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ON G–PSEUDO–CENTRES OF CONVEX BODIES

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Abstract. As is well known, for every convex body A in R^n there is a unique centrally symmetric kernel, that is, a centrally symmetric convex body $C \subset A$ with maximal n-volume. The paper concerns G-kernels of a covex body A for any subgroup G of $O(n)$, i.e. G-invariant convex subsets of A with maximal n-volume. We prove that only for G generated by the central symmetry σ_0 every A has a unique G-kernel. If A is strictly convex, then its G-kernel is unique for every G.

Introduction

In 1950 Fáry and Rédei proved that for every convex body A in R^n there exists a unique centrally symmetric convex body $C \subset A$ with a maximal volume (see [2]). They referred to the set C as the *centrally symmetric kernel of* A. Let $p(A)$ be the symmetry centre of the kernel C. We call $p(A)$ the *pseudo-centre of* A.

The map $p : \mathscr{K}_0^n \to R^n$ defined on the class $\mathscr{K}_0^n$ of all convex bodies in R^n is a selector, i.e., $p(A) \in A$ for every A. Evidently

0.1. *The map p is equivariant under affine automorphisms, i.e., $f(p(A)) = p(f(A))$ for every $f \in GA(n)$.*

0.2. (comp.[2], Satz 5) *If A is a simplex, then $p(A)$ is the centroid of A.*

Of course, in general, for arbitrary subgroup G of $O(n)$, the situation is quite different than for the group $\langle\sigma_0\rangle$ generated by the reflection at 0. For instance, a convex body may contain many balls (i.e. translates of an $O(n)$-invariant body) with a maximal volume.

We shall refer to any G-invariant (up to a translation) convex body contained in A with a maximal volume as a *G-kernel of* A. We prove that $\langle\sigma_0\rangle$ is the only non-trivial subgroup G of $O(n)$ such that every convex body in R^n has a unique G-kernel (Theorem 3.8); however, if A is strictly convex, then A has a unique G-kernel for arbitrary non-trivial G (Theorem 3.9). Our conjecture is that for arbitrary $G \subset O(n)$

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and for every convex body A there is a representative of the affine type of A with a unique G-kernel. We prove this conjecture under some additional assumption on G (Theorem 4.4).

1. Preliminaries

We use the following terminology and notation:

Let $\mathcal{K}^n$ be the class of all convex bodies in R^n, i.e. compact convex subsets of R^n with non-empty interior.

The support function $h_A : S^{n-1} \to \mathrm{R}$ is defined by

$$h_A(u) = \sup\{x \cdot u; x \in A\},$$

where $\cdot$ is the usual scalar product; we write also $h(A, u)$ for $h_A(u)$.

The width of A in direction u is $b(A, u) := h(A, u) + h(A, -u)$ and the thickness of A is $d(A) := \inf\{b(A, u); u \in S^{n-1}\}$. Of course, $\mathrm{diam}(A) = \sup\{b(A, u); u \in S^{n-1}\}$.

It is well known that $d : \mathcal{K}^n \to \mathrm{R}$ is continuous with respect to the Hausdorff limit $\lim_H$.

The unit ball in R^n is B^n and its volume κ_n.

The line passing through a, b $(a \neq b)$ is $\mathrm{aff}(a, b)$. The linear subspace spanned by $(v_1, \ldots, v_k)$ is $\mathrm{lin}(v_1, \ldots, v_k)$.

The relative interior of A with respect to $\mathrm{aff}A$ is $\mathrm{relint}A$.

We use the symbol $\oplus$ for the euclidean direct sum, i.e. the Minkowski sum of subsets of orthogonal subspaces of R^n.

For arbitrary $A, B \subset \mathrm{R}^n$, let

$$\mathrm{dist}(A, B) = \inf\{\|a - b\|; a \in A,\ b \in B\}.$$

Let X be a nonempty convex subset of R^n. A family $\{A_x; x \in X\}$ of subsets of R^n is concave provided that for every $x_0, x_1 \in X$ and $t \in [0, 1]$

$$A_{(1-t)x_0+tx_1} \supset (1 - t)A_{x_0} + tA_{x_1}.$$

As usually, $\mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{SL}(n)$, $\mathrm{GA}(n)$, and $\mathrm{SA}(n)$ are the groups of linear automorphisms, linear isometries, special linear maps (preserving volume), affine automorphisms, and special affine maps (preserving volume) of R^n, respectively.

If $f \in \mathrm{GA}(n)$, then $\det f$ and $\|f\|$ are understood as $\det\bar{f}$ and $\|\bar{f}\|$ for the corresponding linear map $\bar{f}$. Let σ_0 be the reflection at 0 and τ_x the translation by x.

For any group G of transformations of R^n and any $x \in \mathrm{R}^n$, let $G(x)$ be the orbit of x and let

$$G^x = \tau_x G \tau_x^{-1}.$$

Further,

$$\mathrm{fix}\, G := \{x \in \mathrm{R}^n;\ g(x) = x \ \text{for every}\ g \in G\}.$$

A set $C \subset \mathrm{R}^n$ is G-invariant provided that $g(C) = C$ for every $g \in G$. Evidently,

1.1. *C is G^x-invariant if and only if $C-x$ is G-invariant.*

We shall need the following elementary lemma.

1.2. LEMMA. *Let P_n be an n-dimensional parallelepiped in R^n, $n \geqslant 2$, with $(n-1)$-dimensional faces contained in hyperplanes $H_1, ..., H_n, H_1', ..., H_n'$, where H_i and H_i' are parallel for all i. Let x_i be a unit normal vector of H_i. If $\mathrm{dist}(H_i, H_i') = \beta$ and $\sin \sphericalangle(x_i, \mathrm{lin}(x_1, ..., x_{i-1})) \geqslant \alpha > 0$ for $i = 1, ..., n$, then $V_n(P_n) \leqslant \frac{\beta^n}{\alpha^{n-1}}$.*

Proof. We can assume that P_n is the Minkowski sum of n segments:

$$P_n = \Sigma_{i=1}^n \Delta(0, v_i)$$

for some basis $(v_1, ..., v_n)$ of R^n.

Let $\gamma = \sphericalangle(x_n, \mathrm{lin}(x_1, ..., x_{n-1}))$.

Induction on n:

If $n = 2$, then $\gamma = \pi - \sphericalangle(v_1, v_2)$ and

$$V_2(P_2) = \|v_2\|\beta = \frac{\beta^2}{\sin\gamma} \leqslant \frac{\beta^2}{\alpha}.$$

Let $n \geqslant 3$ and assume the assertion holds for $n-1$. Let

$$F = \Sigma_{i=1}^{n-1}\Delta(0, v_i) \quad \text{and} \quad E = (\mathrm{lin} v_n)^\perp.$$

Consider the orthogonal projection $\Pi_E : \mathrm{R}^n \to E$ and let $P_{n-1} = \Pi_E(F)$. Then, evidently, for $i = 1, ..., n-1$, the intersections $E \cap H_i$ and $E \cap H_i'$ are parallel $(n-2)$- dimensional flats containing $(n-2)$-dimensional faces of P_{n-1}. Moreover, $\mathrm{dist}(E \cap H_i, E \cap H_i') = \beta$ and $\sin \sphericalangle(x_i, \mathrm{lin}(x_1, ..., x_{i-1})) \geqslant \alpha$ for $i = 1, ..., n-1$.

Hence, by the inductive assumption,

$$V_{n-1}(P_{n-1}) \leqslant \frac{\beta^{n-1}}{\alpha^{n-2}}.$$

Since

$$V_n(P_n) = \beta V_{n-1}(F), \quad V_{n-1}(F) = V_{n-1}(P_{n-1})\frac{1}{\cos \sphericalangle(x_n, v_n)},$$

and $v_n \perp x_i$ for $i = 1, ..., n$, it follows that

$$\cos \sphericalangle(x_n, v_n) = \sin \sphericalangle(x_n, v_n^\perp) = \sin \sphericalangle(x_n, \mathrm{lin}(x_1, , ..., x_{n-1})) \geqslant \alpha,$$

whence

$$V_n(P_n) \leqslant \frac{\beta}{\alpha} V_{n-1}(P_{n-1}) \leqslant \frac{\beta^n}{\alpha^{n-1}}.$$

This completes the proof. □

2. Invariant convex bodies

Let $n \geqslant 2$. We are interested in subgroups of $\mathrm{GL}(n)$ for which there exist invariant convex bodies in R^n.

2.1. PROPOSITION. *For every $G \subset \mathrm{GL}(n)$ the following are equivalent:*
(i) There exists a G-invariant set $C \in \mathcal{K}_0^n$,
(ii) $G = fG'f^{-1}$ for some $G' \subset \mathrm{O}(n)$ and $f \in \mathrm{GL}(n)$.

Proof. (ii) $\Longrightarrow$ (i): Assume (ii). Let $C = f(B^n)$ and let $g \in G$. Then $g = fg'f^{-1}$ for some $g' \in G'$ and thus

$$g(C) = fg'f^{-1}f(B^n) = f(B^n) = C.$$

(i) $\Longrightarrow$(ii): Let C be G-invariant and let E be the unique ellipsoid with a maximal volume contained in C (see [1] or [4]). Then E is G-invariant and thus E has centre 0, whence $E = f(B^n)$ for some $f \in \mathrm{GL}(n)$. Let $G' := f^{-1}Gf$; then B^n is G'-invariant and, consequently, $G' \subset \mathrm{O}(n)$. □

Evidently,

2.2. *For every $G \subset \mathrm{GL}(n)$ and compact subset C of* R^n

C is G-invariant if and only if C is $\bar{G}$-invariant.

In view of 2.1 and 2.2, we can restrict our consideration to compact subgroups of $\mathrm{O}(n)$.

We shall need the following.

2.3. LEMMA. *Let G be a compact subgroup of* $\mathrm{O}(n)$. *If there is no G-invariant linear subspace of dimension $k \in \{1, ..., n-1\}$, then there exists $\alpha_G > 0$ satisfying the following conditions:*

(i) $d(G(x)) \geqslant \alpha_G$ for every $x \in S^{n-1}$,

(ii) for every $x_1 \in S^{n-1}$ there exist $x_2, ..., x_n \in G(x_1)$ such that $x_1, ..., x_n$ are linearly independent and

$$\sin \sphericalangle(x_i, \mathrm{lin}(x_1, ..., x_{i-1})) \geqslant \frac{1}{2}\alpha_G$$

for $i = 1, ..., n$.

Proof. (i): Since there are no G-invariant subspaces, it follows that

$$\forall x \in S^{n-1}\ d(G(x)) > 0. \tag{2.1}$$

Since G is compact, the function $x \mapsto G(x)$ is continuous, and thus, by the continuity of d, also the function $x \mapsto d(G(x))$ is continuous. Therefore, there exists $\alpha_G > 0$ such that

$$d(G(x)) \geqslant \alpha_G \quad \text{for} \quad \text{every} \quad x \in S^{n-1}.$$

(ii): It suffices to prove that if for some $k \in \{2, ..., n\}$ and $x_1 \in S^{n-1}$

$$x_i \in G(x_1) \text{ for } i \leqslant k-1, \quad x_1, ..., x_{k-1} \text{ are linearly independent} \qquad (2.2)_{k-1}$$

and

$$\sin \sphericalangle(x_i, \operatorname{lin}(x_1, ..., x_{i-1})) \geqslant \frac{1}{2}\alpha_G \quad \text{for } i = 1, ..., k-1, \qquad (2.3)_{k-1}$$

then there exists $x_k \in G(x_1)$ such that $(2.2)_k$ and $(2.3)_k$ hold.

Assume $(2.2)_{k-1}$ and $(2.3)_{k-1}$. Let H and H' be arbitrary two supporting hyperplanes of $G(x_1)$ with normal vectors orthogonal to x_1. Let $L_k := \operatorname{lin}(x_1, ..., x_{k-1})$. Without any loss of generality we can assume that

$$\operatorname{dist}(H, L_k) \geqslant \frac{1}{2}d(G(x_1)).$$

Since $G(x_1)$ is compact, there is an $x_k \in H \cap G(x_1)$. Clearly, $x_1, ...x_k$ are linearly independent and

$$\sin \sphericalangle(x_k, L_k) = \operatorname{dist}(H, L_k) \geqslant d(G(x_1)) \geqslant \frac{1}{2}\alpha_G.$$

□

2.4. PROPOSITION. *Let G be a compact subgroup of $\mathrm{O}(n)$. If there is is no G-invariant linear subspace of dimension $k \in \{1, ..., n-1\}$, then there exists $\lambda_G > 0$ such that*

$$V_n(C) \leqslant \lambda_G \, d(C)^n$$

for every G-invariant $C \in \mathcal{K}_0^n$.

Proof. Let $C \in \mathcal{K}_0^n$ be G-invariant. Then $0 \in C$ and $d(C) > 0$. Hence there exist two parallel supporting hyperplanes H and H' of C such that $\operatorname{dist}(H, H') = d(C)$.

Let x_1 be the unit outer normal vector of H. By Lemma 2.3, there exist $\alpha_G > 0$ and $x_2, ..., x_n \in G(x_1)$ such that $x_1, ..., x_n$ are linearly independent and

$$\sin \sphericalangle(x_n, L_n) \geqslant \frac{1}{2}\alpha_G, \qquad (2.3)_n$$

where $L_n = \operatorname{lin}(x_1, ..., x_{n-1})$.

Choose $g_i \in G$ such that $g_i(x_1) = x_i$, for $i = 1, ..., n$. Let, further,

$$H_i := g_i(H) \text{ and } H_i' := g_i(H').$$

Then $\operatorname{dist}(H_i, H_i') = d(C)$, x_i is a unit normal vector of H_i, and each H_i and H_i' support C.

Let P be the parallelepiped with $(n-1)$-dimensional faces contained in $H_1, \ldots, H_n, H'_1, \ldots, H'_n$. Then, evidently,

$$V_n(C) \leqslant V_n(P).$$

Let $\lambda_G := \left(\frac{2}{\alpha_G}\right)^{n-1}$. Applying now Lemma 1.2 for $\alpha := \frac{1}{2}\alpha_G$ and $\beta := d(C)$, by $(2.3)_n$ we obtain

$$V_n(P) \leqslant \lambda_G \, d(C)^n.$$

□

3. G-pseudo-centres and G-kernels of a convex body

3.1. PROPOSITION. *Let G be any transformation group of* R^n *and let* $A \subset \mathrm{R}^n$. *For every* $C \subset \mathrm{R}^n$ *the following are equivalent:*

(i) C is a maximal G-invariant subset of A,

(ii) $C = \bigcap_{g \in G} g(A)$.

Proof. (ii) $\Longrightarrow$ (i):

Evidently $C \subset A$, since id $\in G$. For every $f \in G$, $f(C) = \bigcap_{g \in G} fg(A) \supset C$ and $f^{-1}(C) = \bigcap_{g \in G} f^{-1}g(A) \supset C$. Thus $f(C) = C$. Hence C is G-invariant.

Moreover, if $C' \subset A$ and C' is G-invariant, then $C' \subset C$; indeed, $C' = g(C') \subset g(A)$ for every $g \in G$, whence $C' \subset \bigcap_{g \in G}(A) = C$. Thus C is maximal.

(i) $\Longrightarrow$ (ii):

Evidently, if $C \subset A$ and $g(C) = C$ for every $g \in G$, then $C \subset \bigcap_{g \in G} g(A)$. Since, by (ii) $\Rightarrow$ (i), this intersection is G-invariant, it follows that

$$C \supset \bigcap_{g \in G} g(A).$$

□

3.2. Definition. For $G \subset \mathrm{O}(n)$, $A \in \mathscr{K}_0^n$, and $x \in A$, let

$$A_{x,G} := \bigcap_{g \in G^x} g(A).$$

If it does not lead to a confusion, we write A_x for $A_{x,G}$.

3.3. PROPOSITION. *For every $G \subset \mathrm{O}(n)$ and $A \in \mathcal{K}_0^n$, the family $(A_{x,G})_{x \in A}$ is concave.*

Proof. For every $g \in G$ and $x \in \mathrm{R}^n$, let

$$g_x := \tau_x g \tau_x^{-1}.$$

Let us first notice that for every $t \in [0, 1]$ and $x_0, x_1 \in A$,

$$(1-t)g_{x_0}(A) + tg_{x_1}(A) = g_{(1-t)x_0+tx_1}(A). \tag{3.1}$$

Indeed, if y belongs to the left-hand side, then

$$y = (1-t)g_{x_0}(a_0) + tg_{x_1}(a_1) \quad \text{for some } a_0, a_1 \in A;$$

thus

$$y = (1-t)(g(a_0 - x_0) + x_0) + t(g(a_1 - x_1) + x_1) = x + g(a - x),$$

where $x = (1-t)x_0 + tx_1$ and $a = (1-t)a_0 + ta_1$; hence y belongs to the right-hand side. This proves $\subset$. The inverse inclusion is obvious; thus (3.1) holds.

For every $g \in G$

$$A_{x_i} \subset g_{x_i}(A) \quad \text{for} \quad i = 0, 1,$$

whence

$$(1-t)A_{x_0} + tA_{x_1} \subset (1-t)g_{x_0}(A) + tg_{x_1}(A).$$

Therefore, by (3.1),

$$(1-t)A_{x_0} + tA_{x_1} \subset \bigcap_{g \in G} g_x(A) = A_x.$$

□

3.4. Definition. For $G \subset \mathrm{O}(n)$ and $A \in \mathcal{K}_0^n$, let

$$P_G(A) := \{p \in A; V_n(A_p) \geqslant V_n(A_x) \text{ for every } x \in A\}.$$

We shall refer to $P_G(A)$ as the set of *G-pseudo-centres of* A.

A convex body $C \subset A$ will be called a *G-kernel of* A if G is G^p-invariant for some $p \in P_G(A)$.

In view of 3.3, for every $G \subset \mathrm{O}(n)$ and $A \in \mathcal{K}_0^n$,

$$P_G(A) \neq \emptyset,$$

i.e., by 3.1, there exists at least one G-kernel of A.

Let us prove a little more.

3.5. PROPOSITION. *For every* $G \subset \mathrm{O}(n)$ *and* $A \in \mathcal{K}_0^n$,

$$P_G(A) \cap \mathrm{int}A \neq \emptyset.$$

Proof. Let $p \in P_G(A)$. Since $A_{p,G} \supset A_{p,\mathrm{O}(n)}$ and $A_{p,\mathrm{O}(n)}$ is a ball, it follows that

$$A_{p,G} \neq \emptyset.$$

Let x_0 be the gravity center of $A_{p,G}$. Then

$$x_0 \in \mathrm{int}A \cap \mathrm{fix}G^p.$$

If $x_0 = p$, then $p \in P_G(A) \cap \mathrm{int}A$. If $x_0 \neq p$, then $x_0, p \in \mathrm{fix}G^p$, whence

$$A_{x_0} = \bigcap_{g \in G} g(A) = A_p.$$

Thus $V_n(A_{x_0}) = V_n(A_p)$ and, therefore, $x_0 \in P_G(A) \cap \mathrm{int}A$. □

The following two statements describe some properties of G-pseudo-centres.

3.6. PROPOSITION. *For every* $A \in \mathcal{K}_0^n$ *the set* $P_G(A)$ *is convex.*

Proof. If $x, y \in P_G(A)$ and $x \neq y$, then $V_n(A_x) = V_n(A_y)$ and thus, by the Brunn-Minkowski inequality ([3],p.309), $V_n(A_z) = V_n(A_x)$ for every $z \in \Delta(x, y)$. Thus $\Delta(x, y) \subset P_G(A)$. □

3.7. PROPOSITION. *Let* $G \subset \mathrm{O}(n)$ *and let* E_1 *and* E_2 *be* G*-invariant linear subspaces of* $\mathbb{R}^n$ *with* $\mathbb{R}^n = E_1 \oplus E_2$. *If* $G_i = \{g|E_i; g \in G\}$ *and* A_i *is a convex body in* E_i *for* $i = 1, 2$, *then*

$$P_G(A_1 \oplus A_2) = P_{G_1}(A_1) \oplus P_{G_2}(A_2).$$

Proof. Let $n_i = \mathrm{dim}E_i$ for $i = 1, 2$ and let $A = A_1 \oplus A_2$. Since $g(A) = g(A_1) \oplus g(A_2)$ for every $g \in G$, it follows that for every $x = x_1 + x_2$ with $x_i \in A_i$, $i = 1, 2$,

$$A_{x_1+x_2,G} = (A_1)_{x_1,G_1} \oplus (A_2)_{x_2,G_2}.$$

Hence,

$$V_n(A_{x,G}) = V_{n_1}\left((A_1)_{x_1,G_1}\right) \cdot V_{n_2}\left((A_2)_{x_2,G_2}\right). \tag{3.2}$$

Let $p \in P_G(A)$. Then $p = p_1 + p_2$ for some $p_i \in A_i$, $i = 1, 2$, and, for every $x_1 \in A_1$,

$$V_n(A_{p_1+p_2,G}) \geqslant V_n(A_{x_1+p_2,G});$$

thus, by (3.2),

$$V_{n_1}(A_{p_1,G_1}) \geqslant V_{n_1}(A_{x_1,G_1}),$$

i.e. $p_1 \in P_{G_1}(A_1)$. Similarly, $p_2 \in P_{G_2}(A_2)$. Hence

$$P_G(A) \subset P_{G_1}(A_1) \oplus P_{G_2}(A_2).$$

Let now $p_i \in P_{G_i}(A_i)$ for $i = 1, 2$ and let $p = p_1 + p_2$. Then, for every $x = x_1 + x_2$ with $x_i \in A_i$,

$$V_{n_i}(A_{p_i,G_i}) \geqslant V_{n_i}(A_{x_i,G_i}),$$

whence, $V_n(A_{p,G}) \geqslant V_n(A_{x,G})$, by (3.2); hence $p \in P_G(A)$.

Thus

$$P_{G_1}(A_1) \oplus P_{G_2}(A_2) \subset P_G(A).$$

□

As was proved by Fáry and Rédei in [2], if $G = \langle \sigma_0 \rangle$, then every convex body A has a unique G-pseudo-centre, $p_G(A)$. Thus, in this particular case we obtain a selector $p_G : \mathcal{K}_0^n \longrightarrow \mathrm{R}^n$.

We shall now prove that the group generated by central symmetry is the only group G with this uniqueness property.

3.8. THEOREM. *Let $G \neq \langle \sigma_0 \rangle$. Then there exists $A \in \mathcal{K}_0^n$ with non-unique G-kernel and thus with*

$$\operatorname{card} P_G(A) > 1.$$

Proof. By the assumption, there exists a line L passing through 0 which is not G-invariant, and thus $g(L) \neq L$ for some $g \in G$.

Let $\beta = \sphericalangle(L, g(L))$; then $\beta \in (0, \frac{\pi}{2}]$. Take $a \in L$ such that

$$\|a\| = \frac{2\sqrt{2}}{\sin \frac{\beta}{2}}. \tag{3.3}$$

Let $b = -a$ and let B be the unit ball in the hyperplane $H = L^\perp$.

Let A be defined by

$$A := B \oplus \Delta(a, b).$$

Then

$$\operatorname{diam}(A \cap g(A)) = \frac{2}{\sin \frac{\beta}{2}} \sqrt{1 + \sin^2 \frac{\beta}{2}}. \tag{3.4}$$

Indeed, let $E_1 = \operatorname{lin}(L \cup g(L))$ and $E_2 = (E_1)^\perp$. Since $\mathrm{R}^n = E_1 \oplus E_2$, it is easy to see that

$$\operatorname{diam}(A \cap g(A)) = \sqrt{4 + \operatorname{diam}(E_1 \cap A \cap g(A))^2}$$

and

$$\operatorname{diam}(E_1 \cap A \cap g(A)) = \frac{2}{\sin \frac{\beta}{2}},$$

which proves (3.4).

In view of (3.3) and (3.4), $\operatorname{diam}(A_{0,G}) \leqslant \operatorname{diam}(A \cap g(A))$. Let $\delta := \|a\| - \operatorname{diam}(A_{0,G})$ and $v := \frac{a-b}{\|a-b\|}$. Then $A_{0,G}$ and $A_{0,G} + \delta \cdot v$ are two different G-kernels of A. □

It is an open problem to characterize the class of convex bodies with exactly one G-kernel for every G. The following theorem gives a partial solution.

3.9. THEOREM. *If A is strictly convex, then for every non-trivial subgroup G of $O(n)$ there exists a unique G-kernel of A.*

Proof. Suppose that C_0 and C_1 are G-kernels of A. By Proposition 3.3 the family $(A_{x,G})_{x\in A}$ is concave; by the Brunn-Minkowski theorem ([3], p.309) it follows that $C_1 = C_0 + v$ for some $v \in R^n$ and all the sets $C_t := (1-t)C_0 + tC_1$ have the same volume for $t \in [0,1]$. By the strong convexity of A,

$$\mathrm{relint}\Delta(c, c+v) \subset \mathrm{int}A$$

for every $c \in C_0$. Hence $C_{\frac{1}{2}} \subset \mathrm{int}A$.

Let $\varepsilon := \mathrm{dist}(C_{\frac{1}{2}}, \mathrm{bd}A)$ and

$$C := C_{\frac{1}{2}} + \varepsilon B^n.$$

Obviously, C is G-invariant and, since $\varepsilon > 0$, it follows that $V_n(C) > V_n(C_i)$, contrary to the assumption. □

Evidently, for any $G \subset O(n)$, if a convex body A has a unique G-pseudocentre, then it has a unique G-kernel. The converse implication in general fails; for example, if G is generated by the symmetry with respect to a line L and $\sigma_L(A) = A$, then the body A is the unique G-kernel of itself but $P_G(A) = A \cap L$.

3.10. PROPOSITION. *If $\mathrm{fix}G = \{0\}$, then for every $A \in \mathcal{K}_0^n$ and every $p_0, p_1 \in P_G(A)$*

$$A_{p_0,G} = A_{p_1,G} \Longrightarrow p_0 = p_1,$$

i.e. the uniqueness of G-kernel implies the uniqueness of G-pseudo-centre.

Proof. We may assume that $p_0 = 0$. Let $p = p_1 \neq 0$. Then there exists $g \in G$ with $g(p) \neq p$. Let us consider the isometry $f := g_p g^{-1}$. Evidently, for every x,

$$f(x) = x + p - g(p),$$

i.e., f is a translation by a non-zero vector.

Since $A_{p,G}$ is invariant under g_p and g, it follows that $f(A_{p,G}) = A_{p,G}$. This contradicts the compactness of A. □

In view of 3.9 and 3.10, if $\mathrm{fix}G = \{0\}$, then every strictly convex body A has a unique G-pseudo-centre, $p_G(A)$.

4. The uniqueness of G-kernel for an affine image

As we have seen, generally a convex body may have many G-kernels (see 3.8). However, our conjecture is that for arbitrary $G \subset O(n)$, the affine class of any convex body has a representative with a unique G-kernel. We prove this conjecture under additional assumption on G, which, in view of 3.10, implies that the uniqueness of G-kernel is equivalent to the uniqueness of G-pseudo-centre.

For any $G \subset \mathrm{O}(n)$, let us consider the function $\phi_G : \mathscr{K}_0^n \longrightarrow \mathrm{R}$ defined by the formula:

$$\phi_G(A) := \frac{\sup_{x \in A} V_n(A_{x,G})}{V_n(A)}. \tag{4.1}$$

We start with two lemmas which hold without any restriction on G.

4.1. LEMMA. *Let $G \subset \mathrm{O}(n)$. For every similarity $f : \mathrm{R}^n \longrightarrow \mathrm{R}^n$,*

$$\phi_G(f(A)) = \phi_G(A).$$

4.2. LEMMA. *For every $G \subset \mathrm{O}(n)$ the function ϕ_G is continuous.*

Proof. In view of 4.1, without any loss of generality we may assume that $V_n(A) = 1$. Then

$$\phi_G(A) = V_n(A_p),$$

where p is an arbitrary point of $P_G(A)$.

By 3.5, we may assume that $p \in \mathrm{int}A$. Thus it suffices to prove that the function $\psi_G : \{(A, x);\ A \in \mathscr{K}_0^n, x \in \mathrm{int}A\} \longrightarrow \mathrm{R}$ defined by the formula

$$\psi_G(A, x) := V_n(A_{x,G}) \tag{4.2}$$

is continuous.

Let $A = \lim_H A_k$ and $x = \lim x_k$, where $A, A_k \in \mathscr{K}_0^n$, $x \in \mathrm{int}A$, and $x_k \in \mathrm{int}A_k$ for $k \in \mathrm{N}$. We replace A and $(A_k)_{k\in\mathrm{N}}$ by A' and $(A'_k)_{k\in\mathrm{N}}$:

$$A' := A - x \quad \text{and} \quad A'_k = A_k - x_k.$$

Then $0 \in A' \cap \bigcap_{k=1}^{\infty} A'_k$, $A' = \lim_H A'_k$, and, by (4.2),

$$\psi_G(A, x) = \psi_G(A', 0) \quad \text{and} \quad \psi_G(A_k, x_k) = \psi_G(A'_k, 0).$$

Hence, it remains to prove that

$$\lim \psi_G(A'_k, 0) = \psi_G(A', 0),$$

i.e.,

$$\lim V_n\left(\bigcap_{g\in G} g(A'_k)\right) = V_n\left(\bigcap_{g\in G} g(A')\right).$$

Since V_n is continuous, it suffices to show that

$$\lim_H \bigcap_{g\in G} g(A'_k) = \bigcap_{g\in G} g(A'). \tag{4.3}$$

There exist $\alpha > 0$ and $\beta > 1$ such that

$$\alpha B^n \subset A' \subset \beta B^n \quad \text{and} \quad \alpha B^n \subset A'_k \subset \beta B^n \quad \text{for every } k.$$

Let $\varepsilon > 0$. Since $A' = \lim A'_k$, there exists $k_o \in \mathrm{N}$ such that

$$A'_k \subset A' + \frac{\alpha\varepsilon}{\beta} \cdot B^n \quad \text{and} \quad A' \subset A'_k + \frac{\alpha\varepsilon}{\beta} \cdot B^n \quad \text{for } k \geqslant k_0.$$

But, it is easy to check that

$$A' + \frac{\alpha\varepsilon}{\beta} \cdot B^n \subset (1 + \frac{\varepsilon}{\beta}) \cdot A'$$

and similarly for A'_k, $k \in \mathrm{N}$.

Thus

$$A'_k \subset (1 + \frac{\varepsilon}{\beta}) \cdot A' \quad \text{and} \quad A' \subset (1 + \frac{\varepsilon}{\beta}) \cdot A'_k \quad \text{for } k \geqslant k_0.$$

Hence, for every $g \in G$,

$$g(A'_k) \subset (1 + \frac{\epsilon}{\beta}) \cdot g(A')$$

and therefore

$$\bigcap_{g \in G} g(A'_k) \subset (1 + \frac{\varepsilon}{\beta}) \bigcap_{g \in G} g(A') \subset \bigcap_{g \in G} g(A') + \varepsilon B^n \quad \text{for } k \geqslant k_0.$$

Similarly,

$$\bigcap_{g \in G} g(A') \subset \bigcap_{g \in G} g(A'_k) + \varepsilon B^n \quad \text{for } k \geqslant k_0.$$

This proves (4.3). □

The next lemma requires an additional assumption on G.

4.3. LEMMA. *Let $G \subset \mathrm{O}(n)$. If there is no G-invariant linear subspace of dimension $k \in \{1, \dots, n-1\}$, then for every $A \in \mathcal{K}_0^n$ and every $\varepsilon > 0$ there exists $\gamma > 0$ such that for every $f \in \mathrm{SA}(n)$*

$$\|f\| > \gamma \Longrightarrow \phi_G(f(A)) < \varepsilon. \tag{4.4}$$

Proof. Let us first notice that it suffices to prove the assertion for the unit n-ball.

Indeed, let it hold for B^n. Take $A \in \mathcal{K}_0^n$ and $\varepsilon > 0$. By 4.1, we may assume that $V_n(A) = 1$. Take $\alpha > 0$ such that $A \subset \alpha \cdot B^n$ and let $\varepsilon' = \frac{\varepsilon}{\alpha^n \kappa_n}$. Then, by the assumption, there exists $\gamma > 0$ such that for every $f \in \mathrm{SA}(n)$ with $\|f\| > \gamma$

$$\phi_G(f(B^n)) < \varepsilon'.$$

Thus

$$\phi_G(f(A)) = V_n((f(A)_{x,G}) \leqslant \alpha^n \kappa_n \cdot \phi_G(f(B^n)) \leqslant \varepsilon,$$

which proves the assertion for arbitrary convex body A.

Hence, we assume $A = B^n$. By Proposition 2.4, there exists $\lambda_G > 0$ such that for every G-invariant $C \in \mathscr{K}_0^n$

$$V_n(C) \leqslant \lambda_G \cdot d(C)^n. \tag{4.5}$$

Take an $\varepsilon > 0$ and let

$$\gamma := \left(\frac{\lambda_G}{\varepsilon}\right)^{\frac{n-1}{n}} \cdot 2^{n-1}. \tag{4.6}$$

We may assume without any loss of generality that $f \in \mathrm{SL}(n)$. Let $\|f\| > \gamma$ and let $a_1, \ldots, a_n$ be the half-axes of the ellipsoid $f(B^n)$, with $a_1 \geqslant \ldots \geqslant a_n$. Then

$$a_n \leqslant (a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}} = \left(V_n\left(f(B^n)\right) \cdot (\kappa_n a_1)^{-1}\right)^{\frac{1}{n-1}},$$

and, since $V_n\left(f(B^n)\right) = \kappa_n$, it follows that $(a_n)^{n-1} \leqslant (a_1)^{-1}$, i.e.,

$$a_1 \leqslant (a_n)^{1-n}. \tag{4.7}$$

But $\|f\| = a_1$; thus, by the assumption, $a_1 > \gamma$, which, together with (4.6) and (4.7), yields

$$2^{n-1} \cdot \left(\frac{\lambda_G}{\varepsilon}\right)^{\frac{n-1}{n}} < (a_n)^{1-n}$$

and, consequently,

$$(2a_n)^n < \frac{\varepsilon}{\lambda_G}. \tag{4.8}$$

Let C be a G-kernel of $f(B^n)$. Then $\phi_G\left(f(B^n)\right) = V_n(C)$, and thus, by (4.5) and (4.8),

$$\phi_G\left(f(B^n)\right) \leqslant \lambda_G\left(d(C)\right)^n \leqslant \lambda_G d\left(f(B^n)\right)^n = \lambda_G(2a_n)^n < \varepsilon.$$

□

4.4. THEOREM. *Let $G \subset \mathrm{O}(n)$. If there is no G-invariant linear subspace of dimension $k \in \{1, \ldots, n-1\}$, then for every $A \in \mathscr{K}_0^n$ there exists an affine automorphism f_0 of R^n such that $f_0(A)$ has a unique G-pseudo-centre.*

Proof. Let $\phi := \phi_G$. Take $A \in \mathscr{K}_0^n$ and $\varepsilon > 0$. By Lemma 4.3, there exists $\gamma > 0$ such that $\phi\left(f(A)\right) < \varepsilon$ whenever $f \in \mathrm{SA}(n)$ and $\|f\| > \gamma$.

By the continuity of ϕ (Lemma 4.2), also the function $\hat{\phi}_A : \mathrm{SA}(n) \longrightarrow \mathrm{R}$ defined by

$$\hat{\phi}_A(f) := \phi\left(f(A)\right)$$

is continuous and, therefore, it attains its maximum in the compact subset $\Phi := \{f \in \mathrm{GA}(n);\ \|f\| \leqslant \gamma,\ |\det f| \leqslant 1\}$ of $\mathrm{GA}(n)$. Let f_0 be a maximizer of $\hat{\phi}_A|\Phi$. We have to show that

$$P_G\left(f_0(A)\right) \quad \text{is a singleton.} \tag{4.9}$$

Let $A' = f_0(A)$ and $p_i \in P_G(A')$ for $i = 0, 1$. Then, by the Brunn-Minkowski Theorem combined with 3.3,

$$(A')_{p_1,G} = (A')_{p_0,G} + v \quad \text{for some } v \in \mathrm{R}^n.$$

and thus, by 3.10, $p_1 = p_0 + v$, because $\mathrm{fix}G = \{0\}$.

Without any loss of generality we may assume that $p_1 = -p_0$.

Suppose that $v \neq 0$ and let $(w_1, ..., w_n)$ be an orthonormal basis of R^n with $w_n = \frac{v}{\|v\|}$. Let f be the linear automorphism with $f(w_i) = w_i$ for $i = 1, ..., n-1$ and $f(w_n) = \alpha \cdot w_n$, where

$$\alpha = \max\left\{\frac{h((A')_0, w_n)}{h((A')_{p_1}, w_n)}, \frac{h((A')_0, -w_n)}{h((A')_{p_0}, -w_n)}\right\}.$$

Then $\alpha < 1$ and $ff_0 \in \Phi$. We shall show that

$$(A')_0 \subset f(A'). \tag{4.10}$$

Let $x = \Sigma_{i=0}^n x_i w_i \in (A')_0$; then $f^{-1}(x) = \Sigma_{i=1}^{n-1} x_i w_i + \frac{x_n}{\alpha} w_n$, whence

$$f^{-1}(x) - x = x_n(\frac{1}{\alpha} - 1) \cdot w_n,$$

and thus

$$f^{-1}(x) \in \Delta(x - |x_n|(\frac{1}{\alpha} - 1) \cdot w_n,\ x + |x_n|(\frac{1}{\alpha} - 1) \cdot w_n). \tag{4.11}$$

Evidently

$$|x_n| \leqslant \max\{h((A')_0, w_n), h((A')_0, -w_n)\}. \tag{4.12}$$

Since $(A')_{p_0} = (A')_0 - \frac{v}{2}$ and $(A')_{p_1} = (A')_0 + \frac{v}{2}$, it follows that

$$h((A')_{p_0}, -w_n) = h((A')_0, -w_n) + \frac{1}{2}\|v\|$$

and

$$h((A')_{p_1}, w_n) = h((A')_0, w_n) + \frac{1}{2}\|v\|.$$

By simple calculation,

$$\frac{1}{\alpha} - 1 = \min\left\{\frac{\|v\|}{2h((A')_0, w_n)}, \frac{\|v\|}{2h((A')_0, -w_n)}\right\}$$
$$= \|v\| \cdot (2\max\{h((A')_0, w_n), h((A')_0, -w_n)\})^{-1}.$$

Hence, by (4.12),

$$\frac{1}{\alpha} - 1 \leqslant \frac{\|v\|}{2|x_n|},$$

which, together with (4.11), implies

$$f^{-1}(x) \in \Delta(x - \frac{v}{2}, x + \frac{v}{2}).$$

Therefore

$$f^{-1}(x) \in \operatorname{conv}((A')_{p_0} \cup (A')_{p_1}) \subset A'.$$

This proves (4.10).

Let now C be any G-kernel of $f(A')$. Since $(A')_0$ is G-invariant, by (4.10) it follows that

$$V_n((A')_0) \leqslant V_n(C).$$

Hence

$$\phi(A') = \frac{V_n((A')_0)}{V_n(A')} = \frac{V_n((A')_0)}{\frac{1}{\alpha}V_n(f(A'))} < \frac{V_n(C)}{V_n(f(A'))} = \phi(f(A')),$$

i.e.,

$$\hat{\phi}_A(ff_0) > \hat{\phi}_A(f_0),$$

contrary to the assumption that f_0 is a maximizer of $\hat{\phi}_A|\Phi$. Hence $\nu = 0$, i.e. $(A')_{p_0,G} = (A')_{p_1,G}$.

Applying now Proposition 3.10, we obtain $p_0 = p_1$. This proves (4.9). □

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