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ON GENERALIZED CAUCHY AND PEXIDER FUNCTIONAL EQUATIONS OVER A FIELD

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Abstract. Let $\mathbb{K}$ be a commutative field and $(P, +)$ be a uniquely 2-divisible group (not necessarily abelian). We characterize all functions $T : \mathbb{K} \to P$ such that the Cauchy difference $T(s+t) - T(t) - T(s)$ depends only on the product st for all $s, t \in \mathbb{K}$. Further, we apply this result to describe solutions of the functional equation $F(s+t) = K(st) \circ H(s) \circ G(t)$, where the unknown functions F, K, H, G map the field $\mathbb{K}$ into some function spaces arranged so that the compositions make sense. Conditions are established under which the equation can be reduced to a corresponding generalized Cauchy equation, and the general solution is given. Finally, we solve the equation $F(s+t) = K(st) + H(s) + G(t)$ for functions F, K, H, G mapping $\mathbb{K}$ into P. The paper generalizes some results from [11], [13] and, up to some extent, from [2].

1. Introduction

Consider the following functional equation

$$F(s+t) = K(st) + H(s) + G(t), \tag{1}$$

where the functions F, K, H and G to be determined map an algebraic structure (with two binary operations) into another one (with one binary operation). The problem of describing the general solution of (1) (under the additional assumption $G = H$) in the class of real functions defined on positive reals had been originally raised by Z. Daróczy (see [8]), and was solved by Gy. Maksa in [14]. In [7], J. A. Baker solved (1) for functions F, K, H, G mapping positive reals into a uniquely 2-divisible abelian group.

B. R. Ebanks, PL. Kannappan and P. K. Sahoo in [11] described all functions T, A mapping a commutative field $\mathbb{K}$ into a uniquely 2-divisible abelian group P such that

$$T(s+t) - T(t) - T(s) = A(st), \qquad s, t \in \mathbb{K}. \tag{2}$$

With this description, they obtained the general solution of (1) for functions F, K, H, G mapping the field $\mathbb{K}$ into the group P. The general solution of (1), for the case of functions F, K, H, G mapping the set of reals into itself, can be also found in [13].

In this paper, among other things, we generalize the results from [11] and [13] presenting general solutions of (1) and (2) for functions F, K, H, G, T, A which

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map a commutative field into a uniquely 2-divisible group (not necessarily abelian). We will consider (1) in an iterative form i.e. for functions F, K, H, G defined on a field $\mathbb{K}$ and taking values in some function spaces, with composition of functions as a binary operation. Then (1) can be rewritten as follows

$$F(s+t) = K(st) \circ H(s) \circ G(t), \qquad s, t \in \mathbb{K}. \tag{GPE}$$

Similarly, (2) takes the form

$$T(s+t) \circ T(t)^{-1} \circ T(s)^{-1} = A(st), \qquad s, t \in \mathbb{K}. \tag{GCE}$$

We shall establish conditions allowing to reduce (GPE) to (GCE), and solve (GPE) for F, K, H, G mapping $\mathbb{K}$ into a group of functions. Since any group can be considered as a group of transformations of a set, the latter result implies a solution of (1) for functions F, K, H, G taking values in an arbitrary abstract group.

The iterative Pexider equation $F(s+t) = H(s) \circ G(t)$, and some its generalizations, over groupoids have been examined in [2], [3], [4], [5], [6].

Some related results concerning equation (2) can be found in [1], [9], [10], [13].

2. Solutions of equation (2)

In this section, taking inspiration from paper [11], we will give a description of all functions A, T satisfying (2), mapping a field into a group (possibly non-abelian).

Throughout the paper, $\mathbb{K}$ means a commutative field.

Let us start with the following

LEMMA 1. *Let P be a group and D be a subgroup of P. Suppose that functions $A : \mathbb{K} \to D$, $T : \mathbb{K} \to P$ satisfy* (2). *Then the function $\varphi : \mathbb{K} \to D$ defined by $\varphi(t) := A(t) - A(0)$ satisfies the equation*

$$\varphi(-s^2) = \varphi(-s^2 - r) + \varphi(r), \qquad s \in \mathbb{K} \setminus \{0\}, \quad r \in \mathbb{K}. \tag{3}$$

Proof. From equation (2), since $\mathbb{K}$ is a commutative field, we get

$$T(t) + T(s) = T(s) + T(t), \qquad s, t \in \mathbb{K}. \tag{4}$$

Hence, by (2) again,

$$A(t) + A(s) = A(s) + A(t), \qquad s, t \in \mathbb{K}. \tag{5}$$

Using (4) it is easy to verify that function $M : \mathbb{K}^2 \to P$ defined by

$$M(s, t) := T(s+t) - T(t) - T(s)$$

satisfies

$$M(s, t) + M(s+t, w) = M(s, t+w) + M(t, w), \qquad s, t, w \in \mathbb{K}.$$

Consequently, by (2), function A satisfies

$$A(st)+A((s+t)w)=A(s(t+w))+A(tw), \qquad s,t,w\in\mathbb{K}. \tag{6}$$

Now, using substitutions given in [11] (Section 2, p. 252) and our formula (5) we shall reduce equation (6) to (3). Put $t=-s$ in (6). The result is that

$$A(-s^2)+A(0)=A(-s^2+sw)+A(-sw), \qquad s,w\in\mathbb{K}. \tag{7}$$

If $s\neq 0$, then setting $w=-r/s$ in (7), we obtain

$$A(-s^2)+A(0)=A(-s^2-r)+A(r), \qquad s\in\mathbb{K}\setminus\{0\}, \quad r\in\mathbb{K}.$$

Hence, by (5), we can write

$$A(-s^2)-A(0)=\{A(-s^2-r)-A(0)\}+\{A(r)-A(0)\},$$

for $s\in\mathbb{K}\setminus\{0\}$, $r\in\mathbb{K}$, which means that $\varphi(t):=A(t)-A(0)$ satisfies equation (3). $\square$

Recall that a group P is said to have 2-*torsion* if $2x=0$ for some element $x\neq 0$ in P. Further, a group P is called *uniquely* 2-*divisible* if the equation $2x=y$ admits a unique solution $x\in P$ for each $y\in P$.

For any prime p, by $\mathbb{F}_p$ we denote the field of p elements.

LEMMA 2. *Suppose that $\mathbb{K}$ is neither $\mathbb{F}_2$, $\mathbb{F}_3$ nor an infinite field of characteristic 2 and P is a group. A map $\varphi:\mathbb{K}\to P$ satisfies* (3) *and*

$$\varphi(t)+\varphi(s)=\varphi(s)+\varphi(t), \qquad s,t\in\mathbb{K}, \tag{8}$$

if and only if φ is an additive function.

LEMMA 3. *Let $\mathbb{K}$ be an infinite field of characteristic 2 and P be a group with no 2-torsion. Then $\varphi:\mathbb{K}\to P$ satisfies* (3) *and* (8) *if and only if $\varphi(t)=0$ for $t\in\mathbb{K}$.*

The above two lemmas can be proved in much the same way as Theorem 1 and Lemma 3 in [11], respectively. There is only one difference. Here we assume condition (8) on function φ, while in [11] the commutativity of group P is supposed. Thus, (8) must be used whenever the commutativity of group P is needed. For the reader convenience, we will sketch the proofs.

Proof of Lemma 2. Let $\mathbb{K}$ be a field with at least four elements, and assume that the characteristic of $\mathbb{K}$ is not equal to 2. Fix $s,t\in\mathbb{K}\setminus\{0\}$. Arguing as in [11] (the proof of Lemma 1), we obtain

$$\varphi(-s^2+t^2)=\varphi(-s^2)-\varphi(-t^2). \tag{9}$$

Now, replacing in (3), r by $-t^2-r$ for $t\in\mathbb{K}\setminus\{0\}$, $r\in\mathbb{K}$ and using (3), (8) and (9), we get

$$\begin{aligned}\varphi(-s^2+t^2+r)&=\varphi(-s^2)-\varphi(-t^2-r)\\&=\varphi(-s^2)-[\varphi(-t^2)-\varphi(r)]=\varphi(-s^2)-\varphi(-t^2)+\varphi(r)\\&=\varphi(-s^2+t^2)+\varphi(r).\end{aligned}$$

Hence, in the same way as in [11], we obtain

$$\varphi(s+t) = \varphi(s) + \varphi(t), \qquad s \in \mathbb{K} \setminus \{-1, 1\}, \quad t \in \mathbb{K}, \tag{10}$$

which yields $\varphi(-s) = -\varphi(s)$ for $s \in \mathbb{K} \setminus \{-1, 1\}$. Further, following [11], we get

$$-\{\varphi(s) + \varphi(-1)\} = -\varphi(s) + \varphi(1) \quad \text{for} \quad s \in \mathbb{K} \setminus \{-1, 0, 1, 2\}.$$

This implies, by (8), $-\varphi(-1) = \varphi(1)$ i.e., (10) holds for $s \in \{-1, 1\}$. Thus, φ is an odd function. Since the rest of the proof does not differ essentially from the proof of Theorem 1 in [11], we omit it. $\square$

Proof of Lemma 3. Arguing as in the proof of Lemma 3 in [11], we get

$$\varphi(s^2) - \varphi(t^2) - \varphi(r) = \varphi(s^2) - \{\varphi(t^2) - \varphi(r)\}$$

for $s, t \in \mathbb{K} \setminus \{0\}$ such that $s + t \neq 0$ and $r \in \mathbb{K}$. Hence, by (8), we have $2\varphi(r) = 0$ for $r \in \mathbb{K}$, which implies $\varphi(r) = 0$ for $t \in \mathbb{K}$. The converse is evident, and the proof is finished. $\square$

Remark 1. (cf. [11], Remark 1) If $\mathbb{K}$ is a field of characteristic 2 and P is a group with no 2-torsion, then the map $\varphi : \mathbb{K} \to P$ is additive if and only if φ is the zero map. In fact, putting $s = t$ in $\varphi(s+t) = \varphi(s) + \varphi(t)$, we obtain $2\varphi(t) = \varphi(0) = 0$.

Now, we can give a description of functions A, T satisfying equation (2).

THEOREM 1. *Assume that $\mathbb{K}$ is neither $\mathbb{F}_2$ nor $\mathbb{F}_3$, P is a group and D is a uniquely 2-divisible subgroup of P. Then the maps $A : \mathbb{K} \to D$ and $T : \mathbb{K} \to P$ satisfy equation* (2) *if and only if they have the form*

$$A(t) = 2A_1(t) + c, \qquad t \in \mathbb{K}, \tag{11}$$

$$T(t) = A_1(t^2) + A_2(t) - c, \qquad t \in \mathbb{K}, \tag{12}$$

where A_1, A_2 are additive functions mapping $\mathbb{K}$ into D, P, respectively, satisfying the equation

$$A_1(s) + A_2(t) = A_2(t) + A_1(s), \qquad s, t \in \mathbb{K} \tag{13}$$

and $c \in D$ is such that

$$c + A_i(t) = A_i(t) + c \qquad \textit{for} \quad t \in \mathbb{K}, \quad i \in \{1, 2\}. \tag{14}$$

Proof. For the proof of the "if" part, note that by the additivity of functions A_1 and A_2 (since $\mathbb{K}$ is a commutative field), we get

$$A_i(s) + A_i(t) = A_i(t) + A_i(s), \qquad i \in \{1, 2\}, \quad s, t \in \mathbb{K}. \tag{15}$$

Using the additivity of A_1, A_2, (13), (14) and (15) the rest of that part of the proof is a simple verification.

We proceed to show the "only if" part. Let us begin, as in the proof of Lemma 1, by observing that T satisfies (4). Consequently, by (2), we get (5) and

$$T(s) + A(t) = A(t) + T(s), \qquad s, t \in \mathbb{K}. \tag{16}$$

Hence, in particular, we have

$$T(s) + A(0) = A(0) + T(s), \qquad s \in \mathbb{K}. \tag{17}$$

Note that, by Lemma 1, $\varphi(t) = A(t) - A(0)$ satisfies (3). Furthermore, it is easy to see by (5), that φ also satisfies (8).

Consider first the case when $\mathbb{K}$ is neither $\mathbb{F}_3$ nor a field of characteristic 2. Then, from Lemma 2 we infer that $\varphi : \mathbb{K} \to D$ is an additive function. Following [11] (see the proof of Theorem 2), let $A_1 : \mathbb{K} \to D$ be an additive map such that $2A_1(t) = \varphi(t)$ for $t \in \mathbb{K}$. Then by the definition of φ, we get (11), where $c = A(0)$.

Combining (5) and (11) we see that

$$-c + A(t) = A(t) - c = 2A_1(t), \qquad t \in \mathbb{K}.$$

Computing $A(t)$ from the above equalities and comparing the obtained formulae, we have

$$2A_1(t) + c = c + 2A_1(t), \qquad t \in \mathbb{K}. \tag{18}$$

Further, substituting (11) into (16) and using (17) (recall that $c = A(0)$), we get

$$T(s) + 2A_1(t) = 2A_1(t) + T(s), \qquad s, t \in \mathbb{K}. \tag{19}$$

Since the characteristic of $\mathbb{K}$ is not equal to 2, by the fact that $2A_1(t) = A_1(2t)$ for $t \in \mathbb{K}$, from (18) and (19) we infer, respectively

$$A_1(s) + c = c + A_1(s), \qquad s \in \mathbb{K}, \tag{20}$$

$$T(s) + A_1(t) = A_1(t) + T(s), \qquad s, t \in \mathbb{K}. \tag{21}$$

Using (11) we can rewrite equation (2) as follows (cf. [11], the proof of Theorem 2):

$$\begin{aligned} T(s+t) - T(t) - T(s) &= 2A_1(st) + c = A_1(2st) + c \\ &= A_1((s+t)^2 - s^2 - t^2) + c = \{A_1((s+t)^2) - c\} - \{A_1(s^2) - c\} \\ &\quad - \{-c + A_1(t^2)\} \end{aligned}$$

for $s, t \in \mathbb{K}$. Now, it is easily seen, by (17), (20) and (21), that the map $A_2 : \mathbb{K} \to P$ defined by

$$A_2(t) := T(t) - A_1(t^2) + c$$

is additive. This provides (12). Finally, it is easy to check that (12), (17), (20) and (21) jointly with the additivity of function A_1, imply (13) and (14).

To end the proof, consider the remaining case when $\mathbb{K}$ is a field of characteristic 2 with at least four elements. Then by Lemma 2 and Remark 1 (in the case when $\mathbb{K}$ is a finite field) or by Lemma 3 (if $\mathbb{K}$ is an infinite field) we infer that $\varphi(t) = 0$ for

$t \in \mathbb{K}$. Consequently, $A(t) = A(0)$ for $t \in \mathbb{K}$. Setting $c = A(0)$, we get (11) with $A_1(t) = 0$ for $t \in \mathbb{K}$.

On account of (2), we obtain

$$T(s+t) - T(t) - T(s) = c \qquad \text{for} \quad s, t \in \mathbb{K}.$$

Hence, by (17), we can write

$$T(s+t) + c = \{T(s) + c\} + \{T(t) + c\}, \qquad s, t \in \mathbb{K},$$

which shows that function $A_2 : \mathbb{K} \to P$ defined by

$$A_2(t) := T(t) + c, \qquad t \in \mathbb{K}$$

is additive. Note that (12) holds with $A_1(t) = 0$ for $t \in \mathbb{K}$. It is evident that (13) and (14) are satisfied. $\square$

Remark 2. It is obvious that in case of an abelian group P conditions (13) and (14) can be dropped. If moreover $P = D$ then our Theorem 1 reduces to the corresponding result from [11] (Theorem 2) (see also [13], Corollary 1).

The next corollary follows easily from Theorem 1 and will be used later.

COROLLARY 1. *Let $\mathbb{K}$, P and D be the same as in Theorem 1. Then an additive function $A : \mathbb{K} \to D$ and $T : \mathbb{K} \to P$ satisfy (2) if and only if they are given by*

$$\begin{aligned} A(t) &= 2A_1(t), \qquad t \in \mathbb{K}, \\ T(t) &= A_1(t^2) + A_2(t), \qquad t \in \mathbb{K}, \end{aligned}$$

where A_1, A_2 are additive functions mapping $\mathbb{K}$ into D, P, respectively, satisfying (13).

Remark 3. In [11] one can find examples (see Examples 1, 2, 3 and Remark 2) showing the sharpness of the results stated in Lemma 2 and Theorem 1.

3. Solutions of (GPE) and (1)

In the sequel, W, X, Y, Z stand for arbitrary nonempty sets. By $In\,(X, Y)$ ($Sur\,(X, Y)$, $Bij\,(X, Y)$) we denote the set of all injections (surjections, bijections) of a set X into (onto) itself. For simplicity of notation, we write $In\,X$, $Sur\,X$, $Bij\,X$ in the case when $X = Y$. As usual Y^X means the set of all functions mapping X into Y. $Ran\,f$ denotes the range of the function f and id_X stands for the identity function on the set X.

By abuse of language, a function φ mapping a field $\mathbb{K}$ into a group of functions $(P, \circ)$ (with composition of functions as the group operation) satisfying

$$\varphi(s+t) = \varphi(s) \circ \varphi(t), \qquad s, t \in \mathbb{K}$$

is said to be additive.

We now can formulate and prove our results concerning the generalized Pexider equation (GPE), which state conditions allowing to reduce (GPE) to (GCE).

THEOREM 2. *Suppose that the functions F, K, H, G mapping $\mathbb{K}$ into Z^W, Z^Y, $Bij\,(X,Y)$, X^W, resp., satisfy* (GPE). *If $K(0) \in In\,(Y,Z)$ and $G(0) \in Sur\,(W,X)$ then there exist functions $a \in Sur\,(W,Z_0)$, $b \in Bij\,(X,Z_0)$, $c \in Bij\,(Y,Z_0)$, $A, T : \mathbb{K} \to Bij\,Z_0$, where $Z_0 := Ran\,K(0)$, such that* (GCE) *holds, A satisfies*

$$A(-s^2) = A(-s^2 - r) \circ A(r) \qquad \text{for} \quad s \in \mathbb{K} \setminus \{0\}, \quad r \in \mathbb{K} \tag{22}$$

and

$$\begin{cases} F(t) &= T(t) \circ a, \\ K(t) &= A(t) \circ c, \\ H(t) &= c^{-1} \circ T(t) \circ b, \\ G(t) &= b^{-1} \circ T(t) \circ a, \qquad t \in \mathbb{K}. \end{cases} \tag{23}$$

Moreover, if $\mathbb{K}$ is neither $\mathbb{F}_2$, $\mathbb{F}_3$ nor an infinite field of characteristic 2 then $A : \mathbb{K} \to (Bij\,Z_0, \circ)$ is an additive function.

Conversely, if $a \in Z_0^W$, $b \in Bij\,(X,Z_0)$, $c \in Bij\,(Y,Z_0)$ for a nonempty set $Z_0 \subset Z$ and $A, T : \mathbb{K} \to Bij\,Z_0$ satisfy (GCE), *then the functions F, K, H, G given by* (23) *satisfy* (GPE).

Proof. To simplify the notation, we put $F_t := F(t)$, $K_t := K(t)$, $H_t := H(t)$, $G_t := G(t)$ for $t \in \mathbb{K}$. Assume that the functions F, K, H, G such that $H_t \in Bij\,(X,Y)$, $t \in \mathbb{K}$, $K_0 \in In\,(Y,Z)$ and $G_0 \in Sur\,(W,X)$ satisfy (GPE). Setting alternately $t = 0$ and $s = 0$ into (GPE), we get respectively

$$F_s = K_0 \circ H_s \circ G_0, \qquad s \in \mathbb{K}, \tag{24}$$

$$F_t = K_0 \circ H_0 \circ G_t, \qquad t \in \mathbb{K}. \tag{25}$$

Comparing the right hand sides of (24) and (25) for $s = t$, by the injectivity of K_0, we obtain

$$H_t \circ G_0 = H_0 \circ G_t, \qquad t \in \mathbb{K}.$$

Hence

$$G_t = H_0^{-1} \circ H_t \circ G_0, \qquad t \in \mathbb{K}. \tag{26}$$

On the other side, by (25), we have

$$G_t = H_0^{-1} \circ K_0^{-1} \circ F_t, \qquad t \in \mathbb{K}. \tag{27}$$

Introduce on W an equivalence relation "ϱ" putting

$$w\,\varrho\,v \qquad \text{if and only if} \qquad G_0(w) = G_0(v)$$

Let g be an invertible mapping such that $g([w]) \in [w]$ for $w \in W$, where $[w]$ stands for the equivalence class containing w. Set

$$\widetilde{G}_0 := G_0 \circ g \quad \text{and} \quad \widetilde{F}_t := F_t \circ g, \quad t \in \mathbb{K}.$$

It is evident that $\widetilde{G}_0$ is a bijection of the factor set W/ϱ of W modulo "ϱ" onto X. Further, since $H_t \in In\,(X, Y)$ for $t \in \mathbb{K}$ and $K_0 \in In\,(Y, Z)$, by (24), we get

$$F_t(w) = F_t(v) \quad \text{iff} \quad G_0(w) = G_0(v)$$

for every $t \in \mathbb{K}$, $v, w \in W$. Thus $\widetilde{F}_t \in In\,(W/\varrho, Z)$ for $t \in \mathbb{K}$.

Now, in view of (24), we can write

$$\widetilde{F}_t = K_0 \circ H_t \circ \widetilde{G}_0, \qquad t \in \mathbb{K},$$

which implies

$$H_t = K_0^{-1} \circ \widetilde{F}_t \circ \widetilde{G}_0^{-1}, \qquad t \in \mathbb{K}. \tag{28}$$

Substituting (28) and (27) into (GPE), we obtain

$$F_{s+t} = K_{st} \circ K_0^{-1} \circ \widetilde{F}_s \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} \circ F_t, \qquad s, t \in \mathbb{K}.$$

Hence

$$\widetilde{F}_{s+t} = K_{st} \circ K_0^{-1} \circ \widetilde{F}_s \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} \circ \widetilde{F}_t, \qquad s, t \in \mathbb{K}. \tag{29}$$

Observe that, by (24), since $H_t \in Sur\,(X, Y)$ for $t \in \mathbb{K}$ and $G_0 \in Sur\,(W, X)$, we get

$$Ran\, F_t = Ran\, K_0 = Z_0, \qquad t \in \mathbb{K}$$

and consequently, $\widetilde{F}_t \in Bij\,(W/\varrho, Z_0)$ for $t \in \mathbb{K}$. Define $T : \mathbb{K} \to Bij\, Z_0$, by

$$T_t := \widetilde{F}_t \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1}, \qquad t \in \mathbb{K}. \tag{30}$$

Using (30) and (29) we can write

$$\begin{aligned} T_{s+t} &= \widetilde{F}_{s+t} \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} = K_{st} \circ K_0^{-1} \circ \widetilde{F}_s \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} \\ &\quad \circ \widetilde{F}_t \circ \widetilde{G}_0^{-1} \circ H_0^{-1} \circ K_0^{-1} = K_{st} \circ K_0^{-1} \circ T_s \circ T_t. \end{aligned}$$

Thus T and K satisfy the following functional equation

$$T_{s+t} = K_{st} \circ K_0^{-1} \circ T_s \circ T_t, \qquad s, t \in \mathbb{K}. \tag{31}$$

Define $A : \mathbb{K} \to Z^{Z_0}$ setting

$$A_t := K_t \circ K_0^{-1}, \qquad t \in \mathbb{K}. \tag{32}$$

Then from (31), we get

$$T_{s+t} = A_{st} \circ T_s \circ T_t, \qquad s, t \in \mathbb{K}, \tag{33}$$

i.e. A and T satisfy (GCE). It is clear that $A_t \in Bij\, Z_0$ for every $t \in \mathbb{K}$.

Now, since from (32) we get $A_0 = id_{Z_0}$, by Lemma 1 (for $P = D = (Bij\, Z_0, \circ)$) we conclude that A satisfies (22).

We are going to show that formulae (23) hold. On account of (24) and (30), we have

$$H_t = K_0^{-1} \circ T_t \circ K_0 \circ H_0, \quad t \in \mathbb{K}. \tag{34}$$

Putting (34) into (26), we get

$$G_t = H_0^{-1} \circ K_0^{-1} \circ T_t \circ K_0 \circ H_0 \circ G_0, \qquad t \in \mathbb{K}. \tag{35}$$

Next, substitution of (35) into (25) gives

$$F_t = T_t \circ K_0 \circ H_0 \circ G_0, \qquad t \in \mathbb{K}. \tag{36}$$

In view of (32), we obtain

$$K_t = A_t \circ K_0, \qquad t \in \mathbb{K}. \tag{37}$$

Finally, setting

$$a := K_0 \circ H_0 \circ G_0, \quad b := K_0 \circ H_0, \quad c := K_0$$

it is easily seen, that $a \in Sur\,(W, Z_0)$, $b \in Bij\,(X, Z_0)$, $c \in Bij\,(Y, Z_0)$, and from (34), (35), (36), (37) (switching to the previous notation) we get formulae (23).

To finish the first part of the proof it remains to show that if $\mathbb{K}$ is neither $\mathbb{F}_2$, $\mathbb{F}_3$ nor an infinite field of characteristic 2, then A is an additive function. To see this, observe that from (33) we get,

$$T_s \circ T_t = T_t \circ T_s, \qquad s, t \in \mathbb{K}.$$

(since, the commutativity of $\mathbb{K}$ yields $T_{s+t} = T_{t+s}$ and $A_{st} = A_{ts}$). Consequently, by (33) again, we obtain

$$A_s \circ A_t = A_t \circ A_s, \qquad s, t \in \mathbb{K}.$$

Applying Lemma 2 we get the additivity of function A.

The converse statement is easy to check. □

From the above proof one can easily infer the following

COROLLARY 2. *Let R be a ring with unity. Suppose that functions F, K, H, G mapping R into Z^W, Z^Y, $Bij\,(X, Y)$, X^W, respectively, satisfy* (GPE). *If $K(0) \in In\,(Y, Z)$ and $G(0) \in Sur\,(W, X)$ then there exist functions $a \in Sur\,(W, Z_0)$, $b \in Bij\,(X, Z_0)$, $c \in Bij\,(Y, Z_0)$, $A, T : \mathbb{K} \to Bij\,Z_0$ ($Z_0 = Ran\,K(0)$), satisfying* (GCE) *and such that* (23) *holds.*

Conversely, if $a \in Z_0^W$, $b \in Bij\,(X, Z_0)$, $c \in Bij\,(Y, Z_0)$ for a nonempty set $Z_0 \subset Z$ and $A, T : R \to Bij\,Z_0$ satisfy (GCE), *then the functions F, K, H, G given by* (23) *satisfy* (GPE).

Remark 4. Observe that in the case when $Y = Z$ and $K(t) = id_Y$ for $t \in R$ (R means a ring with unity) then (GPE) takes the form of the well-known Pexider equation which, by the above corollary, can be reduced to the Cauchy equation. In fact, we have $A(t) = id_Y$ for $t \in R$ (cf. (32)) and (GCE) reduces to the Cauchy equation. This particular result has been obtained in a more general situation in [2] (see Theorem 1), where the Pexider equation on a groupoid has been examined.

Theorem 2 jointly with Lemma 3 and Corollary 1 allow us to prove the following theorem.

THEOREM 3. *Suppose that $\mathbb{K}$ is neither $\mathbb{F}_2$ nor $\mathbb{F}_3$ and D is a uniquely 2-divisible subgroup of the group $(Bij\,Y, \circ)$. Then functions F, K, H, G mapping $\mathbb{K}$ into Y^W, D, $Bij\,(X, Y)$, X^W, respectively, such that $G(0) \in Sur\,(W, X)$ satisfy* (GPE) *if and only if they are given by*

$$\begin{cases} F(t) &= A_1(t^2) \circ A_2(t) \circ a, \\ K(t) &= A_1^2(t) \circ c, \\ H(t) &= c^{-1} \circ A_1(t^2) \circ A_2(t) \circ b, \\ G(t) &= b^{-1} \circ A_1(t^2) \circ A_2(t) \circ a, \qquad t \in \mathbb{K} \end{cases} \tag{38}$$

where $A_1 : \mathbb{K} \to D$, $A_2 : \mathbb{K} \to (Bij\,Y, \circ)$ are additive functions satisfying

$$A_1(s) \circ A_2(t) = A_2(t) \circ A_1(s), \qquad s, t \in \mathbb{K},$$

and $a \in Sur\,(W, Y)$, $b \in Bij\,(X, Y)$, $c \in Bij\,Y$ are constants.

Proof. Suppose that functions F, K, H, G mapping $\mathbb{K}$ into $Y^W, D, Bij\,(X, Y), X^W$, respectively, such that $G(0) \in Sur\,(W, X)$ satisfy (GPE). Then, in view of Theorem 2, there exist $a \in Sur\,(W, Y), b \in Bij\,(X, Y), c \in Bij\,Y$ (note that $Z_0 = Ran\,K(0) = Y$ in this case), $A : \mathbb{K} \to D$ (cf. (32)), $T : \mathbb{K} \to Bij\,Y$ such that (GCE) and (23) hold. Moreover, by Theorem 2 or Lemma 3 (in case of an infinite field of characteristic 2), we infer that A is an additive function. Applying Corollary 1, for $P = (Bij\,Y, \circ)$, we get from (23) the required formulae (38).

The converse is straightforward. □

From Theorem 3, we obtain the general solution of (1). Namely, the subsequent result holds.

COROLLARY 3. *Assume that $\mathbb{K}$ is neither $\mathbb{F}_2$ nor $\mathbb{F}_3$ and $(D, +)$ is a uniquely 2-divisible group. Then the maps $F, K, H, G : \mathbb{K} \to D$ satisfy* (1) *if and only if they are given by*

$$\begin{cases} F(t) &= A_1(t^2) + A_2(t) + a, \\ K(t) &= 2A_1(t) + c, \\ H(t) &= -c + A_1(t^2) + A_2(t) + b, \\ G(t) &= -b + A_1(t^2) + A_2(t) + a, \qquad t \in \mathbb{K} \end{cases} \tag{39}$$

where $A_1, A_2 : \mathbb{K} \to D$ are additive functions satisfying (13) *and $a, b, c \in D$ are constants.*

Proof. By the well-known Cayley theorem (see e.g. [12]) any group can be considered as a group of bijections of a set. Moreover, since D is a uniquely 2-divisible group, the corresponding group of transformations (the group of left (right) translations of D) share the property as well. Thus, there is no loss of generality assuming that D is a uniquely 2-divisible group of bijections of a set onto itself. Now, Theorem 3 (switching to additive notation) yields the statement. □

Remark 5. Corollary 3 generalizes corresponding results from [11] and [13] (Corollary 2 and Theorem 1, respectively), which actually inspired the paper.

Namely, it is easily seen that in the case when D is an abelian group, the general solution of (1), given by (39), reduces to this obtained in [11] and [13]. However, the methods of solution of (1) presented in those papers fail in the case when functions F, K, H, G take values in a non-abelian group.

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