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ON THE HYPERBOLIC PARTIAL DIFFERENCE EQUATIONS AND THEIR OSCILLATORY PROPERTIES

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Abstract. For the solutions hyperbolic partial difference equation

$$\mathrm{D}^2_{/1,2}\, y(m,n) = a(m,n)y(m,n)$$

satisfying some boundary conditions an analytical formula is presented. The solutions are then studied in relation to their oscillatory properties.

In this note we shall study the linear partial difference equations of the form

$$\mathrm{D}^2_{/1,2}\, y(m,n) = a(m,n)y(m,n), \ \ (m,n) \in \mathbf{N}_0^2 \tag{E}$$

where $\mathbf{N}_0 = \{0, 1, \ldots\}$ is the set of nonnegative integers.

In Section 1 a closed analytical formula for the solution of (E) satisfying some boundary conditions is given. In Section 2 we shall study existence of oscillatory solutions, by using the method of separation of variables.

We shall consider real valued sequences of two independent variables, that is the functions $y : \mathbf{N}_0^2 \to \mathbf{R}$.

For the sequence $y = \{y(m,n)\}_{m=0}^{\infty}{}_{n=0}^{\infty}$ we define partial difference operators of the first order:

$$\begin{aligned} \mathrm{D}_{/1}\, y(m,n) &= y(m+1,n) - y(m,n), \quad m,n \in \mathbf{N}_0 \\ \mathrm{D}_{/2}\, y(m,n) &= y(m,n+1) - y(m,n), \quad m,n \in \mathbf{N}_0; \end{aligned}$$

and of the second order

$$\mathrm{D}^2_{/1,2}\, y(m,n) = \mathrm{D}_{/2}(\mathrm{D}_{/1}\, y(m,n)), \quad m,n \in \mathbf{N}_0.$$

It is evident that

$$\mathrm{D}^2_{/1,2}\, y(m,n) = \mathrm{D}^2_{/2,1}\, y(m,n) = y(m+1,n+1) - y(m+1,n) - y(m,n+1) + y(m,n).$$

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1. Explicit form of solutions of the equation (E)

In this Section we introduce some sum operators, similar to those used in [2], which allow us to present every solution of (E) in an explicit form.

For any two sequences $x : \mathbf{N}_0^2 \to \mathbf{R}$, $\omega : \mathbf{N}_0 \to \mathbf{R}$ we define

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i) = \sum_{\substack{m\geqslant i_1>i_2\dots>i_r\geqslant\mu\\ n\geqslant j_1>j_2\dots>j_r\geqslant\nu}} \Big(\prod_{k=1}^{r} x(i_k,j_k)\Big)\omega(i_r)$$

for $\mu, m, \nu, n \in \mathbf{N}_0$, $1 \leqslant r \leqslant \min(m-\mu+1, n-\nu+1)$, $m \geqslant \mu$, $n \geqslant \nu$;

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i) = 0$$

for $r = 0$, or $r > \min(m-\mu+1, n-\nu+1)$, or $m < \mu$ or $n < \nu$.

For example

$$\sum_{\substack{i\in\overline{2,5}\\ j\in\overline{6,7}}}{}_{1}\, x(i,j)\circ\omega(i) = x(2,6)\omega(2)+x(2,7)\omega(2)+x(3,6)\omega(3)+x(3,7)\omega(3)$$
$$+x(4,6)\omega(4)+x(4,7)\omega(4)+x(5,6)\omega(5)+x(5,7)\omega(5),$$

$$\sum_{\substack{i\in\overline{2,5}\\ j\in\overline{6,7}}}{}_{2}\, x(i,j)\circ\omega(i) = x(3,7)x(2,6)\omega(2)+x(4,7)x(2,6)\omega(2)+x(5,7)x(2,6)\omega(2)$$
$$+x(5,7)x(3,6)\omega(3)+x(4,7)x(3,6)\omega(3)+x(5,7)x(4,6)\omega(4),$$

$$\sum_{\substack{i\in\overline{2,5}\\ j\in\overline{6,7}}}{}_{3}\, x(i,j)\circ\omega(i) = 0.$$

The following properties of the operator $\sum_r$ can be observed

$$\sum_{\substack{i\in\overline{\kappa,\kappa}\\ j\in\overline{\nu,n}}}{}_{1}\, x(i,j)\circ\omega(i) = \sum_{j=\nu}^{n} x(\kappa,j)\omega(\kappa), \qquad \sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\kappa,\kappa}}}{}_{1}\, x(i,j)\circ\omega(i) = \sum_{i=\mu}^{m} x(i,\kappa)\omega(i),$$

hence

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{1}\, x(i,j)\circ\omega(i) + \sum_{s=\mu}^{m} x(s,n+1)\omega(s) = \sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n+1}}}{}_{1}\, x(i,j)\circ\omega(i),$$

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{1}\, x(i,j)\circ\omega(i) + \sum_{s=\nu}^{n} x(m+1,s)\omega(m+1) = \sum_{\substack{i\in\overline{\mu,m+1}\\ j\in\overline{\nu,n}}}{}_{1}\, x(i,j)\circ\omega(i);$$

furthermore

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i) = \sum_{n\geqslant j_1>j_2\ldots>j_r\geqslant\nu}\Big(\sum_{i_1=\mu}^{m} x(i_1,j_1)\sum_{i_2=\mu}^{i_1-1} x(i_2,j_2)\ldots\sum_{i_r=\mu}^{i_{r-1}-1} x(i_r,j_r)\omega(i_r)\Big)$$

and

$$\sum_{p=\mu}^{m}\sum_{q=\nu}^{n} x(p,q)\sum_{\substack{i\in\overline{\mu,p-1}\\ j\in\overline{\nu,q-1}}}{}_{r-1}\, x(i,j)\circ\omega(i) = \sum_{p=\mu+1}^{m}\sum_{q=\nu+1}^{n} x(p,q)\sum_{\substack{i\in\overline{\mu,p-1}\\ j\in\overline{\nu,q-1}}}{}_{r-1}\, x(i,j)\circ\omega(i)$$

$$= \sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i),$$

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i) = \sum_{\substack{i\in\overline{\mu,m-1}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i) + \sum_{q=\nu+r-1}^{n} x(m,q)\sum_{\substack{i\in\overline{\mu,m-1}\\ j\in\overline{\nu,q-1}}}{}_{r-1}\, x(i,j)\circ\omega(i)$$

for $r \geqslant 2$.

Similar properties are possessed by the operator

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(j) = \sum_{\substack{m\geqslant i_1>i_2\ldots>i_r\geqslant\mu\\ n\geqslant j_1>j_2\ldots>j_r\geqslant\nu}}\Big(\prod_{k=1}^{r} x(i_k,j_k)\Big)\omega(j_r).$$

For $\omega(n)\equiv 1$ on $\mathbf{N}_0$ we shall write

$$\sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(j) = \sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j)\circ\omega(i) = \sum_{\substack{i\in\overline{\mu,m}\\ j\in\overline{\nu,n}}}{}_{r}\, x(i,j) = \sum_{\substack{m\geqslant i_1>i_2\ldots>i_r\geqslant\mu\\ n\geqslant j_1>j_2\ldots>j_r\geqslant\nu}}\Big(\prod_{k=1}^{r} x(i_k,j_k)\Big).$$

From the above we obtain:

THEOREM 1. *Let $a : \mathbf{N}_0^2 \to \mathbf{R}$, and $\varphi, \psi : \mathbf{N}_0 \to \mathbf{R}$ be functions such that $\varphi(0) = \psi(0)$. Then there exists a unique solution of the problem*

$$\begin{cases} \mathrm{D}^2_{/1,2}\, y(m,n) = a(m,n)y(m,n)\\ y(m,0) = \varphi(m)\\ y(0,n) = \psi(n)\end{cases} \tag{P1}$$

for $m, n \in \mathbf{N}_0$. This solution can be presented in the form

$$y(m,n) = \varphi(m) + \sum_{r=1}^{m-1}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,n-1}}}{}_{r}\, a(i,j)\circ\varphi(i) + \psi(n) + \sum_{r=1}^{n-1}\sum_{\substack{i\in\overline{0,m-1}\\ j\in\overline{1,n-1}}}{}_{r}\, a(i,j)\circ\psi(j)$$

$$+ \big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{\rho}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{1,n-1}}}{}_{r}\, a(i,j) + \big(\varphi(0)a(0,0)-\varphi(0)\big) \tag{1}$$

for all $m, n \in \mathbf{N}$, where $\rho = \min\{m-1, n-1\}$.

Proof. Equation (E) can be written in the following equivalent form:

$$y(m+1, n+1) = y(m+1, n) + y(m, n+1) - y(m, n) + a(m, n)y(m, n). \quad (2)$$

Assuming that (1) holds for all $m \in \{1, 2, \ldots, m_1\}, n \in \{1, 2, \ldots, n_1\}$ we can prove (1) inductively from (2), first for $m = m_1 + 1$ and successively $n = 1, 2, \ldots, n_1$; then for $n = n_1 + 1$ and successively $m = 1, 2, \ldots, m_1$, and finally for $m = m_1 + 1$, $n = n_1 + 1$. The proof is rather technical so to elucidate we present short part of it. Let (1) holds for $y(m, 1)$, that is

$$\begin{aligned} y(m, 1) &= \varphi(m) + \sum_{r=1}^{m-1} \sum_{\substack{i \in \overline{1, m-1} \\ j \in \overline{0, 0}}}{}_{r} \, a(i, j) \circ \varphi(i) + \psi(1) + \big(\varphi(0)a(0, 0) - \varphi(0)\big) \\ &= \varphi(m) + \sum_{i=1}^{m-1} a(i, 0) \circ \varphi(i) + \psi(1) + \big(\varphi(0)a(0, 0) - \varphi(0)\big). \end{aligned}$$

Hence for $y(m+1, 1)$ we get from (2) by initial conditions

$$\begin{aligned} y(m+1, 1) &= y(m+1, 0) + y(m, 1) - y(m, 0) + a(m, 0)y(m, 0) \\ &= \varphi(m+1) + \varphi(m) + \sum_{i=1}^{m-1} a(i, 0) \circ \varphi(i) + \psi(1) + \big(\varphi(0)a(0, 0) - \varphi(0)\big) \\ &\qquad - \varphi(m) + a(m, 0)\varphi(m) \\ &= \varphi(m+1) + \sum_{i=1}^{m} a(i, 0) \circ \varphi(i) + \psi(1) + \big(\varphi(0)a(0, 0) - \varphi(0)\big). \end{aligned}$$

That is (1) valid for $y(m+1, 1)$. Similarly (1) can be verified for $y(1, n)$.

Assume that (1) holds for $y(m+1, k), y(m, k+1), y(m, k)$, where $m > k$. We prove that (1) holds for $y(m+1,k+1)$. By (2) and inductive assumption we obtain

$$
\begin{aligned}
y(m+1,k+1) &= y(m+1,k)+y(m,k+1)-y(m,k)+a(m,k)y(m,k)\\
&= \Bigg\{\varphi(m+1)+\sum_{r=1}^{k}\sum_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k-1}}}{}_{r}\, a(i,j)\circ\varphi(i)+\psi(k)+\sum_{r=1}^{k-1}\sum_{\substack{i\in\overline{0,m}\\ j\in\overline{1,k-1}}}{}_{r}\, a(i,j)\circ\psi(j)\\
&\quad +\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k-1}\sum_{\substack{i\in\overline{1,m}\\ j\in\overline{1,k-1}}}{}_{r}\, a(i,j)+\big(\varphi(0)a(0,0)-\varphi(0)\big)\Bigg\}\\
&\quad +\Bigg\{\varphi(m)+\sum_{r=1}^{m-1}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}}{}_{r}\, a(i,j)\circ\varphi(i)+\psi(k+1)+\sum_{r=1}^{k}\sum_{\substack{i\in\overline{0,m-1}\\ j\in\overline{1,k}}}{}_{r}\, a(i,j)\circ\psi(j)\\
&\quad +\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{1,k}}}{}_{r}\, a(i,j)+\big(\varphi(0)a(0,0)-\varphi(0)\big)\Bigg\}\\
&\quad -\Bigg\{\varphi(m)+\sum_{r=1}^{k}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}}{}_{r}\, a(i,j)\circ\varphi(i)+\psi(k)+\sum_{r=1}^{k-1}\sum_{\substack{i\in\overline{0,m-1}\\ j\in\overline{1,k-1}}}{}_{r}\, a(i,j)\circ\psi(j)\\
&\quad +\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k-1}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{1,k-1}}}{}_{r}\, a(i,j)+\big(\varphi(0)a(0,0)-\varphi(0)\big)\Bigg\}\\
&\quad +a(m,k)\Bigg\{\varphi(m)+\sum_{r=1}^{k}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}}{}_{r}\, a(i,j)\circ\varphi(i)+\psi(k)+\sum_{r=1}^{k-1}\sum_{\substack{i\in\overline{0,m-1}\\ j\in\overline{1,k-1}}}{}_{r}\, a(i,j)\circ\psi(j)\\
&\quad +\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k-1}\sum_{\substack{i\in\overline{1,m-1}\\ j\in\overline{1,k-1}}}{}_{r}\, a(i,j)+\big(\varphi(0)a(0,0)-\varphi(0)\big)\Bigg\}.
\end{aligned}
$$

Notice that suitable upper limits of summations have been changend because of the condition $m > k$. We shall consider the sum

$$\sum_{r=1}^{k} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) + \sum_{r=1}^{m-1} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}} a(i,j)\circ\varphi(i) - \sum_{r=1}^{k} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$+ a(m,k)\varphi(m) + a(m,k)\sum_{r=1}^{k} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$= \sum_{r=1}^{m-1} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}} a(i,j)\circ\varphi(i) + a(m,k)\varphi(m) + a(m,k)\sum_{r=1}^{k} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$+ \mathop{{\sum}_{1}}\limits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) + \mathop{{\sum}_{2}}\limits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) + \ldots + \mathop{{\sum}_{k}}\limits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$- \mathop{{\sum}_{1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) - \mathop{{\sum}_{2}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) - \ldots - \mathop{{\sum}_{k}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$= \sum_{r=1}^{m-1} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}} a(i,j)\circ\varphi(i) + a(m,k)\varphi(m) + a(m,k)\sum_{r=1}^{k} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$+ \sum_{s=0}^{k-1} a(m,s)\varphi(s) + \sum_{s=1}^{k-1} a(m,s) \mathop{{\sum}_{1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}} a(i,j)\circ\varphi(i) + \ldots$$

$$+ \sum_{s=k-1}^{k-1} a(m,s) \mathop{{\sum}_{k-1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}} a(i,j)\circ\varphi(i)$$

$$= \sum_{r=1}^{m-1} \mathop{{\sum}_{r}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}} a(i,j)\circ\varphi(i) + \sum_{s=0}^{k} a(m,s)\varphi(s)$$

$$+ a(m,k) \mathop{{\sum}_{1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) + \sum_{s=1}^{k-1} a(m,s) \mathop{{\sum}_{1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}} a(i,j)\circ\varphi(i) + \ldots$$

$$+ a(m,k) \mathop{{\sum}_{k-1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i) + \sum_{s=k-1}^{k-1} a(m,s) \mathop{{\sum}_{k-1}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}} a(i,j)\circ\varphi(i)$$

$$+ a(m,k) \mathop{{\sum}_{k}}\limits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}} a(i,j)\circ\varphi(i)$$

$$= \sum_{r=1}^{m-1} \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}}^{r} a(i,j)\circ\varphi(i) + \sum_{s=0}^{k} a(m,s)\varphi(s) + \sum_{s=1}^{k} a(m,s) \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}}^{1} a(i,j)\circ\varphi(i)$$

$$+ \ldots + \sum_{s=k-1}^{k} a(m,s) \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}}^{k-1} a(i,j)\circ\varphi(i) + a(m,k) \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}}^{k} a(i,j)\circ\varphi(i)$$

$$= \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}}^{1} a(i,j)\circ\varphi(i) + \sum_{s=0}^{k} a(m,s)\varphi(s) + \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}}^{2} a(i,j)\circ\varphi(i)$$

$$+ \sum_{s=1}^{k} a(m,s) \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}}^{1} a(i,j)\circ\varphi(i) + \ldots + \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}}^{k} a(i,j)\circ\varphi(i)$$

$$+ \sum_{s=k-1}^{k} a(m,s) \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,s-1}}}^{k-1} a(i,j)\circ\varphi(i)$$

$$+ a(m,k) \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k-1}}}^{k} a(i,j)\circ\varphi(i) \left(+ \sum\nolimits_{\substack{i\in\overline{1,m-1}\\ j\in\overline{0,k}}}^{k+1} a(i,j)\circ\varphi(i) \quad \text{if} \quad m > k+1 \right)$$

$$= \sum\nolimits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k}}}^{1} a(i,j)\circ\varphi(i) + \sum\nolimits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k}}}^{2} a(i,j)\circ\varphi(i) + \ldots$$

$$+ \sum\nolimits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k}}}^{k} a(i,j)\circ\varphi(i) + \sum\nolimits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k}}}^{k+1} a(i,j)\circ\varphi(i)$$

$$= \sum_{r=1}^{k+1} \sum\nolimits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k}}}^{r} a(i,j)\circ\varphi(i) = \sum_{r=1}^{m} \sum\nolimits_{\substack{i\in\overline{1,m}\\ j\in\overline{0,k}}}^{r} a(i,j)\circ\varphi(i).$$

In similar way we can prove

$$\sum_{r=1}^{k-1} \sum_{\substack{i\in\overline{0,m} \\ j\in\overline{1,k-1}}}{}_r\, a(i,j)\circ\psi(j) + \sum_{r=1}^{k} \sum_{\substack{i\in\overline{0,m-1} \\ j\in\overline{1,k}}}{}_r\, a(i,j)\circ\psi(j) - \sum_{r=1}^{k-1} \sum_{\substack{i\in\overline{0,m-1} \\ j\in\overline{1,k-1}}}{}_r\, a(i,j)\circ\psi(j)$$

$$+ a(m,k)\psi(k) + a(m,k)\sum_{r=1}^{k-1} \sum_{\substack{i\in\overline{0,m-1} \\ j\in\overline{1,k-1}}}{}_r\, a(i,j)\circ\psi(j) = \sum_{r=1}^{k} \sum_{\substack{i\in\overline{0,m} \\ j\in\overline{1,k}}}{}_r\, a(i,j)\circ\psi(j),$$

and

$$\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k-1} \sum_{\substack{i\in\overline{1,m} \\ j\in\overline{1,k-1}}}{}_r\, a(i,j) + \big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k} \sum_{\substack{i\in\overline{1,m-1} \\ j\in\overline{1,k}}}{}_r\, a(i,j)$$

$$-\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k} \sum_{\substack{i\in\overline{1,m-1} \\ j\in\overline{1,k-1}}}{}_r\, a(i,j) + a(m,k)\big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k-1} \sum_{\substack{i\in\overline{1,m-1} \\ j\in\overline{1,k-1}}}{}_r\, a(i,j)$$

$$+a(m,k)\big(\varphi(0)a(0,0)-\varphi(0)\big) = \big(\varphi(0)a(0,0)-\varphi(0)\big)\sum_{r=1}^{k} \sum_{\substack{i\in\overline{1,m} \\ j\in\overline{1,k}}}{}_r\, a(i,j).$$

Since we consider the case $m > k$, it suffices to replace upper limits of summation (because, for example in the sum $\sum_{r=1}^{m} \sum_{\substack{i\in\overline{1,m} \\ j\in\overline{1,k}}}{}_r\, a(i,j)$ terms with $r > k$ are equal to zero), and (2) holds for $y(m+1,k+1)$.

In the following result we shall consider a problem which will be useful in the next section.

THEOREM 2. *Let $a : \mathbf{N}_0^2 \to \mathbf{R}$ and $\varphi, \gamma : \mathbf{N}_0 \to \mathbf{R}$ be functions such that $\varphi(0) = \gamma(0)$. Then there exists a unique solution of the problem*

$$\begin{cases} \mathrm{D}^2_{/1,2}\, y(m,n) = a(m,n)y(m,n) \\ y(m,0) = \varphi(m) \\ y(n,n) = \gamma(n) \end{cases} \qquad \text{(P2)}$$

for $m, n \in \mathbf{N}_0$.

Proof. It suffices to take in Theorem 1

$$\psi(n) = \gamma(n) - \varphi(n) - \sum_{r=1}^{n-1} \sum_{\substack{i\in\overline{1,n-1}\\ j\in\overline{0,n-1}}}{}_{r}\, a(i,j)\circ\varphi(i) - \sum_{r=1}^{n-1} \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{1,n-1}}}{}_{r}\, a(i,j)\circ\psi(j)$$
$$- \big(\varphi(0)a(0,0) - \varphi(0)\big) \sum_{r=1}^{n-1} \sum_{\substack{i\in\overline{1,n-1}\\ j\in\overline{1,n-1}}}{}_{r}\, a(i,j) - \big(\varphi(0)a(0,0) - \varphi(0)\big)$$

for $n \geqslant 1$, $\psi(0) = \varphi(0)$, and note that from (1) we get $y(n,n) = \gamma(n)$.

A similar theorem can be formulated for the problem

$$\begin{cases} D^2_{/1,2}\, y(m,n) = a(m,n)y(m,n) \\ y(0,n) = \psi(n) \\ y(m,m) = \gamma(m) \end{cases} \tag{P3}$$

for $m, n \in \mathbf{N}_0$.

2. Oscillatory properties

Definition. A sequence $y = \{y(m,n)\}_{m=0}^{\infty}{}_{n=0}^{\infty}$ is said to be nonoscillatory (in relation to 0) if there exist positive integers μ, ν such that

$$y(m,n) > 0 \quad \text{(positive sequence)} \quad \text{for all} \quad m \geqslant \mu, n \geqslant \nu$$

or

$$y(m,n) < 0 \quad \text{(negative sequence)} \quad \text{for all} \quad m \geqslant \mu, n \geqslant \nu.$$

Otherwise the sequence y is called oscillatory.

However this definition calls eventually zero sequence as oscillatory, we exclude this type of sequences from our considerations.

If the sequence y is nonoscillatory then it is nonoscillatory for each section along each arbitrary but fixed $m = \overline{m}$, $\overline{m} \geqslant \mu$ as well as arbitrary but fixed $n = \overline{n}$, $\overline{n} \geqslant \nu$. Moreover, all these sections are of the same sign (positive or negative).

A necessary and sufficient condition for oscillation can be formulated as follows.

THEOREM 3. *A sequence $y = \{y(m,n)\}_{m=0}^{\infty}{}_{n=0}^{\infty}$ of real numbers is oscillatory if and only if there exist increasing to infinity sequences $\mathfrak{m} = \{m_k\}_{k=1}^{\infty}$ and $\mathfrak{n} = \{n_k\}_{k=1}^{\infty}$ such that $\{y(m_k, n_k)\}_{k=1}^{\infty}$ is an oscillatory sequence.*

Example 1. Consider the equation

$$D^2_{/1,2}\, y(m,n) = y(m,n).$$

One of the oscillatory solutions of this equation, namely satisfying

$$y(m,0) = 1 \quad \text{for} \quad m \in \mathbf{N}_0,$$

and

$$y(m,m) = (-1)^m \quad \text{for} \quad m \in \mathbf{N}_0,$$

can be presented in the form

$$y(m,n) = (-1)^n + \sum_{k=1}^{n} \Big\{ (-1)^{n-k} \sum_{j_n=n+1}^{m} \sum_{j_{n-1}=n}^{j_n} \cdots \sum_{j_{n-k+1}=n-k+2}^{j_{n-k+2}} 1 \Big\} \quad \text{for} \quad m \geqslant n, n \in \mathbf{N}.$$

For example

$$y(m,3) = (-1)^3 + \sum_{k=1}^{3} \Big\{ (-1)^{3-k} \sum_{j_3=4}^{m} \cdots \sum_{j_{4-k}=5-k}^{j_{5-k}} 1 \Big\}$$

$$= (-1)^3 + (-1)^2 \sum_{j_3=4}^{m} 1 + (-1)^1 \sum_{j_3=4}^{m} \sum_{j_2=3}^{j_3} 1 + (-1)^0 \sum_{j_3=4}^{m} \sum_{j_2=3}^{j_3} \sum_{j_1=2}^{j_2} 1.$$

Assuming that

$$\sum_{k=1}^{n} \Big\{ (-1)^{n-k} \sum_{j_n=n+1}^{m} \sum_{j_{n-1}=n}^{j_n} \cdots \sum_{j_{n-k+1}=n-k+2}^{j_{n-k+2}} 1 \Big\}$$

$$= \sum_{k=1}^{n} \Big\{ (-1)^{n-k} \sum_{j_n=n+1}^{s} \sum_{j_{n-1}=n}^{j_n} \cdots \sum_{j_{n-k+1}=n-k+2}^{j_{n-k+2}} 1 \Big\} \Bigg|_{s=m}$$

the above formula remains true for $m \leqslant n$, if furthermore we take $\sum_{k=1}^{n} \ldots = 0$ for $n = 0$ also for $y(m,0)$, $m \in \mathbf{N}_0$.

From Theorem 2 it follows immediately:

THEOREM 4. *Let $a : \mathbf{N}_0^2 \to \mathbf{R}$. For any function $\varphi : \mathbf{N}_0 \to \mathbf{R}$ there exists a family of oscillatory solutions of the problem*

$$\begin{cases} \mathrm{D}^2_{/1,2}\, y(m,n) = a(m,n)y(m,n) \\ \qquad y(m,0) = \varphi(m) \end{cases} \tag{P4}$$

for $m, n \in \mathbf{N}_0$

Proof. To get the desired result we can take in the Theorem 2 any arbitrary oscillatory sequence ψ.

In fact we can obtain the same result as in the Theorem 4 for the problem

$$\begin{cases} \mathrm{D}^2_{/1,2}\, y(m,n) = a(m,n)y(m,n) \\ y(m,0) = \varphi(m) \\ y(0,\alpha(n)) = \gamma(m) \end{cases} \tag{P5}$$

where $\alpha : \mathbf{N}_0 \to \mathbf{N}_0$ is strictly increasing and such that $\operatorname{card}\{\mathbf{N}_0 \setminus \alpha(\mathbf{N}_0)\} = \aleph_0$.

THEOREM 5. *Let* $a : \mathbf{N}_0^2 \to \mathbf{R}_+$, *and* $\varphi, \psi : \mathbf{N}_0 \to \mathbf{R}_+$ *be functions such that* $\varphi(0) = \psi(0)$. *Then the solution of the problem* (P1) *is positive on* $\mathbf{N}_0^2$.

In what follows we shall study solutions of the equation (E) of a particular type, namely those which can be presented as a product of two functions of one independent variable. We use the method of separation of variables (see [1] for details) and examine the solutions thus obtained.

THEOREM 6. *Let* $\alpha, \beta : \mathbf{N}_0 \to \mathbf{R}$. *Then there exist solutions of the equation*

$$\mathrm{D}^2_{/1,2}\, y(m,n) = \alpha(m)\beta(n)y(m,n), \quad (m,n) \in \mathbf{N}_0^2 \tag{E1}$$

which can be written as

$$y(m,n) = y(0,0)\left(\prod_{i=0}^{m-1}\left(1 + c\alpha(i)\right)\right)\left(\prod_{j=0}^{n-1}\left(1 + \beta(j)/c\right)\right), \tag{3}$$

where c is any real nonzero constant.

Proof. Suppose that the solution of (E1) can be presented as the product $y(m,n) = u(m)v(n)$ for all $(m,n) \in \mathbf{N}_0^2$. Then $\mathrm{D}^2_{/1,2}\, y(m,n) = \left(\Delta u(m)\right)\left(\Delta v(n)\right)$, where $\Delta u(m) = u(m+1) - u(m)$. Assuming that the suitable operations are admissible, we get

$$\frac{\Delta u(m)}{\alpha(m)u(m)} = \frac{\beta(n)v(n)}{\Delta v(n)} \; (= c).$$

From this we obtain two first order difference equations in one variable

$$\begin{cases} \Delta u(m) = c\alpha(m)u(m), & m \in \mathbf{N}_0 \\ \Delta v(n) = (1/c)\beta(n)v(n), & n \in \mathbf{N}_0. \end{cases}$$

The solutions of these equations are

$$u(m) = u(0)\prod_{i=0}^{m-1}\left(1 + c\alpha(i)\right), \quad m \in \mathbf{N}_0$$

and

$$v(n) = v(0)\prod_{j=0}^{n-1}\left(1 + \beta(j)/c\right), \quad n \in \mathbf{N}_0.$$

Therefore

$$\begin{aligned} u(m)v(n) &= \left(u(0)\prod_{i=0}^{m-1}\left(1 + c\alpha(i)\right)\right)\left(v(0)\prod_{j=0}^{n-1}\left(1 + \beta(j)/c\right)\right) \\ &= y(0,0)\left(\prod_{i=0}^{m-1}\left(1 + c\alpha(i)\right)\right)\left(\prod_{j=0}^{n-1}\left(1 + \beta(j)/c\right)\right) \end{aligned}$$

should be solution of (E1). By direct substitution to (E1) we check validity of this statement.

From the formula (3) we can deduce existence of solutions of (E1) which posses some interesting properties. For example in relation to oscillation we get

THEOREM 7. *Let $\alpha, \beta : \mathbf{N}_0 \to \mathbf{R}$. The equation* (E1) *has an oscillatory solution in the form given by the formula* (3) *if and only if the sequence α or β possesses a subsequence bounded away from zero.*

Proof. Necessity. Suppose for contrary both $\lim_{m\to\infty} \alpha(m) = 0$ and $\lim_{n\to\infty} \beta(n) = 0$. Hence for any c there exist $m(c) \in \mathbf{N}$ and $n(c) \in \mathbf{N}$ such that $1 + c\alpha(m) > 0$ for all $m \geqslant m(c)$ and $1 + \beta(n)/c > 0$ for all $n \geqslant n(c)$. So for all $m \geqslant m(c)$, $n \geqslant n(c)$ there is

$$\operatorname{sgn}\left\{y(0,0)\left(\prod_{i=0}^{m-1}(1+c\alpha(i))\right)\left(\prod_{j=0}^{n-1}(1+\beta(j)/c)\right)\right\}$$
$$= \operatorname{sgn}\left\{y(0,0)\left(\prod_{i=0}^{m(c)-1}(1+c\alpha(i))\right)\left(\prod_{j=0}^{n(c)-1}(1+\beta(j)/c)\right)\right\}.$$

Sufficiency. Suppose that there exists a sequence $\{m_k\}_{k=1}^{\infty}$ and a constant $\varepsilon < 0$ such that $\alpha(m_k) < \varepsilon$ for all $k \in \mathbf{N}$. Then for $c > \dfrac{-1}{\varepsilon}$ we have

$$1 + c\alpha(m_k) < 1 + c\varepsilon < 0 \quad \text{for all} \quad k \in \mathbf{N}.$$

Therefore the sequence $\left\{\prod_{i=0}^{m-1}(1+c\alpha(i))\right\}_{m=1}^{\infty}$ is oscillatory, and hence the same behaviour characterises the sequences $\left\{y(0,0)\left(\prod_{i=0}^{m-1}(1+c\alpha(i))\right)\left(\prod_{j=0}^{v-1}(1+\beta(j)/c)\right)\right\}_{m=1}^{\infty}$ for any fixed but arbitrary $v \in \mathbf{N}$. Hence the solution $y(m,n)$ given by the formula (3) is oscillatory. Similar proofs can be made in other cases that is: $\alpha(m_k) > \varepsilon > 0$, $\beta(n_k) < \varepsilon < 0$, $\beta(n_k) > \varepsilon > 0$.

Example 2. Let in (E1) $\alpha(m) = a$ (constant), $\beta(n) = 1$. Then the solutions of (E1) given by (3) are

$$y(m,n) = y(0,0)(1+ca)^m(1+1/c)^n.$$

Let in (E1) $\alpha(m) = \dfrac{1}{m+1}$, $\beta(n) = \dfrac{1}{n+1}$. Then $y(m,n) = y(0,0)\prod_{i=0}^{m-1}\left(1 + \frac{c}{i+1}\right)\prod_{j=0}^{n-1}\left(1+\frac{1}{c(j+1)}\right)$.

Since

$$\prod_{i=0}^{m-1}\left(1+\tfrac{c}{i+1}\right) = \prod_{i=0}^{m-1}\tfrac{1}{i+1}(i+1+c) = \frac{1}{m!}\frac{\Gamma(c+m)}{\Gamma(c)} \quad \text{for} \quad c \neq -k,\ k \in \mathbf{N},$$

$$\prod_{j=0}^{n-1}\left(1+\tfrac{1}{c(j+1)}\right) = \prod_{j=0}^{n-1}\tfrac{1}{j+1}\left(j+1+\tfrac{1}{c}\right) = \frac{1}{n!}\frac{\Gamma(n+1/c)}{\Gamma(1/c)} \quad \text{for} \quad c \neq \tfrac{1}{k},\ k \in \mathbf{N},$$

then

$$y(m,n) = y(0,0)\frac{1}{m!}\frac{1}{n!}\frac{\Gamma(m+c)\Gamma(n+1/c)}{\Gamma(c)\Gamma(1/c)}$$

for $(m, n) \in \mathbf{N}^2, c \neq 0, c \neq -k, c \neq -1/k, k \in \mathbf{N}$.

For $c = -k$, $k \in \mathbf{N}$ there is $y(m, n) = 0$ for $m \geqslant k$, similarly for $c = -1/k$, $k \in \mathbf{N}$ we get $y(m, n) = 0$ for $n \geqslant k$.

Let in (E1) $\alpha(m) = m$, $\beta(n) = n$. Then the solution given by (3) for $c \neq 0$, $c \neq -k, c \neq -\frac{1}{k}, k \in \mathbf{N}$ is of the form

$$y(m, n) = y(0, 0)c^{m-n}\frac{\Gamma(m - 1 + 1/c)\Gamma(n - 1 + c)}{\Gamma(1/c)\Gamma(c)} \quad \text{for} \quad m \geqslant 1,\ n \geqslant 1.$$

Furthermore $y(m, n) = 0$ for $n \geqslant k + 1$ in the case $c = -k, k \in \mathbf{N}$ and $y(m, n) = 0$ for $m \geqslant k + 1$ in the case $c = -1/k, k \in \mathbf{N}$.

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