,\ 1/w^2)$, the orthocenter $H(\varrho(\sigma,\ \tau\,\mu^*))$, and the incenter $I(-\tau,\ -\sigma\,\mu^*)$ satisfy it.

Just as with the Kiepert hyperbola, the polynomial j_F which represents the Feuerbach hyperbola is rather complicated. In barycentric and trilinear coordinates its equations are far simpler. More precisely, in barycentric coordinates the Feuerbach hyperbola has the equation $m_F = 0$, where $m_F = \mathbb{S}\, u\, \delta_a^2\, \zeta_a\, (u^2 - \mu_a)\, y\, z$ or $m_F = \mathbb{S}\, a\, d_a\, s_a\, y\, z$.

8. Proof of Theorem 2 for $X = K$, $i = 1$, and $j = 5$

Let $UVW = [ABC,\ PQR,\ K_1[h]]$ and $LMN = [ABC,\ UVW,\ K_5[k]]$. We know $\widetilde{U}$, $\widetilde{V}$, and $\widetilde{W}$ from the proof of Theorem 1, so that it is not difficult to see that $\widetilde{L} = \widetilde{U} + I\, k\, \delta_a^3\, \mu_a^*$, $\widetilde{M} = \varphi(\widetilde{L})$, and $\widetilde{N} = \psi(\widetilde{L})$.

Let us note that there exist at most two values of k when points L, M, and N are collinear. These values have rather complicated form.

The orthology condition for triangles $P^a P^b P^c$ and LMN is

$$(P^a P^b P^c,\ LMN) = 2\, I\, (\langle 6,\ 1,\ 1\rangle k + 2\,\mu\, h)\, j_K\, j_0\, \mu_2^*\, \mathbb{P}\, n_a^*,$$

This shows that LMN is $(\mathscr{A}, \gamma_K)$-generating for all k except the value $-2\,h\,s_2^*$ and at most two more values for which points L, M, and N are collinear.

9. Proof of Theorem 3 for $X = K$ and $i = 1$

Let $UVW = [ABC,\ PQR,\ v[h]]$ and $LMN = [ABC,\ UVW,\ K_1[k]]$. It is easy to check that $\widetilde{U} = \widetilde{P} + I\,h\,\zeta_a\,\mu^*\,\mathbb{P}\,\delta_a$, $\widetilde{V} = \varphi(\widetilde{U})$, and $\widetilde{W} = \psi(\widetilde{U})$. It follows that $\widetilde{L} = \widetilde{U} + I\,k\,\delta_a$, $\widetilde{M} = \varphi(\widetilde{L})$, and $\widetilde{N} = \psi(\widetilde{L})$. Once again there exist at most two values of k when points L, M, and N are collinear. These values have complicated expressions in terms of side lengths. Finally, $(P^aP^bP^c,\ LMN) = 4\,I\,k\,j_K\,j_0\,\mu^*\,\mathbb{P}\,n_a^*$.

10. Proof of Theorem 5 for $X = F$ and $i = e$

Assume $\widetilde{A} = u^2$, $\widetilde{B} = v^2$, and $\widetilde{C} = w^2$. Recall [14] that $\widetilde{A_e} = \tau_a$, $\widetilde{B_e} = \tau_b$, and $\widetilde{C_e} = \tau_c$. Let U, V, W denote vertices of the triangle $[ABC,\ A_eB_eC_e,\ v[h]]$. It is easy to check that $\widetilde{U} = \tau_a + I\,h\,\zeta_{2a}\,\mu_2^*\,\mathbb{P}\,\delta_{2a}$, $\widetilde{V} = \varphi(\widetilde{U})$, and $\widetilde{W} = \psi(\widetilde{U})$.

Once again there exist at most two values of h when points U, V, and W are collinear. These are $2^*\,S^*\,(-m \pm \sqrt{m\,(s^3 - 4\,s\,t + 9\,m)})\,\mathbb{P}\,s_a^*$. Finally, $(P^aP^bP^c,\ UVW)$ is equal to $4\,j_F\,j_0\,\mu^*\,\mathbb{P}\,\delta_a\,\varrho(n_a)^*$.

11. Proof of Theorem 7

Let $\widetilde{Q} = (x, y)$. Then $\widetilde{Q^a} = (\mu_a xy - x^2 + \zeta_a x - 2\mu_a)(x + \mu_a y - \zeta_a)^*$, $\widetilde{Q^b} = \varphi(\widetilde{Q^a})$, and $\widetilde{Q^c} = \psi(\widetilde{Q^a})$. It follows that $(P^aP^bP^c,\ Q^aQ^bQ^c) = 4(xy - 1)j_Q j_0 \mathbb{P}\delta_a n_a^*(x + \mu_a y - \zeta_a)^*$, where $j_Q = ap^2 + bq^2 + cp + dq + e$,

$$a{=}\mu\,y^2{-}x{-}\tau\,y{+}\sigma,\ b{=}\mu\,(\sigma\,x{-}x^2{+}\mu\,y{-}\tau),\ c{=}x^2{-}\mu\,\sigma\,y^2{+}\zeta_a\,\zeta_b\,\zeta_c\,y{-}\sigma_2{-}\tau,$$

$$d{=}\tau\,x^2{-}\zeta_a\,\zeta_b\,\zeta_c\,x{-}\mu_2\,y^2{+}\tau_2{+}\mu\,\sigma,\quad e{=}(\sigma_2{+}\tau)\,x{-}\sigma\,x^2{+}\mu\,\tau\,y^2{-}(\tau_2{+}\mu\,\sigma)\,y.$$

It is obvious that $j_Q = 0$ is an equation of a conic. One can easily check that it goes through A, B, C, H, and Q.

12. Outline of proof of Theorem 8

Let $\widetilde{D} = (x,\ a)$, $\widetilde{E} = (y,\ b)$, and $\widetilde{F} = (z,\ c)$. The expression $(P^aP^bP^c,\ DEF)$ has two major factors. The first is j_0 (the equation of the circumcircle). The second is the polynomial of degree two in p and q. We assume that its coefficients are equal to the product of a constant with corresponding coefficients of the polynomial representing γ_X. This gives six equations for variables x, a, y, b, z, c. Let the solutions have index 0 and let $\widetilde{D_0} = x_0, \dots$. Finally, we check that the triangles

$$[ABC,\ D_0E_0F_0,\ v[h]] \qquad \text{and} \qquad [ABC,\ D_0E_0F_0,\ X_i[h]]$$

($i = 1, \ldots, n_X$) are ($\mathscr{A}$, γ_X)-generating for all real numbers h except at most one value.

13. Concluding remarks and an introduction to the appendix

A careful reader should have noticed that we gave proofs of a very few of our results and that no case of both Theorems 4 and 6 was proved. One reason for this is that even with complex numbers we quickly get rather complicated expressions with lots of absolute values that are difficult to handle and hard to write down. In order to overcome these difficulties we must position a base triangle ABC so that lengths of sides are rational functions of three parameters. Only in this way we can avoid square-root and absolute value functions that are creating problems.

In an appendix to this paper we shall describe how one can (with the help from a computer) do all this calculations using only elementary analytic geometry in the plane. We shall limit ourselves to proofs of a few cases of Theorems 4 and 6 hoping that the reader will be able to infer proofs of all our claims following the same technique.

14. Appendix – Preliminaries

For an expression f, let $[f]$ denote a triple $(f, \varphi(f), \psi(f))$, where $\varphi(f)$ and $\psi(f)$ are cyclic permutations of f. For example, if $f = \sin A$ and $g = b + c$, then

$$[f] = (\sin A, \sin B, \sin C) \qquad \text{and} \qquad [g] = (b + c, c + a, a + b).$$

Let T denote a function that maps each triple $[a]$ of real numbers to a number

$$T([a]) = (a + b + c)(b + c - a)(a - b + c)(a + b - c).$$

We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0, 0)$, the vertex B is on the x-axis and has coordinates $(r\,e, 0)$, and the vertex C has coordinates $(g\,q\,r/w, 2\,f\,g\,r/w)$, where $e = f + g$, $w = f\,g - 1$, $p = f^2 + 1$, $q = f^2 - 1$, $s = g^2 + 1$, $t = g^2 - 1$, $u = f^4 + 1$, and $v = g^4 + 1$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (i. e., that angles A and B are acute).

Nice features of this placement are that all central points from Table 1 in [10] have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{r\,f\,s}{w}, \qquad b = \frac{r\,g\,p}{w}, \qquad c = r\,e,$$

$$f = \frac{(b+c)^2 - a^2}{\sqrt{T([a])}}, \quad g = \frac{(a+c)^2 - b^2}{\sqrt{T([a])}}, \quad r = \frac{\sqrt{T([a])}}{2\,(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears. More precisely, if a point P with coordinates x and y has projections P_a, P_b, and P_c onto the side lines BC, CA, and AB and $\lambda = PP_a/PP_b$ and $\mu = PP_b/PP_c$, then

$$x = \frac{e\,g\,(p\,\mu + q)\,r}{f\,s\lambda\,\mu + g\,p\,\mu + e\,w}, \qquad y = \frac{2\,e\,f\,g\,r}{f\,s\lambda\,\mu + g\,p\,\mu + e\,w}.$$

This formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first trilinear coordinate. For example, we write $X_6[a]$ to indicate that the symmedian point X_6 has trilinears equal to $a : b : c$. Then we use the above formulas with $\lambda = a/b$ and $\mu = b/c$ to get the coordinates

$$\left(\frac{(f\,q\,t + 2\,g\,u)\,e\,g\,r}{2\,(f^2\,v + f\,g\,q\,s + g^2\,u)}, \frac{f\,g\,e^2\,w\,r}{f^2\,v + f\,g\,q\,s + g^2\,u}\right).$$

of X_6 in our coordinate system.

Let UVW denote a function which to a triple $(P,\ Q,\ R)$ of points $P(x,\ x')$, $Q(y,\ y')$, and $R(z,\ z')$ and a triple $(h_a,\ h_b,\ h_c)$ of real numbers associates a triple $(U,\ V,\ W)$ of points, where

$$U\left(x + \frac{2\,g\,h_a}{s},\ x' + \frac{t\,h_a}{s}\right), \quad V\left(y - \frac{2\,f\,h_b}{p},\ y' + \frac{q\,h_b}{p}\right), \quad W(z,\ z' - h_c),$$

On the other hand, for triangles PQR and XYZ, let $OR(PQR,\ XYZ)$ be the sum

$$\mathbb{S}\,[p\,(y - z) + p'\,(z' - y')],$$

where $P(p,\ p')$, $Q(q,\ q')$, $R(r,\ r')$, $X(x,\ x')$, $Y(y,\ y')$, and $Z(z,\ z')$. Observe that triangles PQR and XYZ are orthologic if and only if $OR(PQR,\ XYZ) = 0$.

Let $P(x,\ y)$ be a point. Then the vertices of the antipedal triangle $P^aP^bP^c$ are

$$P^a\left(\frac{2\,g\,w\,y^2 - w\,t\,x\,y + 2\,r\,g\,e\,w\,x + r\,(2\,w + s)(f\,t - 2\,g)\,y - 2\,r^2\,g\,e^2\,w}{w\,(2\,g\,x + t\,y - 2\,r\,g\,e)}, \frac{(r\,e - x)(2\,g\,w\,y - t\,w\,x + r\,g\,(w^2 - e^2))}{w\,(2\,g\,x + t\,y - 2\,r\,g\,e)}\right),$$

$$P^b\left(\frac{(w\,q\,x + 2\,f\,w\,y - r\,g\,p^2)\,y}{w\,(q\,y - 2\,f\,x)}, \frac{(w\,q\,x + 2\,f\,w\,y - r\,g\,p^2)\,x}{w\,(q\,y - 2\,f\,x)}\right),$$

$$P^c\left(r\,e - x,\ \frac{x^2 - r\,e\,x}{y}\right).$$

15. Appendix – Proof of Theorem 4 for $X = K$ and $j = 1$

Since triangles ABC and PQR are homothetic, there is a point $T(\bar{x}, \bar{y})$ and a real number ξ different from -1 such that with $\eta = (\xi + 1)^*$,

$$P(\eta\,\xi\,\bar{x},\ \eta\,\xi\,\bar{y}),\ Q(\eta\,(\xi\,\bar{x}+e\,r),\ \eta\,\xi\,\bar{y}),\ R(\eta\,(\xi\,\bar{x}+g\,w^*\,q\,r),\ \eta\,(\xi\,\bar{y}+2\,f\,g\,w^*\,r)).$$

Let h be a real number. Let $(U, V, W) = UVW(P, Q, R, K_1[h])$. Then U, V, and W are vertices of the triangle $[ABC, PQR, K_1[h]]$. Since $K_1[h] = (fhrsw^*, ghprw^*, ehr)$, it follows that $U(P_x + 2fghrw^*, P_y + fhrtw^*)$, $W(R_x, R_y - ehr)$, and $V(Q_x - 2fghrw^*, Q_y + ghqrw^*)$, where P_x and P_y denote the first and the second coordinate of P.

Let k be a real number. Let $(L, M, N) = UVW(U, V, W, v[k])$. Then L, M, and N are vertices of the triangle $[ABC, UVW, v[k]]$. Since $v[k] = 2efgkr^3w^{*2}(qs, pt, e^2 - w^2)$, it follows that

$$L(U_x + 4efg^2kqr^3w^{*2},\ U_y + 2efgkqr^3tw^{*2}),$$
$$M(V_x - 4ef^2gkr^3tw^{*2},\ V_y + 2efgkqr^3tw^{*2}),$$
$$N(W_x,\ W_y + 2efgkr^3(w^2 - e^2)w^{*2}).$$

The orthology condition is $OR(P^aP^bP^c, LMN) = hm_0m_Kq_a^*q_b^*y^*w^{*2}$, where

$$q_a = 2g(er - x) - ty,\ q_b = qy - 2fx,\ m_0 = 2w(x^2 + y^2) - 2erwx + r(w^2 - e^2)y,$$

$$m_K = 2fg(f - g)(w + 2)(x^2 - erx - y^2) - \\ 2(g^2u + fgqt + f^2t^2 - 6f^2g^2)xy + egr(2gu + fqt - 8f^2g)y.$$

Notice that $m_0 = 0$ is the equation of the circumcircle of ABC because it is equivalent to the equation $(x - 2^*er)^2 + (y + 4^*(w^2 - e^2)rw^*)^2 - 16^*p^2r^2s^2w^{*2} = 0$ of the circle with center at the circumcenter $O(2^*er, -4^*(w^2 - e^2)rw^*)$ and the radius equal to the circumradius 4^*prsw^*. Also, $m_K = 0$ is the equation of the Kiepert hyperbola of ABC since the vertices $A(0, 0)$, $B(eh, 0)$, and $C(gqrw^*, 2fgrw^*)$, the orthocenter $H(gqrw^*, 2^*qrtw^*)$, and the centroid $G(3^*r(ft + 2gq), 3^*2fgrw^*)$ satisfy it. This shows that LMN is $(\mathscr{A}, \gamma_K)$-generating for all real numbers $h \neq 0$ and k.

16. Appendix – Proof of Theorem 4 for $X = J$ and $j = 2$

We first determine P, Q, and R as above. Let h be a real number. Let U, V, and W be vertices of the triangle $[ABC, PQR, J_2[h]]$. Since

$$J_2[h] = 2hw((2w + s)^*, (2w + p)^*, w^*(w + 2)^*),$$

it follows that $U(P_x + 4ghws^*(2w + s)^*, P_y + 2htws^*(2w + s)^*)$,

$$V(Q_x - 4fhwp^*(2w + p)^*, Q_y + 2hqwp^*(2w + p)^*),\ W(R_x, R_y - 2h(w + 2)^*).$$

Let k be a real number. Let $(L, M, N) = UVW(U, V, W, v[k])$. Then L, M, and N are vertices of the triangle $[ABC, UVW, v[k]]$. It follows that

$$L(U_x + 4efg^2kqr^3w^{*2}, U_y + 2efgkqr^3tw^{*2}),\ N(W_x, W_y + 2efgk(w^2 - e^2)r^3w^{*2}),$$

and $M(V_x - 4ef^2gkr^3tw^{*2}, V_y + 2efgkqr^3tw^{*2})$.

The orthology condition is

$$OR(P^aP^bP^c, LMN) = 2efghm_0m_Jq_a^*q_b^*y^*w^*(w+2)^*p^*s^*(2w+p)^*(2w+s)^*,$$

where

$$m_J = 2w(w+2)(f-g)(w^2-e^2)(y^2+erx-x^2)+w(2g^2u+2f^2v+20f^2g^2-3uv)xy \\ + r(2gv(f^6-1)+ft^3u-2f^2(v+4g^2)(ft+gq))y.$$

Observe that $m_J = 0$ is the equation of the Jarabek hyperbola of ABC since the vertices the orthocenter, and the circumcenter $O(2^*er, -4^*(w^2 - e^2)rw^*)$ satisfy it. This shows that LMN is $(\mathscr{A}, \gamma_J)$-generating for all real numbers $h \neq 0$ and k.

17. Appendix – Proof of Theorem 4 for $X = F$ and $j = 2$

We first determine P, Q, and R as above. Let h be a real number. Let UVW be the triangle $[ABC, PQR, F_2[h]]$. Since $F_2[h] = 2^*r^*h(f^*, g^*, we^*)$, it follows that

$$U(P_x + f^*ghr^*s^*, P_y + 2^*f^*hr^*s^*t), \quad V(Q_x - fg^*hp^*r^*, Q_y + 2^*g^*hp^*qr^*),$$

and $W(R_x, R_y - 2^*e^*hr^*w)$.

Let k be a real number. Let $(L, M, N) = UVW(U, V, W, v[k])$. Then L, M, and N are vertices of the triangle $[ABC, UVW, v[k]]$. It follows that

$$L(U_x + 4efg^2kqr^3w^{*2}, U_y + 2efgkqr^3tw^{*2}), \quad N(W_x, W_y + 2efgk(w^2 - e^2)r^3w^{*2}),$$

and $M(V_x - 4ef^2gkr^3tw^{*2}, V_y + 2efgkqr^3tw^{*2})$. The orthology condition is

$$2^*(e^2 + f^2g^2 - 1)hm_0m_Fp^*q_a^*q_b^*r^*s^*w^*y^*,$$

where

$$m_F = 2(f-g)(x^2 - erx - y^2) + (4 - q - t + w(2-w))xy + r(fqs - 2gp)y.$$

Observe that $m_F = 0$ is the equation of the Feuerbach hyperbola of ABC since the vertices, the orthocenter, and the incenter $I(fr, r)$ satisfy it. This shows that LMN is $(\mathscr{A}, \gamma_F)$-generating for all real numbers $h \neq 0$ and k.

18. Appendix – Proof of Theorem 6 for $X = F$, $i = Oi$, $j' = 1$, and $j = 2$

An arbitrary point X different from the circumcenter O on the line IO joining the incenter with the circumcenter can be represented as a point with coordinates

$$X(2^*\eta r(e\xi + 2f), -4^*\eta r((w^2 - e^2)\xi - 4w)w^*)$$

for a real number ξ different from -1, where $\eta = (\xi + 1)^*$. Projections P, Q, and R of X onto the sidelines BC, CA, and AB are

$$P(2^*\eta rs^*w^*(s(ft + 2gq)\xi + 2(fs + 2g)w), \eta grs^*w^*(fs\xi + 2gw)),$$

$$Q(2^*\eta(gp\xi + 2fw)p^*qrw^*, \eta f(gp\xi + 2fw)p^*rw^*), \quad R(2^*\eta r(e\xi + 2f), 0).$$

Let h be a real number. Let $(U, V, W) = UVW(P, Q, R, F_1[h])$. Then U, V, and W are vertices of the triangle $[ABC, PQR, F_1[h]]$. It follows that

$$U(P_x + 2ghs^*, P_y + hs^*t), \quad V(Q_x - 2fhp^*, Q_y + hqp^*), \quad W(R_x, -h),$$

where P_x and P_y denote the first and the second coordinate of P.

Let k be a real number. Let $(L, M, N) = UVW(U, V, W, F_2[k])$. Then L, M, and N are vertices of the triangle $[ABC, UVW, F_2[k]]$. Hence,

$$L(U_x + f^*gkr^*s^*, U_y + 2^*f^*kr^*s^*t), \quad M(V_x - fg^*kp^*r^*, V_y + 2^*g^*kp^*qr^*),$$

and $N(W_x, W_y - 2^*e^*kr^*w)$.

The orthology condition is $2^*\eta m_0 m_1 m_F p^* q_a^* q_b^* r^* s^* w^* y^*$, where

$$m_1 = (\xi + 1)[4efghr + (e^2 + f^2g^2 - 1)k] + 4efgr^2,$$

This shows that LMN is $(\mathscr{A}, \gamma_F)$-generating for all real numbers h and k which satisfy the relation $m_1 \neq 0$.

REFERENCES

[1] H. S. M. Coxeter, *Some applications of trilinear coordinates*, Linear Algebra and Its Applications **226-228** (1995), 375–388.

[2] H. M. Cundy and C. F. Parry, *Some cubic curves associated with a triangle*, Journal of Geometry, **53** (1995), 41–66.

[3] Philip J. Davis, *The rise, fall, and possible transfiguration of triangle geometry: A mini-history*, American Mathematical Monthly **102** (1995), 205–214.

[4] R. Deaux, *Introduction to the geometry of complex numbers*, Ungar Publishing Co., New York, 1956.

[5] H. L. Dorwart, *The Neuberg cubic: A nostalgic look*, California Mathematics **3** (1978), 31–38.

[6] R. H. Eddy and R. Fritsch, *The conics of Ludwig Kiepert: A comprehensive lesson in the geometry of the triangle*, Mathematics Magazine, **67** (1994), 188–205.

[7] R. H. Eddy and J. B. Wilker, *Plane mappings of isogonal-isotomic type*, Soochow Journal of Math. **18** (1992), 123–126.

[8] W. Gallatly, *The Modern Geometry of the Triangle*, Francis Hodgson, London, 1910.

[9] L. Hahn, *Complex numbers and geometry*, Mathematical Association of America, Washington, 1994.

[10] Clark Kimberling, *Central points and central lines in the plane of a triangle*, Mathematics Magazine, **67** (1994), 163–187.

[11] M. Mandart, *Sur l'hyperbole de Feuerbach*, Mathesis (1893), 81-89.
[12] H. Martini, *On the theorem of Napoleon and related topics*, Math. Semesterber. **43** (1996), 47–64.
[13] D. Merriell and P. J. Kelly, *Concentric polygons*, Amer. Math. Monthly **71** (1964), 37–41.
[14] Frank Morley and F. V. Morley, *Inversive Geometry*, Chelsea Publ. Co., New York, 1954.
[15] Jean Baptiste Joseph Neuberg, *Mémoire sur la tétraèdre*, Supplément V, Mathesis 1885.
[16] Jean Baptiste Joseph Neuberg, *Sur la parabole de Kiepert*, Annales de la Société scientifique de Bruxelles, (1909-1910), 1–11.
[17] J. Neuberg, *Cours de Géométrie Analytique*, A. Hermann, Paris, 1908.
[18] A. Oldknow, *Computer aided research into triangle geometry*, Math. Gaz. **80** (1996), 263–274.
[19] D. Pedoe, *A Course of Geometry*, Cambridge University Press, Cambridge, 1970.
[20] Guido M. Pinkernell, *Cubic curves in the triangle plane*, Journal of Geometry **55** (1996), 141–161.
[21] Pablo Rubio, *Cubic lines relative to a triangle*, Journal of Geometry **34** (1989), 152–171.
[22] H. Schwerdtfeger *Geometry of Complex Numbers*, Oliver and Boyd, Toronto, 1962.
[23] I. M. Yaglom, *Complex Numbers in Geometry*, Academic Press, New York, 1968.
[24] P. Yff, *Two families of cubics associated with a triangle*, In Eves' Circles (Joby Milo Anthony, editor), MAA, Washington, 1993.

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