Interdisciplinary Description of Complex Systems 10(2), 57-65, 2012

SUSTAINABLE EFFICIENCY OF FAR-FROM-EQUILIBRIUM SYSTEMS

Michel Moreau[1, *], Bernard Gaveau[2] and Leonard S. Schulman[3]

[1]Laboratory for Theoretical and Condensed Matter Physics, University Pierre et Marie Curie
Paris, France

[2]Department of Mathematics, University Pierre et Marie Curie
Paris, France

[3]Department of Physics, Clarkson University
Potsdam, The United States of America

Regular article *Received:* 12. January 2011. *Accepted:* 20. May 2011.

ABSTRACT

The Carnot efficiency of usual thermal motors compares the work produced by the motor to the heat received from the hot source, neglecting the perturbation of the cold source: thus, even if it may be appropriate for industrial purposes, it is not pertinent in the scope of sustainable development and environment care. In the framework of stochastic dynamics we propose a different definition of efficiency, which takes into account the entropy production in all the irreversible processes considered and allows for a fair estimation of the global costs of energy production from heat sources: thus, we may call it "sustainable efficiency". It can be defined for any number of sources and any kind of reservoir, and it may be extended to other fields than conventional thermodynamics, such as biology and, hopefully, economics.

Both sustainable efficiency and Carnot efficiency reach their maximum value when the processes are reversible, but then, power production vanishes. In practise, it is important to consider these efficiencies out of equilibrium, in the conditions of maximum power production. It can be proved that in these conditions, the sustainable efficiency has a universal upper bound, and that the power loss due to irreversibility is at less equal to the power delivered to the mechanical, external system.

However, it may be difficult to deduce the sustainable efficiency from experimental observations, whereas Carnot's efficiency is easily measurable and most generally used for practical purposes. It can be shown that the upper bound of sustainable efficiency implies a new higher bound of Carnot efficiency at maximum power, which is higher than the so-called Curzon-Ahlborn bound of efficiency at maximum power.

KEY WORDS

efficiency, sustainable efficiency, maximum power, entropy

CLASSIFICATION

JEL: Q50

PACS: 05.70.-a, 05.70.Ln

*Corresponding author, η: moreau@lptl.jussieu.fr; +33 1 4427 5947;
LPTMC (UMR 7600), University Pierre et Marie Curie, CNRS, case 121, 75252 Paris Cedex 05, France

INTRODUCTION: A SHORT STORY OF EFFICIENCY

CARNOT EFFICIENCY

From the beginnings of thermodynamics, efficiency in work production from heat exchanges played a basic role. In the particular case of a cyclic motor operating between two reservoirs with temperatures T_1 and T_2 ($T_1 > T_2$) the celebrated Carnot efficiency γ_C is defined arbitrarily, but logically, as the ratio of the total work W produced per cycle and of the heat Q_1 provided in each cycle by the high temperature reservoir.

In 1824, Carnot stated that the efficiency of a thermal motor only depends on the temperatures of the sources. He published his results at his own expenses in a small booklet "*Réflexions sur la Puissance Motrice du Feu et sur les Machines Propres à Développer cette Puissance*" [1], although the equivalence of heat and energy was not recognized yet. This deeply innovative work was underestimated by other scientists untill Clapeyron (1834), Clausius (1850) and Thomson (1851) pointed out its importance. Then, Carnot's result was written as the famous inequality

$$\gamma_C \leq 1 - \frac{T_2}{T_1},$$

ANOTHER DEFINITION OF EFFICIENCY

Carnot efficiency gives very different status to the heat sources, because the importance of environment (usually the low temperature source), is often underestimated in practical applications. However, this should not be the case from a modern point of view, and Carnot efficiency could be inappropriate for taking into account the damages due to the irreversible changes implied in work production from heat sources.

Other definitions can be proposed to take these damages into account. For instance, the total "price" of the thermal exchanges should consider not only the price of the heat provided to the hot reservoir (the furnace), but also the cost of the heat rejected into the lower reservoir (the environment).

A simple, naive way to do this could be to define the „ecological" efficiency

$$\gamma_E = \frac{|W|}{|Q_1| + |Q_2|} = \frac{Q_1 + Q_2}{Q_1 - Q_2} \leq \frac{T_1 - T_2}{T_1 + T_2},$$

Such an efficiency does not differ essentially from Carnot definition, since γ_E is just an increasing function of γ_C. Nevertheless, it gives a better measure of efficiency from the point of view of sustainable development.

EFFICIENCY AT MAXIMUM POWER

Yvon (1955)

It was soon realized that actual engines cannot work with Carnot maximum efficiency, since then power production vanishes. In practise, it is important to obtain the highest possible power from a given fuel expense. The first physicist who analysed this problem precisely and gave it a response seems to be Jacques Yvon (who is mainly known because in 1935 he formulated the principle of the celebrated B.B.G.K.Y. hierarchy of statistical mechanics). At the International Conference on Peaceful Uses of Atomic Energy, Genova, 1955, Yvon presented a talk intitled "*The Saclay Reactor: Two years experience on heat transfer by means of compressed gas*" where he gave the efficiency at maximum power (including a coefficient due to non thermodynamic losses) [2]

$$\gamma_C \sim 1 - \sqrt{\frac{T_2}{T_1}} .$$

This talk appeared in the proceedings of the conference, but the derivation is only sketched (although its principle is clearly given), and it was not published elsewhere.

Novikov (1958)

Since then, this formula was known among nuclear physicists and engineers [3, 4]. To our knowledge, Novikov was the first one to publish a complete derivation (including other ingredients necessary for application to an atomic plant) of the Yvon result, which could thus be called the Yvon-Novikov efficiency. At this stage, all the derivations were based on a very simple reasoning, taking into account that the highest temperature of the system should be less than the hot source temperature, in order that the diffusive heat flux received by the engine can be finite.

Others

Twenty years after Yvon's work, Curzon and Ahlborn gave a more complete derivation of his result by considering the temperature differences between both the hot and the cold sources and the highest and lowest temperatures of the system, respectively, and, under specific conditions, the durations of the exchanges [5]. Maximizing the power produced with respect to these temperatures differences yields the Yvon-Novikov efficiency. This is why this efficiency is now often denoted γ_{CA} and called the Curzon-Ahlborn (CA) efficiency

$$\gamma_{CA} = 1 - \sqrt{\frac{T_2}{T_1}} .$$

In 2005, Van den Broek obtained the same result from linear irreversible thermodynamics for a small temperature difference $T_1 - T_2$ [6]. Then he extended the formula to an engine controlled by an external parameter which makes him operate with a cascade of intermediary temperatures, as explained below.

Later, many authors extended these results to more general cases (see for instance [7]) and the upper bound γ_{CA} has been generalized to complex situations and chemical engines (see for instance [8-11]).

Experimental study of actual motors indeed showed that the observed upper bound of their Carnot efficiency is often much closer to γ_{CA} than to the Carnot limit [5]. It can be remarked, however, that the industrial efficiency includes many non thermodynamic losses, which may be difficult to evaluate, so that the thermodynamic efficiency cannot be estimated very accurately. The old experimental results given by Curzon and Ahlborn bear on a very small number of examples, but more recent and systematic measures are reported by Esposito et al. [12]. These values do not always respect the CA bound and suggest that this bound may not be universal. We will see that this conclusion is supported by very recent developments.

RECENT ADVANCES

With the advances on nanomotors and the development of stochastic thermo-dynamics, the study of motors efficiency became a very active field in the last years. In 2010, Gaveau and coworkers introduced the new concept of sustainable efficiency, presented in the next section [13]. As summarized below, they showed that the Carnot efficiency of a bi-thermal motor in a stationary state has the upper bound

$$\bar{\gamma}_C = \frac{T_1 - T_2}{T_1 + T_2}.$$

T_1 and T_2 being the hot and cold temperatures, respectively. This value is always larger than the Yvon-Novikov, or Curzon-Ahlborn value.

Shortly after the publication of this result, Esposito et al. showed that the Carnot efficiency at maximum power of a cyclic, bi-thermal motor also admits $\bar{\gamma}_C$ as an upper bound [12]. On a long time interval, the probability distribution of a state of such a system is stationary, and one recovers the previous result for this special class of stationary motors. Equivalently, a statistical ensemble of cyclic motors obeys a stationary states distribution, with the same conclusion.

Parallely, recent studies of solvable microscopic cycles [14-16] showed that their Carnot efficiency at maximum power can be quite different and larger than the CA value. Thus the CA bound at maximum power does not apply in all situations.

CYCLIC MOTORS, STATIONARY MOTORS

Carnot efficiency, as well as most efficiencies mentioned previously, concern thermodynamic cycles: this is convenient in engineering since most industrial motors work in this way. In biology, however, many cycles can be identified, but they exist at the molecular level. At a mesoscopic scale, the processes often appear to be quasi-stationary and the usual thermodynamic efficiencies may be inappropriate. The observed, mesoscopic stationary state can be interpreted as the average state of an ensemble of cyclic systems.

In the next section, we consider the definition of sustainable efficiency for a stochastic, stationary system, making explicit use of the power dissipation due to irreversibility. This definition can be applied with an arbitrary number of sources, without giving a special status to one of them, and it can be convenient for the concerns of sustainable development. For these reasons, it may be more appropriate than Carnot efficiency in biomechanics and in modern nano-technologies. The new definition is obtained from the formalism of stochastic thermodynamics.

STOCHASTIC THERMODYNAMICS, SUSTAINABLE EFFICIENCY AND CARNOT EFFICIENCY AT MAXIMUM POWER

STOCHASTIC DYNAMICS AND ENTROPY PRODUCTION

We consider the stochastic dynamics of a discrete system s undergoing a discrete time process. The elementary time step t is taken as time unit. The dynamics is defined by the stochastic matrix $\boldsymbol{R} \equiv \{R_{xy}\}$, where $R_{xy} \equiv p(x, t + t \mid y, t)$ is the transition probability from y to x in time t.

For each elementary transition $y \to x$, we have [17]

$$\frac{R_{xy}}{R_{yx}} = \exp(\delta s^{tot}_{xy}), \qquad R_{xy} R_{yx} \neq 0,$$

where $\delta s^{tot}{}_{xy}$ is the total entropy variation of s and other systems, a relation equivalent with the time reversal asymmetry relation of Gallavotti and Cohen [18, 19]. During an elementary transition $y \to x$, the system s can receive heat $\mathrm{d}q_{xy}$ from *at most one of the reservoirs* at temperature $T_{xy} = 1/b_{xy}$ and work $\mathrm{d}w_{xy}$.

The energy variation of system s is

$$\mathrm{d}_{xy}e = \mathrm{d}q_{xy} + \mathrm{d}w_{xy} \, ,$$

where $e = e(x)$ is energy of system s in state x, and the mesoscopic entropy variation of s is $\mathrm{d}_{xy}s$, so that

$$\frac{R_{xy}}{R_{yx}} = \exp(\delta s_{xy}^{\mathrm{tot}}) = \exp(\delta_{xy}s - \beta_{xy}\delta q_{xy}) \, .$$

ENTROPY PRODUCTION AND POWER DISSIPATION

The probability current J_{xy} for transition $y \to x$ is defined by

$$J_{xy} = R_{xy}p^0(y) - R_{yx}p^0(x) .$$

with $p^0(x)$ being the stationary (non equilibrium) distribution. The stationary entropy production per unit time can be written

$$D_S = \frac{1}{2}\sum_{x,y} J_{xy}(\delta s^{\mathrm{tot}})_{xy} = \frac{1}{2}\sum_{x,y} D_{xy} \, ,$$

with

$$D_{xy} = J_{xy}\ln\frac{R_{xy}p^0(y)}{R_{yx}p^0(x)} \geq 0.$$

The power received by s is

$$P = \frac{1}{2}\sum_{x,y} J_{xy}\delta w_{xy} = \frac{1}{2}\sum_{\substack{x,y,\\ \beta_{xy}>0}}\left[T_{xy}D_{xy} - T_{xy}J_{xy}\delta_{xy}(s + \phi)\right],$$

where $\phi = -\ln p^0(x)$ is the information entropy (or stochastic potential) of distribution $p^0(x)$.

The power dissipated by the irreversible processes can be defined as

$$D_W = \frac{1}{2}\sum_{\substack{x,y,\\ \beta_{xy}>0}} T_{xy}D_{xy} \geq 0. \, ,$$

It vanishes if and only if $J_{xy} = 0$ for all x, y, i.e. at equilibrium.

SUSTAINABLE EFFICIENCY

The power released by the system can be written

$$-P = -D_W + \frac{1}{2}\sum_{x,y} T_{xy}J_{xy}\delta_{xy}(s + \phi) \leq \frac{1}{2}\sum_{x,y} T_{xy}J_{xy}\delta_{xy}(s + \phi) \equiv A. \, ,$$

Here $D_w > 0$ is the power loss due to irreversibility. It is the energetic equivalent of the entropy creation, or order loss, per unit time. A can be interpreted as the resources consumption per unit time. It is the power that would be produced in the absence of dissipation. If system s exchanges other quantities than energy with the reservoirs (such as particles numbers N, …), A becomes

$$A = \frac{1}{2}\sum_{x,y} J_{xy}\left\{T_{xy}\delta_{xy}(s + \phi) + \mu_{xy}\delta_{xy}N + ...\right\}.$$

Whatever the interpretation of A, it is logical to define a new efficiency γ_S by comparing the power produced to its value in absence of dissipation

$$\gamma_S = -\frac{P}{A} = 1 - \frac{D_W}{A}.$$

We can tentatively call γ_S "sustainable efficiency" because, by the interpretation of A, it measures the relative importance of the entropy losses in all the processes, so it is directly related to the concerns of sustainable development.

SUSTAINABLE EFFICIENCY AT MAXIMUM POWER

From the definition of γ_S it can be shown [13] that

$$\gamma_S = 1 - \frac{D_W}{A} \le \frac{1}{2}.$$

In fact, near equilibriuim $-A$ is roughly a linear function of the currents J_{xy} (if the stochastic potential is kept fixed), whereas the power dissipation D_W can be approximated by a quadratic function of the currents near detailed balance conditions. Thus, if the currents are varied, the maximum power $-P = A - D_W$ is obtained when $D_W = A/2$, so that, near reversibility conditions, the power dissipated is equal to the power produced.

It can be shown by more sophisticated arguments that the approximate value $\gamma_S \approx 1/2$, valid near equilibrium, is an upper bound of γ_S out of equilibrium. Thus, we obtain the surprising conclusion that at maximum power production, the power dissipated is at least equal to the power produced!

CARNOT EFFICIENCY FAR FROM EQUILIBRIUM

In spite of its disadvantages in certain cases, Carnot efficiency is easily measurable and it is widely used in engineering. In practice, it remains a universal tool for motors operating between two heat sources, even though sustainable efficiency can be preferable from a theoretical point of view. Clearly, γ_C and γ_S cannot be exactly related, since they do not involve the same quantities. However, one can write, with the previous notations

$$\gamma_C = 1 - \frac{T_2}{T_1} - T_2\frac{D_S}{\dot{Q}_1},$$

which explicitly relates γ_C with the entropy dissipation D_S. From the definition of γ_S it results that

$$T_2\frac{D_S}{\dot{Q}_1} = \left(1 - \frac{T_2}{T_1}\right)T_2 \cdot \frac{D_0 + D_1 + D_2}{\dfrac{T_1D_1 + T_2D_2}{1-\gamma_s} + T_2D_0 + (T_2 - T_1)D_1},$$

D_i being the entropy dissipation with source i where i = 0 is the mechanical system, i = 1 the hot source, and i = 2 the cold source).

UPPER BOUND FOR CARNOT EFFICIENCY AT MAXIMUM POWER

In the previous formula, D_0 is the entropy dissipation during exchanges with mechanical system. It is expected that this dissipation vanishes if the motor is conveniently operated (in particular, if it is correctly lubricated), so that these exchanges are reversible. Then, using the fact that $\gamma_S \le 1/2$, we have

$$T_2\frac{D_S}{\dot{Q}_1} \ge \left(1 - \frac{T_2}{T_1}\right)\frac{1}{2} \cdot \frac{D_1 + D_2}{T_1D_1 + T_2D_2 + \dfrac{(T_2 - T_1)D_1}{2}},$$

It can be shown that the left hand side reaches its minimum value when $D_2/D_1 \to 0$, so that eventually

$$\gamma_C \leq \frac{T_1 - T_2}{T_1 + T_2} \equiv \bar{\gamma}_C .$$

As mentioned previously $\bar{\gamma}_C$ is a new upper bound for Carnot efficiency at maximum power, obviously smaller than the classical Carnot upper bound. It has been shown [12] that this upper bound also applies for cyclic thermal motors.

It can be noticed that, by chance, $\bar{\gamma}_C$ coincides with the upper bound of the naive "ecologic" efficiency γ_E defined in the introduction.

THE NEW UPPER BOUND AND THE CA BOUND

The upper bound $\bar{\gamma}_C$ can only be attained in the limits $D_0 \to 0$ (which can be approached experimentally, using a convenient lubrication, since it concerns mechanical exchanges), $D_2/D_1 \to 0$ (which is not so realistic, because thermal exchanges are not so easily controlled) and $\bar{\gamma}_C \to 1/2$ (implying that the system is close to detailed balance conditions). These conditions show how an actual motor should be designed in order to increase the upper bound $\bar{\gamma}_C$. On the other hand it is easily seen that

$$\bar{\gamma}_N \equiv 1 - \sqrt{\frac{T_2}{T_1}} \leq \bar{\gamma}_C = \frac{T_1 - T_2}{T_1 + T_2},$$

the equality holding if and only if $T_2 = T_1$.

Thus, the new upper bound allows larger values for the Carnot efficiency at maximum power than the CA upper bound γ_{CA}. This remark is supported by the experimental measures, but it cannot be asserted that the observed power stations actually operate in the conditions of maximum power production, although they should presumably be close to these conditions.

CONCLUSION

We introduced a new definition for efficiency for systems at stationary state. This sustainable efficiency γ_S gives a "fair" estimation of the losses due to irreversibility and should be well adapted for the concerns of sustainable development. We proved that at maximum power production, the sustainable efficiency is lower than or equal to 1/2, so that the power produced cannot exceed the power dissipated. Although the sustainable efficiency is not easily measured, knowing its expression can hopefully allow one limiting the irreversible losses when designing a motor. As a practical result, the previous, universal upper bound of γ_S implies a new upper bound for Carnot efficiency at maximum power, intermediary between the Carnot upper bound and the Yvon-Novikov, or Curzon-Ahlborn bound. The derivation makes clear the theoretical conditions for approaching the new upper bound.

The notion of sustainable efficiency can be applied for any number of sources, and can be extended to other situations, for instance to chemical motors. Then it should be compared with other classical results, obtained for instance by Sieniutycz, (see [8-10] and references therein, and [11]). This is why sustainable efficiency should be especially useful in biophysics.

More generally, it can be used in any field where entropy, or a similar quantity, can be defined, provided that another quantity similar to work, measures the "benefit" obtained from the system. Although this situation only applies in special cases, it can lead to interesting new perspectives, hopefully completing the pioneering work that Amelkin, Martinas and Tsirlin devoted to thermodynamics and microeconomics in 2002.

REFERENCES

[1] Carnot, S.: *Réflexions sur la Puissance Motrice du Feu set sur les Machines Propres à Développer cette Puissance*.
Book published at the author's expenses, Paris, 1824,
[2] Yvon J.: *The Saclay reactor: two years experience on heat transfer by means of a compressed gas*.
Proceedings of the International Conference on Peaceful Uses of Atomic Energy, Geneva, p.387, 1995,
[3] Novikov, I.I.: *The Efficiency of Atomic Power Stations*.
Journal of Nuclear Energy II **7**(1-2), 125-128, 1958,
[4] Chambadal, P.: *Les Centrales Nucléaires*.
Armand Colin, Paris, 1957,
[5] Curzon, F.L. and Ahlborn, B.: *Efficiency of a Carnot engine at maximum power output*.
American Journal of Physics **43**(1), 22-24, 1975,
[6] Van den Broeck, C.: *Thermodynamic Efficiency at Maximum Power*.
Physical Review Letters **95**(19), 190602, 2005,
[7] Schmiedl, T. and Seifert, U.: *Efficiency at maximum power: An analytically solvable model for stochastic heat engines*.
Europhysics Letters **81**(2), 20003, 2008,
[8] Sieniutycz, S.: *Hamilton-Jacobi-Bellman theory of dissipative thermal avalaibility.*
Physical Review E **56**(5), 5051-5064, 1997,
[9] Sieniutycz, S.: *Hamilton-Jacobi-Bellman framework for optimal control in multistage energy systems*.
Physics Reports **326**(4), 165-258, 2000,
[10] Sieniutycz, S.: *Dynamics bounds for power and efficiency of non-ideal energy converters under non-linear tranfer laws*.
Energy **34**(3), 334-340, 2009,
[11] Tsirlin, A.M.; Kazakov, V.; Kan, N.-M. and Trushkov, V.V.: *Thermodynamic analysis and thermodynamic efficiency of chemical reactors*.
Journal of Physical Chemistry B **110**(5), 2338-2342, 2006,
[12] Esposito, M.; Kawai, R.; Lindenberg, K. and Van den Broeck, C.: *Efficiency at maximum power of low dissipation Carnot engines*.
http://arxiv.org/abs/1008.2464v1, 2010,
[13] Gaveau, B.; Moreau, M. and Schulman, L.S.: *Stochastic Thermodynamics and Sustainable Efficiency in Work Production*.
Physical Review Letters **105**(6), 060601, 2010a,
[14] Esposito, M.; Lindenberg, K. and Van den Broeck, C.: *Thermoelectric efficiency at maximum power in a quantum dot*.
Europhysics Letters **85**(6), 60010, 2009,
http://arxiv.org/abs/0808.0216v1,
[15] Esposito, M.; Lindenberg, K. and Van den Broeck, C.: *Universality of Efficiency at Maximum Power*.
Physical Review Letters**102**(13), 130602, 2009,
[16] Gaveau, B.: Moreau, M. and Schulman, L.S.: *Constrained maximal power in small engines*.
Physical Review E **82**(5), 051109, 2010b,
[17] Gaveau, B.; Moreau, M. and Schulman, L.S.: *Generalized Clausius relation and power dissipation in nonequilibrium stochastic systems*.
Physical Review E **79**(1), 010102, 2009,
[18] Gallavotti, G. and Cohen, E.G.D.: *Dynamical Ensembles in Nonequilibrium Statistical Mechanics*.
Physical Review Letters **74**(14), 2694-2697, 1995,

[19] Gallavotti, G. and Cohen, E.G.D.: *Dynamical ensembles in stationary states*. Journal of Statistical Physics **80**(5-6), 931-970, 1995.

ODRŽIVA UČINKOVITOST SUSTAVĀ DALEKO OD RAVNOTEŽE

M. Moreau[1], B. Gaveau[2] and L.S. Schulman[3]

[1]Laboratorij za teorijsku fiziku i fiziku čvrstoga stanja, Sveučilište Pierre i Marie Curie
Pariz, Francuska

[2]Odsjek za matematiku, Sveučilište Pierre i Marie Curie
Pariz, Francuska

[3]Odsjek za fiziku, Sveučilište Clarkson
Potsdam, Sjedinjene američke države

SAŽETAK

The Carnot efficiency of usual thermal motors compares the work produced by the motor to the heat received from the hot source, neglecting the perturbation of the cold source: thus, even if it may be appropriate for industrial purposes, it is not pertinent in the scope of sustainable development and environment care. In the framework of stochastic dynamics we propose a different definition of efficiency, which takes into account the entropy production in all the irreversible processes considered and allows for a fair estimation of the global costs of energy production from heat sources: thus, we may call it "sustainable efficiency". It can be defined for any number of sources and any kind of reservoir, and it may be extended to other fields than conventional thermodynamics, such as biology and, hopefully, economics.

Both sustainable efficiency and Carnot efficiency reach their maximum value when the processes are reversible, but then, power production vanishes. In practise, it is important to consider these efficiencies out of equilibrium, in the conditions of maximum power production. It can be proved that in these conditions, the sustainable efficiency has a universal upper bound, and that the power loss due to irreversibility is at less equal to the power delivered to the mechanical, external system.

However, it may be difficult to deduce the sustainable efficiency from experimental observations, whereas Carnot's efficiency is easily measurable and most generally used for practical purposes. It can be shown that the upper bound of sustainable efficiency implies a new higher bound of Carnot efficiency at maximum power, which is higher than the so-called Curzon-Ahlborn bound of efficiency at maximum power.

KLJUČNE RIJEČI

učinkovitost, održiva učinkovitost, maksimalna snaga, entropija