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CURVES IN n–DIMENSIONAL k–ISOTROPIC SPACE

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Abstract. In this paper we develop the theory of curves in n-dimensional k-isotropic space I_n^k. We derive explicit expressions and geometrical interpretations for the curvatures of a curve.

1. Introduction

The n-dimensional k-isotropic space I_n^k was introduced by H. Vogler and H. Wresnik in [17]. We follow the notations and the terminology used in that paper. The special cases of I_2^1, I_3^1, I_3^2 were thoroughly studied in [2], [3], [4], [9], [10] [12], [13], [14], [15], [16]. The case of I_n^1 was introduced in [11], and studied in [1] and [5]. The theory of curves in n-dimensional flag space I_n^{n-1} was studied in [7] and in [8]. A general approach to the theory of curves in Cayley/Klein spaces is given in [6].

In this paper we develop the theory of curves in I_n^k. We construct the Frenet frame of an admissible curve and calculate the explicit expressions of the curvatures of such a curve. We derive also the geometrical interpretation of these curvatures and investigate the curves having some of their curvatures equal to zero. Finally we describe the conditions, in terms of curvatures, if a curve lies in an l-isotropic m-plane.

Let A denote an n-dimensional affine space and V its corresponding vector space. The space V is decomposed in a direct sum

$$V = U_1 \oplus U_2 \tag{1}$$

such that $\dim U_2 = k$, $\dim U_1 = n - k$. Let $B_2 = \{\mathbf{b}_{n-k+1}, \ldots, \mathbf{b}_n\}$ be a basis for the subspace U_2. In U_2 a flag of vector spaces $U_2 := C_1 \supset \ldots \supset C_l \supset C_{l+1} \supset \ldots \supset C_k := [\mathbf{b}_n]$, $C_l = [\mathbf{b}_{n-k+l}, \ldots \mathbf{b}_n]$ is defined. According to it we distinguish the following classes of vectors: the Euclidean vectors as the vectors in $V \setminus U_2$ and the isotropic vectors of degree l or l-isotropic vectors, $l = 1, \ldots k$, as the vectors in U_2, $\mathbf{x} = \sum_{m=1}^{k} x_{n-k+m}\mathbf{b}_{n-k+m}$, for which holds

$$x_{n-k+1} = \ldots = x_{n-k+l-1} = 0, x_{n-k+l} \neq 0.$$

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By $\pi_i : V \to U_i$, $i = 1, 2$, we denote the canonical projections.
The scalar product $\cdot : U_1 \times U_1 \to \mathbf{R}$ is extended in the following way on the whole V by

$$\mathbf{x} \cdot \mathbf{y} = \pi_1(\mathbf{x}) \cdot \pi_1(\mathbf{y}). \tag{2}$$

Therefore the isotropic vectors are orthogonal (scalar product vanishes) to all other vectors, especially also to themselves.

For $\mathbf{x} \in V$ we define its isotropic length by $||\mathbf{x}|| := |\pi_1(\mathbf{x})|$. But if $\mathbf{x}$ is an l-isotropic vector, then its isotropic length is 0, and therefore we introduce as isotropic length the lth- range of $\mathbf{x}$, i.e. $[\mathbf{x}]_l := x_{n-k+l}$, $l = 1, \ldots, k$.

The group of motions of I_n^k is given by the matrix

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}, \tag{3}$$

where A is an orthogonal $(n-k, n-k)$-matrix, det$A$=1, B a real $(k, n-k)$-matrix and C a real lower triangular (k, k)-matrix such that $c_{n-k+l}^{n-k+l} = 1$.

2. Hyperplanes in I_n^k

We distinguish the following classes of hyperplanes in I_n^k. We say that a hyperplane in I_n^k given by an equation

$$u_0 + u_1x_1 + \ldots + u_nx_n = 0$$

is of type l or l- isotropic, $l = 0, \ldots k$, if $u_{n-l} \neq 0$ and $u_{n-l+1} = \ldots = u_n = 0$. Especially, for $l = 0$ we say that a hyperplane is non-isotropic and for $l = k$ that it is completely isotropic.

PROPOSITION 1. *Let H be an l-isotropic hyperplane, $l = 0, \ldots, k-1$. Then there are no $(k-l)$-isotropic vectors in H. Furthermore, there exists a basis consisting of $n-k$ Euclidean vectors and of one of m-isotropic vectors, $m = 1, \ldots, k$, $m \neq k-l$, but also a basis consisting of $n-l-1$ Euclidean vectors and of one of m-isotropic vectors, $m = k-l+1, \ldots, k$.*
In every basis of H the number of Euclidean vectors varies from $n-k$ to $n-l-1$; there are at most $k-m$ m-isotropic vectors, if $m \leqslant k-l-1$, and at most $k-m+1$ m-isotropic vectors, if $m \geqslant k-l+1$.

Proof. Let H be an l-isotropic hyperplane given by

$$u_0 + u_1x_1 + \ldots u_{n-l}x_{n-l} = 0, \quad u_{n-l} \neq 0.$$

Then its equation can be written in the following form

$$\begin{vmatrix} x_1 & \dots & x_{n-k} & \dots & x_{n-l-1} & x_{n-l}+\frac{u_0}{u_{n-l}} & x_{n-l+1} & \dots & x_n \\ u_{n-l} & \dots & 0 & \dots & 0 & -u_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & u_{n-l} & \dots & 0 & -u_{n-k} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & u_{n-l} & -u_{n-l-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{vmatrix} = 0. \quad (4)$$

From (4) it can be seen that there are no $(k-l)$-isotropic vectors in an l-isotropic hyperplane, $l = 0, \dots, k-1$. Furthermore, it can also be seen that there exist the mentioned bases for H; the first follows directly from (4), the others by making linear combinations of the vectors of the first mentioned basis.

COROLLARY 1. *In a non-isotropic hyperplane there are no k-isotropic vectors. Furthermore, there exists a basis consisting of* $n-1$ *Euclidean vectors, but also a basis consisting of* $n-k$ *Euclidean vectors and of one of m-isotropic vectors,* $m = 1, \dots, k-1$.
In every basis the number of Euclidean vectors varies from $n-k$ *to* $n-1$*, there are at most* $k-m$ *m-isotropic vectors,* $m = 1, \dots, k-1$.

COROLLARY 2. *In a completely isotropic hyperplane exist all m-isotropic directions,* $m = 1, \dots, k$.
There exists a basis consisting of $n-k-1$ *Euclidean vectors and of one of m-isotropic vectors,* $m = 1, \dots, k$. *Generally, every basis consists of* $n-k-1$ *Euclidean vectors, and of at most* $k-m+1$ *m-isotropic vectors,* $m = 1, \dots, k$.

3. Curves in I_n^k

Definition 1. Let $I \subseteq \mathbf{R}$ be an open interval and $\varphi : I \to I_n^k$ a vector function given in affine coordinates by

$$\vec{OX}(t) = (x_1(t), \dots, x_n(t)) := \mathbf{x}(t),$$

where $\varphi(t) = X$ is a point in A.
The set of points $c \in I_n^k$ is called a C^r- curve, $r \geqslant 1$, if there is an open interval $I \subseteq \mathbf{R}$ and a C^r-mapping $\varphi : I \to I_n^k$ such that $\varphi(I) = c$.
A C^r-curve is regular if $\dot{\mathbf{x}}(t) \neq 0, t \in I$.

A C^r-curve is simple if it is regular and φ is injective.

One can easily see that the notions of C^r-curve, regular C^r-curve and simple C^r-curve are invariant under the group of motions of I_n^k.

Definition 2. A point $P_0(t_0)$ of a regular C^n-curve is called an inflection point of order l, $l = 2, \dots, n-1$, if the set of vectors

$$\{\dot{\mathbf{x}}(t_0), \dots, \mathbf{x}^{(l-1)}(t_0)\}$$

is linearly independent and the set of vectors

$$\{\dot{\mathbf{x}}(t_0), \dots, \mathbf{x}^{(l)}(t_0)\}$$

is linearly dependent.
If a curve has no inflection points of any order l, $l = 2, \dots, n-1$, it is said to be non-degenerated.

The notion of an inflection point of order l is a geometrical notion i.e. it does not depend on parametrization and is invariant under the group of motions. Moreover, it is a differential invariant of order l.

4. Osculating planes

Definition 3. Let c be a simple C^r-curve given by $\mathbf{x} = \mathbf{x}(t)$ and $P(t) \in c$ an inflection point of order r. The osculating m-plane, $m = 1, \dots, r-1$, at the point P is m-dimensional plane in I_n^k spanned by the vectors $\dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(m)}(t)$.

If c is a non-degenerated simple C^n-curve, then the osculating hyperplane of c at $P(t)$ is the hyperplane spanned by $\dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n-1)}(t)$. Its equation is given by

$$det(\mathbf{x} - \mathbf{x}(t), \dot{\mathbf{x}}(t), \dots, \mathbf{x}^{(n-1)}(t)) = 0, \tag{5}$$

where $\mathbf{x}$ denotes a position vector of an arbitrary point of the osculating hyperplane.

PROPOSITION 2. *Let $c : I \to I_n^k$ be a simple $C^{(l+1)}$-curve on which all of the points are inflection points of order $l+1$, $l = 1, \dots, n-1$. Then there exists an l-plane which contains the curve c.*

Definition 4. A curve c is said to be an admissible C^r-curve, $r \geqslant n-1$, if $\pi_1(c)$ is non-degenerated and c is a simple, non-degenerated C^r-curve without l-isotropic osculating hyperplanes, $l = 1, \dots, k$.

THEOREM 1. *A C^r-curve c, $r \geqslant n-1$, is admissible if and only if*

$$\begin{vmatrix} \dot{x}_1(t) & \dots & \dot{x}_{n-1}(t) \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)}(t) & \dots & x_{n-1}^{(n-1)}(t) \end{vmatrix} \neq 0, \quad t \in I, \tag{6}$$

$$\begin{vmatrix} \dot{x}_1(t) & \dots & \dot{x}_{n-k}(t) \\ \vdots & \vdots & \vdots \\ x_1^{(n-k)}(t) & \dots & x_{n-k}^{(n-k)}(t) \end{vmatrix} \neq 0, \quad t \in I. \tag{7}$$

An admissible curve has neither l-isotropic tangents nor l-isotropic osculating m-planes, $l = 1, \dots, k$, $m = 2, \dots, n-1$.

Proof. If c is admissible, then the statement obviously holds.
Conversely, if (6) holds, then c is non-degenerated. Furthermore c is regular because otherwise it would be $\dot{\mathbf{x}}(t) = 0, t \in I$, and so the first row of the determinant (6) would consists of zeros. If c has l-isotropic tangents, then the first row of the determinant (7) would be zero. In every l-isotropic m-plane, $l = 1, \dots, k$, there is k-isotropic direction. Therefore if c has osculating l-isotropic m-plane, (6) would be zero.

5. Frenet's equations of a curve in I_n^k

Definition 5. Let $c : [a, b] \to I_n^k$ be an admissible curve. Then

$$s := \int_a^b ||\dot{\mathbf{x}}|| dt = \int_a^b |\pi_1(\dot{\mathbf{x}})| dt$$

is called the isotropic arc length of the curve c from $\mathbf{x}(a)$ to $\mathbf{x}(b)$.

One can notice that the isotropic arc length of an admissible curve c coincides with the Euclidean arc length of the projection $\pi_1(c)$ to the basic space.

PROPOSITION 3. *Every admissible C^r-curve c can be reparametrized by the arc length s and s is the arc length on c exactly when* $||\dot{\mathbf{x}}(s)|| = 1$.

Let $c : I \to I_n^k$ be a curve parametrized by the arc length. Notice that c is also admissible. Now we can define the n-frame $\{\mathbf{t}_1(s), \dots, \mathbf{t}_n(s)\}$ of a curve c in a point $\mathbf{x}(s)$. It should be an orthonormal basis of V like it is defined in [17].
By applying the Gram-Schmidt orthogonalization process to the set

$$\{\mathbf{x}', \dots, \mathbf{x}^{(n-k)}\}$$

we get the orthonormal set of vectors $\{\mathbf{t}_1, \dots, \mathbf{t}_{n-k}\}$

$$\begin{aligned} \mathbf{t}_1 &:= \mathbf{x}' \\ \mathbf{b}_m &:= \mathbf{x}^{(m)} - \textstyle\sum_{i=1}^{m-1} (\mathbf{x}^{(m)} \cdot \mathbf{t}_i)\mathbf{t}_i \\ \mathbf{t}_m &:= \frac{\mathbf{b}_m}{||\mathbf{b}_m||}, \quad m = 2, \dots, n-k. \end{aligned}$$

One can see that the frame $\{\pi_1(\mathbf{t}_1), \dots, \pi_1(\mathbf{t}_{n-k})\}$ is the Frenet $(n-k)$-frame of the curve $\pi_1(c)$.
If we put $\bar{U}_1 = [\mathbf{t}_1, \dots, \mathbf{t}_{n-k}]$, then $\bar{U}_1 \cap U_2 = \{0\}$, and therefore we have the following decomposition $V = \bar{U}_1 \oplus U_2$. Now we should define the basis of U_2

consisting of one unit 1-isotropic vector, ..., one unit k-isotropic vector. Let us suppose that $x_{n-k+1}^{(n-k+1)}(s) \neq 0$. If $x_{n-k+1}^{(n-k+1)}(s) = 0$ then there must exist some other coordinate x_{n-k+i} such that $x_{n-k+i}^{(n-k+1)}(s) \neq 0$ and we can form the vector $\mathbf{t}_{n-k+1}$ by it. Now we define

$$\mathbf{t}_{n-k+1} := (\underbrace{0, \dots 0}_{n-k}, 1, \frac{x_{n-k+2}^{(n-k+1)}}{x_{n-k+1}^{(n-k+1)}}, \dots, \frac{x_{n}^{(n-k+1)}}{x_{n-k+1}^{(n-k+1)}}).$$

Obviously $\mathbf{t}_{n-k+1}$ is an unit 1-isotropic vector.
Let us also define

$$\kappa_{n-k+1}(s) = \left(\frac{x_{n-k+2}^{(n-k+1)}(s)}{x_{n-k+1}^{(n-k+1)}(s)} \right)' .$$

If $\kappa_{n-k+1}(s) \neq 0$, we can put

$$\mathbf{t}_{n-k+2} := (\underbrace{0, \dots 0, 0}_{n-k+1}, 1, \frac{\left(\frac{x_{n-k+3}^{(n-k+1)}}{x_{n-k+1}^{(n-k+1)}} \right)'}{\kappa_{n-k+1}}, \dots, \frac{\left(\frac{x_{n}^{(n-k+1)}}{x_{n-k+1}^{(n-k+1)}} \right)'}{\kappa_{n-k+1}})$$

which is an unit 2-isotropic vector. Now we introduce

$$\kappa_{n-k+2}(s) = \left(\frac{\left(\frac{x_{n-k+3}^{(n-k+1)}(s)}{x_{n-k+1}^{(n-k+1)}(s)} \right)'}{\kappa_{n-k+1}(s)} \right)' .$$

Continuing the process, under the assumptions $\kappa_{n-k+2}(s) \neq 0, \dots, \kappa_{n-k+j}(s) \neq 0$, we define the $(j+1)$-isotropic vector

$$\mathbf{t}_{n-k+j+1} = (\underbrace{0, \dots, 0}_{n-k+j}, 1, \left((x_{n-k+j+2}^{(n-k+1)} : x_{n-k+1}^{(n-k+1)})' : \kappa_{n-k+1} \right)' : \dots : \kappa_{n-k+j}, \dots,$$

$$\dots, \left((x_{n}^{(n-k+1)} : x_{n-k+1}^{(n-k+1)})' : \kappa_{n-k+1} \right)' : \dots : \kappa_{n-k+j})$$

and

$$\kappa_{n-k+j+1} = \left(\left(\left((x_{n-k+j+2}^{(n-k+1)} : x_{n-k+1}^{(n-k+1)})' : \kappa_{n-k+1} \right)' : \kappa_{n-k+2} \right)' : \dots : \kappa_{n-k+j} \right)' ,$$

$$j = 1, \dots k-2.$$

The last vector is equal to

$$\mathbf{t}_n = (0, \dots 0, 1).$$

Obviously the following theorem is true.

THEOREM 2. **(Frenet's Equations)**
Let c be an admissible curve in I_n^k parametrized by the arc length and let $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$ be its Frenet n-frame. Then there exist functions $\kappa_1, \ldots, \kappa_{n-1} : I \to \mathbf{R}$ such that the following equations hold

$$\begin{aligned} \mathbf{t}_1{}' &= \kappa_1 \mathbf{t}_2, \\ \mathbf{t}_i{}' &= -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, \quad i = 2, \ldots, n-k, \\ \mathbf{t}_{n-k+j}{}' &= \kappa_{n-k+j}\mathbf{t}_{n-k+j+1}, \quad j = 1, \ldots, k-1, \\ \mathbf{t}_n{}' &= 0. \end{aligned}$$

6. Explicit expressions of the curvatures of a curve in I_n^k

Let us derive now the explicit expressions of the curvatures of an admissible curve c parametrized by its arc length. Since κ_i, $i = 1, \ldots, n-k-1$, are the curvatures of the projection $\pi_1(c)$ of the curve c, we have

$$\kappa_i^2(s) = \frac{\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(i-1)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(i+1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(i)})}, \quad i = 1, \ldots, n-k-1,$$

where Γ denotes Gram's determinant with a scalar product defined in (2).
The expressions for the curvatures $\kappa_{n-k+1}, \ldots, \kappa_{n-1}$ are given by the construction of the Frenet frame in the previous section. We can obtain the explicit expression for the curvature κ_{n-k} in the following way. Using Frenet's equations we get

$$\begin{aligned} \mathbf{x}' &= \mathbf{t}_1 \\ \mathbf{x}^{(i)} &= a_{i1}\mathbf{t}_1 + \ldots + a_{ii-1}\mathbf{t}_{i-1} + \kappa_1 \cdots \kappa_{i-1}\mathbf{t}_i, \quad i = 2, \ldots, n. \end{aligned}$$

Therefore it holds

$$\begin{aligned} det(\mathbf{x}', \ldots, \mathbf{x}^{(n)}) &= \kappa_1^{n-1} \ldots \kappa_{n-1} \\ det(\pi_1(\mathbf{x}'), \ldots, \pi_1(\mathbf{x}^{(n-k)})) &= \kappa_1^{n-k-1} \cdots \kappa_{n-k-1}. \end{aligned}$$

Now we have

$$\kappa_{n-k}^k = \frac{det(\mathbf{x}', \ldots, \mathbf{x}^{(n)})}{det(\pi_1(\mathbf{x}'), \ldots, \pi_1(\mathbf{x}^{(n-k)}))^{k+1}\kappa_{n-k+1}^{k-1} \cdots \kappa_{n-1}}.$$

By substituting the expressions for κ_i, $i = 1, \ldots, n-k-1$, and by noticing that

$$det(\pi_2(\mathbf{x}^{(n-k+1)}), \ldots, \pi_2(\mathbf{x}^{(n)})) = (x_{n-k+1}^{(n-k+1)})^k \kappa_{n-k+1}^{k-1} \cdots \kappa_{n-1}$$

we get the following expression for κ_{n-k}

$$\kappa_{n-k}^k = \frac{det(\mathbf{x}', \ldots, \mathbf{x}^{(n)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k-1)})^{k/2}(x_{n-k+1}^{(n-k+1)})^k}{det(\pi_1(\mathbf{x}'), \ldots, \pi_1(\mathbf{x}^{(n-k)}))^{k+1}det(\pi_2(\mathbf{x}^{(n-k+1)}), \ldots, \pi_2(\mathbf{x}^{(n)}))}. \tag{8}$$

Let us notice that for the curvatures $\kappa_{n-k}, \ldots, \kappa_{n-1}$ we can also derive the following explicit expressions. It is easy to show that

$$\begin{vmatrix} x_1' & \cdots & x_{n-k+j} \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+j)} & \cdots & x_{n-k+j}^{(n-k+j)} \end{vmatrix} = \kappa_1^{n-k+j-1} \cdots \kappa_{n-k+j-1} \tag{9}$$

$$j = 2, \ldots, k-1,$$

holds. By using (9) and by considering that

$$(\kappa_1 \cdots \kappa_{n-k-1})^2 = \frac{\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k)})}{\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k-1)})}$$

we get

$$\kappa_{n-k}^2 = \frac{\begin{vmatrix} x_1' & \cdots & x_{n-k+1}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+1)} & \cdots & x_{n-k+1}^{(n-k+1)} \end{vmatrix}^2 \Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k-1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k)})} \tag{10}$$

and by induction

$$\kappa_{n-k+j} = \frac{\begin{vmatrix} x_1' & \cdots & x_{n-k+j+1}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+j+1)} & \cdots & x_{n-k+j+1}^{(n-k+j+1)} \end{vmatrix} \begin{vmatrix} x_1' & \cdots & x_{n-k+j-1}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+j-1)} & \cdots & x_{n-k+j-1}^{(n-k+j-1)} \end{vmatrix}}{\begin{vmatrix} x_1' & \cdots & x_{n-k+j}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+j)} & \cdots & x_{n-k+j}^{(n-k+j)} \end{vmatrix}^2}, \tag{11}$$

$$j = 1, \ldots, k-1.$$

Let us now suppose that V is endowed with a scalar product $\cdot : V \times V \to \mathbf{R}$ such that its restriction to U_1 coincides with the already defined scalar product $\cdot : U_1 \times U_1 \to \mathbf{R}$. We shall use the same notation for the scalar product on V as for the degenerated scalar product defined in (2). Let us also introduce the following notation. Let $\Gamma_{n-k+i}(\mathbf{y}_1, \ldots, \mathbf{y}_m)$, $i = 1, \ldots, k$, denote the Gram's determinant of the projections of the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_m$ onto the $(n-k+i)$-dimensional subspace of V spanned by the first $n-k+i$ coordinate vectors and $\Gamma_{n-k}(\mathbf{y}_1, \ldots, \mathbf{y}_m) = \Gamma(\mathbf{y}_1, \ldots, \mathbf{y}_m)$. Then the expression (10) can be written as

$$\kappa_{n-k}^2 = \frac{\Gamma_{n-k+1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+1)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k-1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k)})} \tag{12}$$

and the expressions (11) as

$$\kappa_{n-k+j}^2 = \frac{\Gamma_{n-k+j+1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+1)})\Gamma_{n-k+j-1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j-1)})}{\Gamma_{n-k+j}^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j)})}, \qquad (13)$$
$$j = 1, \ldots, k-1.$$

We can prove the following theorem.

THEOREM 3. *Let* $\kappa_1, \ldots, \kappa_{n-1} : I \to \mathbf{R}$ *be differentiable functions different from 0 such that* $\kappa_1, \ldots, \kappa_{n-k-2} > 0$. *Then there exists, up to isotropic motions, a unique admissible curve c parametrized by the arc length such that* $\kappa_1, \ldots, \kappa_{n-1}$ *are its curvatures.*

Proof. Under these assumptions, there exists, up to an Euclidean motion, a unique projection $\pi_1(c)$ of the curve c in the Euclidean space U_1 parametrized by the arc length such that $\kappa_1, \ldots, \kappa_{n-k-1}$ are its curvatures. Furthermore, (9) implies

$$\begin{vmatrix} x_1' & \cdots & x_{n-k+1} \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+1)} & \cdots & x_{n-k+1}^{(n-k+1)} \end{vmatrix} = \kappa_1^{n-k} \cdots \kappa_{n-k}.$$

Expansion by the last column of this determinant gives a linear differential equation with differentiable coefficients for the function $x_{n-k+1}(s)$ which enables us to find that function. By similar reasoning, for already found functions $x_1, \ldots, x_{n-k+j-1}$, the expression (9) enables us to find the functions x_{n-k+j}, $j = 2, \ldots, k-1$. Therefore, the existence of the curve c is proved.
In order to show that a curve c is unique up to an isotropic motion, we can see at first that $y_1(s) = 1, y_2(s) = x_1(s), \ldots, y_{n-k+j} = x_{n-k+j-1}(s)$ form the fundamental solutions for the corresponding homogeneous differential equation of the equation (9). If $x_{n-k+j}^p(s)$ is a particular solution of (9), then the general solution of (9) is given by

$$x_{n-k+j}(s) = C \cdot 1 + C_1 x_1(s) + \ldots + C_{n-k+j-1} x_{n-k+j-1}(s) + x_{n-k+j}^p(s).$$

Therefore, every curve which is obtained by an isotropic motion from the curve $\mathbf{x}(s) = (x_1(s), \ldots, x_{n-k}(s), x_{n-k+1}^p(s), \ldots, x_n^p(s))$ satisfies the conditions of the theorem.

7. Geometrical interpretations of the curvatures

Using explicit expressions of the curvatures obtained in the previous section we can show that the following propositions hold.

PROPOSITION 4. *Let c be an admissible* C^n*-curve. Then*

$$|\kappa_{n-1}(s_0)| = lim_{s\to 0}\left|\frac{\theta}{s}\right|$$

where θ denotes the angle between the osculating hyperplanes at the points $\mathbf{x}(s_0)$ and $\mathbf{x}(s+s_0)$ and s is the parameter of the arc length.

Proof: Since the osculating hyperplanes of an admissible curve c at the points $\mathbf{x}(s_0)$ and $\mathbf{x}(s+s_0)$ are non-isotropic, their angle is given by

$$|\theta| = \left| \frac{\begin{vmatrix} x_1'(s+s_0) & \dots & x_{n-2}'(s+s_0) & x_n'(s+s_0) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s+s_0) & \dots & x_{n-2}^{(n-1)}(s+s_0) & x_n^{(n-1)}(s+s_0) \end{vmatrix}}{\begin{vmatrix} x_1'(s+s_0) & \dots & x_{n-1}'(s+s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s+s_0) & \dots & x_{n-1}^{(n-1)}(s+s_0) \end{vmatrix}} - \frac{\begin{vmatrix} x_1'(s_0) & \dots & x_{n-2}'(s_0) & x_n'(s_0) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s_0) & \dots & x_{n-2}^{(n-1)}(s_0) & x_n^{(n-1)}(s_0) \end{vmatrix}}{\begin{vmatrix} x_1'(s_0) & \dots & x_{n-1}'(s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s_0) & \dots & x_{n-1}^{(n-1)}(s_0) \end{vmatrix}} \right|.$$

Using the Taylor expansion of $x_i^{(k)}(s+s_0) = x_i^{(k)}(s_0) + x_i^{(k+1)}(s_0)s + \cdots$, $k = 1,\dots,n-1$, $i = 1,\dots,n$, we get that

$$lim_{s\to 0}\left|\frac{\theta}{s}\right| =$$

$$\left| \frac{\begin{vmatrix} x_1'(s_0) & \dots & x_{n-2}'(s_0) & x_n'(s_0) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(n-2)}(s_0) & \dots & x_{n-2}^{(n-2)}(s_0) & x_n^{(n-2)}(s_0) \\ x_1^{(n)}(s_0) & \dots & x_{n-2}^{(n)}(s_0) & x_n^{(n)}(s_0) \end{vmatrix} \begin{vmatrix} x_1'(s_0) & \dots & x_{n-1}'(s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s_0) & \dots & x_{n-1}^{(n-1)}(s_0) \end{vmatrix}}{\begin{vmatrix} x_1'(s_0) & \dots & x_{n-1}'(s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s_0) & \dots & x_{n-1}^{(n-1)}(s_0) \end{vmatrix}^2} \right.$$

$$\left. - \frac{\begin{vmatrix} x_1'(s_0) & \dots & x_{n-2}'(s_0) & x_n'(s_0) \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s_0) & \dots & x_{n-2}^{(n-1)}(s_0) & x_n^{(n-1)}(s_0) \end{vmatrix} \begin{vmatrix} x_1'(s_0) & \dots & x_{n-1}'(s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-2)}(s_0) & \dots & x_{n-1}^{(n-2)}(s_0) \\ x_1^{(n)}(s_0) & \dots & x_{n-1}^{(n)}(s_0) \end{vmatrix}}{\begin{vmatrix} x_1'(s_0) & \dots & x_{n-1}'(s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)}(s_0) & \dots & x_{n-1}^{(n-1)}(s_0) \end{vmatrix}^2} \right|.$$

Some calculation shows that the numerator of this expression is equal to

$$det(\mathbf{x}', \ldots, \mathbf{x}^{(n)}) \begin{vmatrix} x_1'(s_0) & \ldots & x_{n-2}'(s_0) \\ \vdots & \vdots & \vdots \\ x_1^{(n-2)}(s_0) & \ldots & x_{n-1}^{(n-2)}(s_0) \end{vmatrix}$$

which, comparing by (11) for $j = k - 1$, implies the statement of the proposition.

For the curvatures $\kappa_{n-k}, \ldots, \kappa_{n-2}$ we have the following interpretation.

PROPOSITION 5. *Let c be an admissible $C^{(n)}$-curve. Then*

$$|\kappa_{n-k+j}(s_0)| = lim_{s\to 0}|\frac{\omega}{s}|, \quad j = 0, \ldots, k-2$$

where ω denotes the angle between the $(k-j-1)$-isotropic hyperplanes at the points $\mathbf{x}(s_0)$ and $\mathbf{x}(s+s_0)$ spanned by the vectors $\mathbf{t}_1, \ldots, \mathbf{t}_{n-k+j}, \mathbf{b}_{n-k+j+2}, \ldots, \mathbf{b}_n$, s is the parameter of the arc length, and $\mathbf{b}_{n-k+j+2}, \ldots, \mathbf{b}_n$ are the vectors of the orthonormal basis for U_2.

Proof. For the curvatures $\kappa_{n-k+1}, \ldots, \kappa_{n-2}$ the proof is analogues to the proof of the previous proposition, if we consider the projection of the curve c to the $(n-k+j+1)$-dimensional space spanned by the first $(n-k+j+1)$ coordinate vectors.
For the curvature κ_{n-k} we consider $(k-1)$-isotropic hyperplanes spanned by $\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}, \mathbf{b}_{n-k+2}, \ldots, \mathbf{b}_n$ at the points $\mathbf{x}(s_0)$ and $\mathbf{x}(s+s_0)$. First let us notice that for the formally introduced Euclidean normal vector $\mathbf{u} = (u_1, \ldots, u_n) = \mathbf{t}_1 \wedge \ldots \wedge \mathbf{t}_{n-k} \wedge \mathbf{b}_{n-k+2} \ldots \wedge \mathbf{b}_n$ of such a hyperplane we have $\pi_1(\mathbf{u}') = \kappa_{n-k}\pi_1(\mathbf{t}_{n-k})$ and therefore $||\mathbf{u}'|| = |\kappa_{n-k}|$. Now we have

$$lim_{s\to 0}\frac{\omega^2}{s^2} = lim_{s\to 0}\left[\frac{u_1(s+s_0)-u_1(s_0)}{s}\right]^2 + \cdots + \left[\frac{u_{n-k}(s+s_0)-u_{n-k}(s_0)}{s}\right]^2 =$$

$$= ||\mathbf{u}'||^2 = |\kappa_{n-k}|^2$$

which completes the proof.

Furthermore, by using the explicit expressions for the curvatures, we can show that the following propositions hold.

PROPOSITION 6. *The only admissible C^n-curves for which $\kappa_{n-1} \equiv 0$ holds are the non-degenerated C^n-curves in non-isotropic hyperplanes.*

Proof. Let us first remark that $\kappa_{n-1} \equiv 0$ if and only if

$$det(\mathbf{x}', \ldots, \mathbf{x}^{(n)}) = 0, \quad \begin{vmatrix} x_1' & \ldots & x_{n-1}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-1)} & \ldots & x_{n-1}^{(n-1)} \end{vmatrix} \neq 0. \tag{14}$$

Now let c be a curve in a non-isotropic hyperplane. Then by an isotropic motion we obtain that c lies in a hyperplane $x_n = 0$. Therefore c is given by

$$\mathbf{x}(s) = (x_1(s), \ldots, x_{n-1}(s), 0)$$

from which (14) follows.

Conversely, let us show that c lies in its osculating hyperplane at an arbitrary point $\mathbf{x}(s)$ and that that hyperplane is non-isotropic. The equation of the osculating hyperplane at the point $\mathbf{x}(s)$ is given by

$$det(\mathbf{x} - \mathbf{x}(s), \mathbf{t}_1(s), \ldots, \mathbf{t}_{n-1}(s)) = 0.$$

We can formally introduce its Euclidean normal vector by $\mathbf{t}_1(s) \wedge \cdots \wedge \mathbf{t}_{n-1}(s)$ and by using the Frenet's equations and the assumption $\kappa_{n-1} \equiv 0$ we can show that this vector is a constant vector. Indeed, differentiation yields

$$\begin{aligned} (\mathbf{t}_1(s) \wedge \cdots \wedge \mathbf{t}_{n-1}(s))' &= \mathbf{t}_1(s) \wedge \ldots \wedge \mathbf{t}_{n-2}(s) \wedge \kappa_{n-1}(s)\mathbf{t}_n \\ &= 0. \end{aligned}$$

Therefore, all the osculating hyperplanes are parallel. Let us show now that they are all equal. It is enough to show that

$$det(\mathbf{x}(s), \mathbf{t}_1(s), \ldots, \mathbf{t}_{n-1}(s))$$

is constant. This follows also by differentiating the previous determinant. So, c lies in its osculating hyperplane. From the condition (14) follows that this hyperplane is non-isotropic.

Analogously, the following geometrical interpretations for the curvatures $\kappa_{n-k}, \ldots, \kappa_{n-2}$ hold.

PROPOSITION 7. *Let c be a simple $C^{(n-k+j+1)}$-curve. Then $\kappa_{n-k+j} \equiv 0$ if and only if c is a curve in an $(k-j-1)$-isotropic hyperplane, $j = 0, \ldots, k-2$.*

Proof. Let us first notice that from (10) and (11) follows that $\kappa_{n-k+j} \equiv 0$ if and only if

$$\begin{vmatrix} x_1' & \cdots & x_{n-k+j+1}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+j+1)} & \cdots & x_{n-k+j+1}^{(n-k+j+1)} \end{vmatrix} = 0, \quad \begin{vmatrix} x_1' & \cdots & x_{n-k+j}' \\ \vdots & \vdots & \vdots \\ x_1^{(n-k+j)} & \cdots & x_{n-k+j}^{(n-k+j)} \end{vmatrix} \neq 0.$$

Then the proof proceeds analogously to the proof of the Proposition 6 if we consider the projection of the curve c onto the $(n-k+j+1)$- dimensional subspace of V spanned by the first $n-k+j+1$ coordinate vectors. We can conclude that this projection lies in a non-isotropic $(n-k+j)$-plane which means that c lies in an $(k-j-1)$-isotropic hyperplane.

Furthermore, we know that $\kappa_m \equiv 0$, $m < n-k$, if and only if the projection $\pi_1(c)$ of c is a curve in a m-plane in the basic subspace U_1. That is exactly the case when c lies in a k-isotropic $(m+k)$-plane in V. By using this fact and the previous propositions we may understand better the nature of a degenerated curve c. This can be described by introducing the supplementary curvatures.

We shall distinguish several cases.

Case 1. If $\kappa_m \equiv 0$, $m < n-k$, then c is a curve in a k-isotropic $(m+k)$-plane spanned by vectors $\mathbf{x}', \ldots, \mathbf{x}^{(m+k)}$. We construct the Frenet $(m+k)$-frame in the same way as we did it for non-degenerated curves. We obtain m Euclidean vectors $\mathbf{t}_1, \ldots, \mathbf{t}_m$ and one 1-isotropic vector $\mathbf{t}_{m+1}, \ldots,$ one k-isotropic vector $\mathbf{t}_{m+k}$. Now, there exist functions $\kappa_1, \ldots, \kappa_{m-1}, \kappa_m^{(1)}, \ldots, \kappa_{m+k-1}^{(1)} : I \to \mathbf{R}$ such that the following Frenet's equations are satisfied

$$\begin{aligned}
\mathbf{t}_1' &= \kappa_1 \mathbf{t}_2, \\
\mathbf{t}_i' &= -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, && i = 2, \ldots, m-1, \\
\mathbf{t}_m' &= -\kappa_{m-1}\mathbf{t}_{m-1} + \kappa_m^{(1)} \mathbf{t}_{m+1}, \\
\mathbf{t}_{m+j}' &= \kappa_{m+j}^{(1)} \mathbf{t}_{m+j+1}, && j = 1, \ldots, k-1, \\
\mathbf{t}_{m+k}' &= 0.
\end{aligned}$$

For the supplementary curvatures $\kappa_m^{(1)}, \ldots, \kappa_{m+k-1}^{(1)}$ we can obtain explicit expressions in the same way as we did it for non-degenerated curves. For the higher curvatures $\kappa_{m+1}^{(1)}, \ldots, \kappa_{m+k-1}^{(1)}$ we get

$$\kappa_{m+i+1}^{(1)} = \left(\left(\left((x_{n-k+i+2}^{(m+1)} : x_{n-k+1}^{(m+1)})' : \kappa_{m+1}^{(1)} \right)' : \kappa_{m+2}^{(1)} \right)' : \ldots : \kappa_{m+i}^{(1)} \right)',$$

$$i = 0, \ldots, k-2,$$

or (by supposing that V is unitarian)

$$\left(\kappa_{m+i+1}^{(1)}\right)^2 = \frac{\Gamma_{n-k+i+1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+i+2)})\Gamma_{n-k+i-1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+i)})}{\Gamma_{n-k+i}^2(\mathbf{x}', \ldots, \mathbf{x}^{(m+i+1)})}, \tag{15}$$

$$i = 0, \ldots, k-2.$$

For the next curvature $\kappa_m^{(1)}$ we get

$$\left(\kappa_m^{(1)}\right)^2 = \frac{\Gamma_{n-k+1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+1)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(m-1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(m)})}.$$

Using Propositions 6, 7 we can conclude as follows.

PROPOSITION 8. *Let c be a simple $C^{(m+k)}$-curve such that $\kappa_m \equiv 0$, $m < n-k$. Then $\kappa_{m+i}^{(1)} \equiv 0$ if and only if c is a curve in a $(k-i-1)$-isotropic $(m+k-1)$-plane, $i = 0, \ldots, k-1$.*

Now we can proceed by supposing $\kappa_m = \kappa_m^{(1)} \equiv 0$. Then c lies in a $(k-1)$-isotropic $(m+k-1)$-plane spanned by m Euclidean vectors $\mathbf{t}_1, \ldots, \mathbf{t}_m$, one 2-isotropic vector $\mathbf{t}_{m+1}, \ldots$, one k-isotropic vector $\mathbf{t}_{m+k-1}$. We introduce supplementary curvatures $\kappa_m^{(2)}, \ldots, \kappa_{m+k-2}^{(2)} : I \to \mathbf{R}$ such that the following Frenet's equations hold

$$\begin{aligned} \mathbf{t}_1{}' &= \kappa_1 \mathbf{t}_2, & \\ \mathbf{t}_i{}' &= -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, & i = 2, \ldots, m-1, \\ \mathbf{t}_m{}' &= -\kappa_{m-1}\mathbf{t}_{m-1} + \kappa_m^{(2)}\mathbf{t}_{m+1}, & \\ \mathbf{t}_{m+j}{}' &= \kappa_{m+j}^{(2)}\mathbf{t}_{m+j+1}, & j = 1, \ldots, k-2, \\ \mathbf{t}_{m+k-1}{}' &= 0. & \end{aligned}$$

By proceeding inductively under the assumptions $\kappa_m \equiv \kappa_m^{(1)} \equiv \ldots \equiv \kappa_m^{(l-1)} \equiv 0$ we obtain supplementary curvatures $\kappa_{m+i}^{(l)}$, $l = 1, \ldots, k, i = 0, \ldots, k-l$, for which we obtain the following explicit expressions. For the higher curvatures $\kappa_{m+1}^{(l)}, \ldots, \kappa_{m+k-l}^{(l)}$ we get

$$\kappa_{m+i+1}^{(l)} = \left(\left(\left((x_{n-k+i+l+1}^{(m+1)} : x_{n-k+l}^{(m+1)})' : \kappa_{m+1}^{(l)} \right)' : \kappa_{m+2}^{(l)} \right)' : \ldots : \kappa_{m+i}^{(l)} \right)',$$

$$i = 0, \ldots, k-l,$$

or (by supposing that V is unitarian)

$$\left(\kappa_{m+i+1}^{(l)}\right)^2 = \frac{\Gamma_{n-k+i+l+1,1,\ldots,l-1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+i++2)})\Gamma_{n-k+i+l-1,1,\ldots,l-1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+i)})}{\Gamma^2_{n-k+i+l,1,\ldots,l-1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+i+1)})},$$

$$i = 0, \ldots, k-l.$$

and for the next curvature $\kappa_m^{(l)}$ we obtain

$$\left(\kappa_m^{(l)}\right)^2 = \frac{\Gamma_{n-k+l,1,\ldots,l-1}(\mathbf{x}', \ldots, \mathbf{x}^{(m+1)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(m-1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(m)})},$$

where $\Gamma_{n-k+i,1,\ldots l}(y_1, \ldots, y_m)$, $i = 1, \ldots, k$, $l = 1, \ldots, k-1$, denotes the Gram's determinant of the projections of the given vectors onto the $(n-k+i-l)$-dimensional subspace of V spanned by the first $n-k+i$ coordinate vectors except the first isotropic, $\ldots$, l-th isotropic direction.

Furthermore, the following theorem holds.

THEOREM 4. *Let c be a simple $C^{(m+k)}$-curve such that $\kappa_m \equiv \kappa_m^{(1)} \equiv \ldots \equiv \kappa_m^{(l-1)} \equiv 0$, $m < n-k$, $l = 1, \ldots, k-1$. Then $\kappa_{m+i}^{(l)} \equiv 0$ if and only if c is a curve in a $(k-l-i)$-isotropic $(m+k-l)$-plane.*

COROLLARY 3. *Let c be a simple $C^{(m+k)}$-curve, $m < n-k$. Then c is a curve in a non-isotropic m-plane if and only if $\kappa_m \equiv \kappa_m^{(1)} \equiv \ldots \equiv \kappa_m^{(k)} \equiv 0$.*

Case 2. Let us now consider the case $\kappa_{n-k} \equiv 0$. By Proposition 7 it means that c lies in a $(k-1)$-isotropic hyperplane spanned by vectors $\mathbf{x}', \ldots, \mathbf{x}^{(n-1)}$. Constructing the Frenet $(n-1)$-frame in the same way as we did it for non-degenerated curves, we obtain $n-k$ Euclidean vectors $\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}$, one 2-isotropic vector $\mathbf{t}_{n-k+1}, \ldots$, one k-isotropic vector $\mathbf{t}_{n-1}$. We introduce supplementary curvatures $\kappa^{(1)}_{n-k}, \ldots, \kappa^{(1)}_{n-2}$: $I \to \mathbf{R}$ such that the following Frenet's equations are true

$$\begin{array}{rcll} \mathbf{t}_1{}' & = & \kappa_1 \mathbf{t}_2, & \\ \mathbf{t}_i{}' & = & -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, & i = 2, \ldots, n-k-1, \\ \mathbf{t}_{n-k}{}' & = & -\kappa_{n-k-1}\mathbf{t}_{n-k-1} + \kappa^{(1)}_{n-k}\mathbf{t}_{n-k+1}, & \\ \mathbf{t}_{n-k+j}{}' & = & \kappa^{(1)}_{n-k+j}\mathbf{t}_{n-k+j+1}, & j = 1, \ldots, k-2, \\ \mathbf{t}_{n-1}{}' & = & 0. & \end{array}$$

We can obtain the explicit expressions for the supplementary curvatures. For the higher curvatures $\kappa^{(1)}_{n-k+1}, \ldots, \kappa^{(1)}_{n-2}$ we have

$$\kappa^{(1)}_{n-k+i} = \left(\left(\left((x^{(n-k+1)}_{n-k+i+2} : x^{(n-k+1)}_{n-k+2})' : \kappa^{(1)}_{n-k+1}\right)' : \kappa^{(1)}_{n-k+2}\right)' : \ldots : \kappa^{(1)}_{n-k+i-1}\right)',$$

$$i = 1, \ldots, k-2,$$

or

$$\left(\kappa^{(1)}_{n-k+i}\right)^2 =$$

$$\frac{\Gamma_{n-k+i+2,1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i+1)})\Gamma_{n-k+i,1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i-1)})}{\Gamma^2_{n-k+i+1,1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i)})},$$

$$i = 1, \ldots, k-2,$$

and for the next curvature $\kappa^{(1)}_{n-k}$ we get

$$\left(\kappa^{(1)}_{n-k}\right)^2 = \frac{\Gamma_{n-k+2,1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+1)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k-1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k)})}.$$

Furthermore, the following proposition holds.

PROPOSITION 9. *Let c be a simple $C^{(n-1)}$-curve such that $\kappa_{n-k} \equiv 0$. Then $\kappa^{(1)}_{n-k+i} \equiv 0$ if and only if c lies in a $(k-i-2)$-isotropic $(n-2)$-plane, $i = 0, \ldots, k-2$.*

Let us suppose now that $\kappa_{n-k} \equiv \kappa^{(1)}_{n-k} \equiv 0$. Then c lies in a $(k-2)$-isotropic $(n-2)$-plane spanned by $n-k$ Euclidean vectors $\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}$, one 3-isotropic vector $\mathbf{t}_{n-k+1}, \ldots$, one k-isotropic vector $\mathbf{t}_{n-2}$. We introduce supplementary curvatures

$\kappa^{(2)}_{n-k}, \ldots, \kappa^{(2)}_{n-3} : I \to \mathbf{R}$ such that the following Frenet's equations hold

$$\begin{aligned} \mathbf{t}_1{}' &= \kappa_1 \mathbf{t}_2, & \\ \mathbf{t}_i{}' &= -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, & i = 2, \ldots, n-k-1, \\ \mathbf{t}_{n-k}{}' &= -\kappa_{n-k-1}\mathbf{t}_{n-k-1} + \kappa^{(2)}_{n-k}\mathbf{t}_{n-k+1}, & \\ \mathbf{t}_{n-k+j}{}' &= \kappa^{(2)}_{n-k+j}\mathbf{t}_{n-k+j+1}, & j = 1, \ldots, k-2, \\ \mathbf{t}_{n-2}{}' &= 0. & \end{aligned}$$

By proceeding inductively under the assumptions $\kappa_{n-k} \equiv \kappa^{(1)}_{n-k} \equiv \ldots \equiv \kappa^{(l-1)}_{n-k} \equiv 0$ we obtain supplementary curvatures $\kappa^{(l)}_{n-k+i}$, $l = 1, \ldots, k-1$, $i = 0, \ldots, k-l-1$, for which the following explicit expressions hold. For the higher curvatures $\kappa^{(1)}_{n-k+1}, \ldots, \kappa^{(1)}_{n-2}$ we have

$$\kappa^{(l)}_{n-k+i} = \left(\left(\left((x^{(n-k+1)}_{n-k+i+l+1} : x^{(n-k+1)}_{n-k+l+1})' : \kappa^{(l)}_{n-k+1}\right)' : \kappa^{(l)}_{n-k+2}\right)' : \ldots : \kappa^{(l)}_{n-k+i-1}\right)',$$

$$i = 1, \ldots, k-l-1$$

or

$$\left(\kappa^{(l)}_{n-k+i}\right)^2 = \frac{\Gamma_{n-k+i+l+1,1,\ldots,l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i+1)})\Gamma_{n-k+i+l-1,1,\ldots,l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i-1)})}{\Gamma^2_{n-k+i+l,1,\ldots,l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i)})},$$

$$i = 1, \ldots, k-2.$$

For the next curvature $\kappa^{(l)}_{n-k}$ we get

$$\left(\kappa^{(l)}_{n-k}\right)^2 = \frac{\Gamma_{n-k+l+1,1,\ldots,l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+1)})\Gamma(\mathbf{x}', \ldots, \mathbf{x}^{(n-k-1)})}{\Gamma^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k)})}.$$

Now the following statements hold.

THEOREM 5. *Let c be a simple $C^{(n-1)}$-curve such that $\kappa_{n-k} \equiv \kappa^{(1)}_{n-k} \equiv \ldots \equiv \kappa^{(l-1)}_{n-k} \equiv 0$, $l = 1, \ldots, k-1$. Then $\kappa^{(l)}_{n-k+i} \equiv 0$ if and only if c is a curve in a $(k-l-i-1)$-isotropic $(n-l-1)$-plane, $i = 1, \ldots, k-l-1$.*

COROLLARY 4. *Let c be a simple $C^{(n-1)}$-curve. Then c is a curve in a non-isotropic $(n-k)$-plane if and only if $\kappa_{n-k} \equiv \kappa^{(1)}_{n-k} \equiv \ldots \equiv \kappa^{(k-1)}_{n-k} \equiv 0$.*

Case 3. Finally, let us consider the case when $\kappa_{n-k+j} \equiv 0, j = 1, \ldots, k-2$, holds. By Proposition 7 it follows that c lies in a $(k-j-1)$-isotropic hyperplane spanned by vectors $\mathbf{x}', \ldots, \mathbf{x}^{(n-1)}$. By constructing the Frenet's $(n-1)$-frame we get $n-k$ Euclidean vectors $\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}$, one 1-isotropic vector $\mathbf{t}_{n-k+1}$, ..., one j-isotropic vector $\mathbf{t}_{n-k+j}$, one $(j+2)$-isotropic vector $\mathbf{t}_{n-k+j+1}$, ..., one k-isotropic vector $\mathbf{t}_{n-1}$. Since the geometry of the $(k-j-1)$-isotropic hyperplane,

$j = 1, \ldots, k-2$ coincides with the geometry of the space I_{n-1}^{k-1}, we introduce supplementary curvatures $\kappa_{n-k+j}^{(1)}, \ldots, \kappa_{n-2}^{(1)} : I \to \mathbf{R}$ such that the following Frenet's equations hold

$$\begin{array}{rcll} \mathbf{t}_1{}' & = & \kappa_1 \mathbf{t}_2, & \\ \mathbf{t}_i{}' & = & -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, & i = 2, \ldots, n-k, \\ \mathbf{t}_{n-k+i}{}' & = & \kappa_{n-k+i}\mathbf{t}_{n-k+i+1}, & i = 1, \ldots, j-1, \\ \mathbf{t}_{n-k+l}{}' & = & \kappa_{n-k+l}^{(1)}\mathbf{t}_{n-k+l+1}, & l = j, \ldots, n-2, \\ \mathbf{t}_{n-1}{}' & = & 0. & \end{array}$$

In the same way as before we obtain the explicit expressions for the supplementary curvatures. We get

$$\kappa_{n-k+j+i}^{(1)} =$$

$$\left(\left(\left(\left((x_{n-k+j+i+2}^{(n-k+1)} : x_{n-k+1}^{(n-k+1)})' : \kappa_{n-k+1}\right)' : \ldots : \kappa_{n-k+j-1}\right)' : \kappa_{n-k+j}^{(1)}\right)' :$$

$$\ldots : \kappa_{n-k+j+i-1}^{(1)}\Big)',$$

$$i = 0, \ldots, k-j-2,$$

or

$$\left(\kappa_{n-k+j}^{(1)}\right)^2 =$$

$$\frac{\Gamma_{n-k+j+2,j+1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+1)})\Gamma_{n-k+j}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j-1)})}{\Gamma_{n-k+j}^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j)})},$$

$$\left(\kappa_{n-k+j+1}^{(1)}\right)^2 =$$

$$\frac{\Gamma_{n-k+j+3,j+1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+2)})\Gamma_{n-k+j}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j)})}{\Gamma_{n-k+j+2,j+1}^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+1)})},$$

$$\left(\kappa_{n-k+j+i}^{(1)}\right)^2 =$$

$$\frac{\Gamma_{n-k+j+i+2,j+1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+i+1)})\Gamma_{n-k+j+i,j+1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+i-1)})}{\Gamma_{n-k+j+i+1,j+1}^2(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+i)})},$$

$$i = 2, \ldots, k-j-2.$$

Furthermore, the following proposition is true.

PROPOSITION 10. *Let c be a simple $C^{(n-1)}$-curve such that $\kappa_{n-k+j} \equiv 0$, $j = 1, \ldots, k-2$. Then $\kappa^{(1)}_{n-k+j+i} \equiv 0$ if and only if c lies in a $(k-j-i-2)$-isotropic $(n-2)$-plane, $i = 0, \ldots, k-j-2$.*

Let us now suppose that $\kappa_{n-k+j} \equiv \kappa^{(1)}_{n-k+j} \equiv 0$. Then c lies in a $(k-j-2)$-isotropic $(n-2)$-plane spanned by $n-k$ Euclidean vectors $\mathbf{t}_1, \ldots, \mathbf{t}_{n-k}$, one 1-isotropic vector $\mathbf{t}_{n-k+1}$, ..., one j-isotropic vector $\mathbf{t}_{n-k+j}$, one $(j+3)$-isotropic vector $\mathbf{t}_{n-k+j+1}, \ldots$, one k-isotropic vector $\mathbf{t}_{n-2}$. Again we introduce supplementary curvatures $\kappa^{(2)}_{n-k}, \ldots, \kappa^{(2)}_{n-3} : I \to \mathbf{R}$ such that the following Frenet's equations hold

$$\begin{array}{rcll} \mathbf{t}_1' &=& \kappa_1 \mathbf{t}_2, & \\ \mathbf{t}_i' &=& -\kappa_{i-1}\mathbf{t}_{i-1} + \kappa_i \mathbf{t}_{i+1}, & i = 2, \ldots, n-k, \\ \mathbf{t}_{n-k+i}' &=& \kappa_{n-k+i}\mathbf{t}_{n-k+i+1}, & i = 1, \ldots, j-1, \\ \mathbf{t}_{n-k+l}' &=& \kappa^{(2)}_{n-k+l}\mathbf{t}_{n-k+l+1}, & l = j, \ldots, n-3, \\ \mathbf{t}_{n-2}' &=& 0. & \end{array}$$

By proceeding inductively under the assumptions $\kappa_{n-k+j} \equiv \kappa^{(1)}_{n-k+j} \equiv \ldots \equiv \kappa^{(l-1)}_{n-k+j} \equiv 0$ we obtain supplementary curvatures $\kappa^{(l)}_{n-k+j+i}$, $l = 1, \ldots, k-1, i = 0, \ldots, k-j-l-1$, for which the following explicit expressions hold

$$\kappa^{(l)}_{n-k+j+i} =$$

$$\left(\left(\left(\left(\left(x^{(n-k+1)}_{n-k+j+i+l+1} : x^{(n-k+1)}_{n-k+1}\right)' : \kappa_{n-k+1}\right)' : \ldots : \kappa_{n-k+j-1}\right)' : \kappa^{(l)}_{n-k+j}\right)' :$$

$$\ldots : \kappa^{(l)}_{n-k+j+i-1}\Big)',$$

$$i = 0, \ldots, k-j-l-1,$$

or

$$\left(\kappa^{(l)}_{n-k+j}\right)^2 =$$

$$\frac{\Gamma_{n-k+j+l+1,j+1,\ldots,j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+1)})\Gamma_{n-k+j+l-1}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j-1)})}{\Gamma^2_{n-k+j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j)})},$$

$$\left(\kappa^{(l)}_{n-k+j+1}\right)^2 =$$

$$\frac{\Gamma_{n-k+j+l+2,j+1,\ldots,j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+2)})\Gamma_{n-k+j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j)})}{\Gamma^2_{n-k+j+l+1,j+1,\ldots,j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+1)})},$$

$$\left(\kappa^{(l)}_{n-k+j+i}\right)^2 =$$

$$\frac{\Gamma_{n-k+j+i+l+1,j+1,\ldots,j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+i+1)})}{\Gamma^2_{n-k+j+i+l,j+1,\ldots,j+l}(\mathbf{x}', \ldots, \mathbf{x}^{(n-k+j+i)})}.$$

$$\cdot\Gamma_{n-k+j+i+l-1,j+1,\dots,j+l}(\mathbf{x}',\dots,\mathbf{x}^{(n-k+j+i-1)})$$

$$i=2,\dots,k-j-l-1.$$

The following statements hold.

THEOREM 6. *Let c be a simple $C^{(n-1)}$-curve such that $\kappa_{n-k+j}\equiv\kappa^{(1)}_{n-k+j}\equiv\dots\equiv\kappa^{(l-1)}_{n-k+j}\equiv 0$, $l=1,\dots,k-j-1$. Then $\kappa^{(l)}_{n-k+j+i}\equiv 0$ if and only if c is a curve in a $(k-j-l-i-1)$-isotropic $(n-l-1)$-plane, $i=0,\dots,k-j-l-1$.*

COROLLARY 5. *Let c be a simple $C^{(n-1)}$-curve. Then c is a curve in a non-isotropic $(n-k+j)$-plane if and only if $\kappa_{n-k+j}\equiv\kappa^{(1)}_{n-k+j}\equiv\dots\equiv\kappa^{(k-j-1)}_{n-k+j}\equiv 0$.*

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