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A NOTE ON STEFFENSEN'S AND IYENGAR'S INEQUALITY

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Abstract. Taking Steffensen's inequality as a starting point, we obtain some generalizations of classical Iyengar's inequality.

1. Introduction

If f is a differentiable on $[a, b]$ and $|f'(x)| \leqslant M$, then classical Iyengar's inequality [2] states that it holds

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leqslant \frac{M(b-a)^2}{4} - \frac{1}{4M}(f(b)-f(a))^2. \qquad (1)$$

Recently, Agarwal and Dragomir [1] applied Steffensen's inequality to obtain inequality which is stronger than Iyengar's one:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)(b-a)} \qquad (2)$$

where $m \leqslant f'(x) \leqslant M$ for all $x \in [a, b]$. In fact, (2) reduces to (1) if one takes $-m = M = \sup_{a \leqslant x \leqslant b} |f'(x)|$.

Inequality (1) has been generalized in various ways. From the point of this note, the following one will interest us ([3], Theorem 1).

THEOREM 1. *Let the function $f : [a, b] \to \mathbf{R}$ satisfies the following conditions*
(i) $f^{(n-1)} \in C_\alpha$ (with constant M and $0 < \alpha \leqslant 1$) for some $n \in \mathbf{N}$;
(ii) $f^{(k)}(a) = f^{(k)}(b) = 0$, $(k = 1, \ldots, n-1)$.
Then it holds

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{M(b-a)^{\alpha+n-1}}{(\alpha+n)^{(n)}}\left(\zeta^{\alpha+n-1} - \frac{q}{2}[1+(\alpha+n-1)(2\zeta-1)]\right), \qquad (3)$$

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where ζ *is the real root of the equation*

$$\zeta^{\alpha+n-1} - (1-\zeta)^{\alpha+n-1} = q$$

and

$$q = \frac{(\alpha+n-1)^{(n-1)}}{M(b-a)^{\alpha+n-1}}(f(b)-f(a)), \quad p^{(m)} = p(p-1)\cdots(p-m+1), \quad m \in \mathbf{N}.$$

We shall use Steffensen's inequality to obtain result similar, but simpler than (3).

2. Main results

The starting point is Hayashi's modification of the well-known Steffensen's inequality:

THEOREM 2. *Let* $F : [a,b] \to \mathbf{R}$ *be a nonincreasing mapping on* $[a,b]$ *and* $G : [a,b] \to \mathbf{R}$ *an integrable mapping on* $[a,b]$ *with* $0 \leqslant G(x) \leqslant A$, *for all* $x \in [a,b]$. *Then, the following inequality holds*

$$A\int_{b-\lambda}^{b} F(x)dx \leqslant \int_a^b F(x)G(x)dx \leqslant A\int_a^{a+\lambda} F(x)dx, \tag{4}$$

where $\lambda = \dfrac{1}{A}\displaystyle\int_a^b G(x)dx.$

We shall suppose that $f : I \subseteq \mathbf{R} \to \mathbf{R}$ satisfies following conditions:

1° $f \in C^{(n-1)}(I^\circ)$, $[a,b] \subseteq I^\circ$, and

$$m \leqslant f^{(n)}(x) \leqslant M, \qquad \forall x \in [a,b]; \tag{5}$$

2° $f^{(k)}(a) = f^{(k)}(b)$ for odd $k \geqslant 1$, $k < n$ and $f^{(k)}(a) = -f^{(k)}(b)$ for even $k \geqslant 2$, $k < n$.

THEOREM 3. *Let n be an even number and let f satisfies conditions* 1° *and* 2°. *Then it holds*

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) - \frac{M+m}{2^{n+1}(n+1)!}(b-a)^{n+1}\right| \leqslant \frac{M-m}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+1}\left[1-\left(\frac{|\Delta|}{M-m}\right)^{n+1}\right] \tag{6}$$

where

$$\Delta = \frac{2}{b-a}\left(f^{(n-1)}(a) - 2f^{(n-1)}\left(\frac{a+b}{2}\right) + f^{(n-1)}(b)\right). \tag{7}$$

Proof. Denote $c = (a+b)/2$. We shall apply Theorem 2 for

$$F(x) = \begin{cases} (x-c)^n, & a \leqslant x \leqslant c, \\ -(x-c)^n, & c \leqslant x \leqslant b. \end{cases} \tag{8}$$

and

$$G(x) = \begin{cases} M - f^{(n)}(x), & a \leqslant x \leqslant c, \\ f^{(n)}(x) - m, & c \leqslant x \leqslant b. \end{cases} \tag{9}$$

Then it holds

$$0 \leqslant G(x) \leqslant M - m =: A$$

and

$$\begin{aligned} \lambda &= \frac{1}{A}\left(\int_a^c (M - f^{(n)}(x))dx + \int_c^b (f^{(n)}(x) - m)dx\right) \\ &= \frac{b-a}{2}\left(1 + \frac{\Delta}{M-m}\right), \end{aligned} \tag{10}$$

where Δ is defined in (7).

Two cases are possible: (a) $\Delta \leqslant 0$; and (b) $\Delta > 0$.

Case (a). In the case $\Delta \leqslant 0$ it holds $\lambda \leqslant (b-a)/2$. Therefore, the left term of Hayashi inequality (4) is

$$\begin{aligned} \alpha_1 &= (M-m)\int_{b-\lambda}^b F(x)dx = (M-m)\int_{b-\lambda}^b \left[-(x-c)^n\right]dx \\ &= -\frac{M-m}{n+1}\left(\frac{b-a}{2}\right)^{n+1}\left[1+\left(\frac{\Delta}{M-m}\right)^{n+1}\right]. \end{aligned} \tag{11}$$

It is easy to see that

$$\alpha_2 = (M-m)\int_a^{a+\lambda} F(x)dx = -\alpha_1.$$

In the case (b), it holds $\lambda > (b-a)/2$ and similar calculation gives

$$\alpha_1 = -\frac{M-m}{n+1}\left(\frac{b-a}{2}\right)^{n+1}\left[1-\left(\frac{\Delta}{M-m}\right)^{n+1}\right]. \tag{12}$$

Now, we need only to calculate the middle term in Steffensen's inequality, using integration by parts and properties 2° of function f:

$$\begin{aligned} I &= \int_a^c (x-c)^n(M - f^{(n)}(x))dx - \int_c^b (x-c)^n(f^{(n)}(x) - m)dx \\ &= -n!\left[\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}(M+m)\right]. \end{aligned}$$

This proves the theorem.

COROLLARY 1. *If f satisfies 1° and 2°, then for even $n \in \mathbf{N}$ it holds*

$$\begin{aligned} &\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \\ &\qquad \leqslant \frac{\max\{|M|,|m|\}}{2^n(n+1)!}(b-a)^{n+1} - \frac{|\Delta|^{n+1}(b-a)^{n+1}}{2^{n+1}(M-m)^n(n+1)!}. \end{aligned} \tag{13}$$

THEOREM 4. *Let $n \geqslant 1$ be an odd number and let f satisfies conditions* $1°$ *and* $2°$. *Then it holds*

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leqslant \frac{M-m}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+1}\left[1-\left(\frac{M+m-2\Delta}{M-m}\right)^{n+1}\right], \tag{14}$$

where

$$\Delta = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}. \tag{15}$$

Proof. We use the same idea as in Theorem 3. But, in this case is sufficient to take

$$F(x) = -(x-c)^n \tag{16}$$

and

$$G(x) = f^{(n)}(x) - m \tag{17}$$

Therefore, $A = M - m$ again and since f satisfies $2°$, we have

$$\lambda = \frac{1}{M-m}\int_a^b (f^{(n)}(x)-m)dx = \frac{\Delta - m}{M-m}(b-a),$$

$$\alpha_1 = -(M-m)\int_{b-\lambda}^{b}(x-c)^n dx = -\frac{M-m}{n+1}\left(\frac{b-a}{2}\right)^{n+1}\left[1-\left(\frac{M+m-2\Delta}{M-m}\right)^{n+1}\right].$$

Since $\alpha_2 = -\alpha_1$ and the middle term in Steffensen's inequality is equal to

$$-\int_a^b (x-c)^n[f^{(n)}(x)-m]dx = -n!(x-c)f(x)\Big|_a^b + n!\int_a^b f(x)dx = n!\left(\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right)$$

the theorem follows.

Remark. Let f satisfies conditions of Theorem 1 with $\alpha = 1$. As a corollary of Theorem 1, it is proved in [3] that

$$\left|\int_a^b f(x)dx\right| \leqslant \frac{M(b-a)^{n+1}}{2^n(n+1)!}. \tag{18}$$

holds, if an additional condition $f(a) = f(b) = 0$ is assumed. But, such an inequality holds under weaker conditions $1°$ and $2°$. If n is even, this follows from Corollary 1, and for odd n it is sufficient to take $M = \sup|f^{(n)}(x)|$ and $m = -M$ in Theorem 4. Therefore, for all $n \in \mathbf{N}$ we have:

COROLLARY 2. *If f satisfies* $1°$ *and* $2°$ *then it holds*

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leqslant \frac{M(b-a)^{n+1}}{2^n(n+1)!}. \tag{19}$$

where $M = \sup|f^{(n)}(x)|$, $x \in [a,b]$.

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