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PROPER n–SHAPE CATEGORIES

Katsuro Sakai, Tsukuba, Japan

Abstract. In this paper, it is shown that the proper n-shape category of Ball-Sher type is isomorphic to a subcategory of the proper n-shape category defined by proper n-shapings. It is known that the latter is isomorphic to the shape category defined by the pair $(\mathscr{H}_p^n, \mathscr{H}_p^n\mathrm{Pol})$, where $\mathscr{H}_p^n$ is the category whose objects are locally compact separable metrizable spaces and whose morphisms are the proper n-homotopy classes of proper maps, and $\mathscr{H}_p^n\mathrm{Pol}$ is the full subcategory of $\mathscr{H}_p^n$ whose objects are spaces having the proper n-homotopy type of polyhedra. In case $n = \infty$, this shows the relation between the original Ball-Sher's category and the proper shape category defined by proper shapings. We also discuss the proper n-shape category of spaces of dimension $\leqslant n+1$.

1. Introduction

In this paper, all spaces are assumed to be separable metrizable. By $[X, Y]_p$, we denote the set of the proper homotopy classes of proper maps from X to Y. The *proper shape category* $\mathscr{S}_p$ is defined as the category whose objects are locally compact spaces and whose morphisms from X to Y are natural transformations from $\pi_Y = [Y, -]_p$ to $\pi_X = [X, -]_p$ (which are called *proper shapings*) [3], where $\pi_X = [X, -]_p$ is the functor from the proper homotopy category of polyhedra[1] to the category of sets. Originally, Ball and Sher defined in [4] proper shape of locally compact spaces by modifying Borsuk's definition of shape of compacta in [5]. Just like shape theory, the proper shape category $\mathscr{S}_p$ has various descriptions. In [3], it was shown that they are equivalent to each other except for Ball-Sher's category $\mathscr{S}_{\mathrm{BS}}$.[2] However, it is not known whether this category $\mathscr{S}_p$ is isomorphic to Ball-Sher's category $\mathscr{S}_{\mathrm{BS}}$. For shape theory, refer to [12].

It is said that proper maps $f, g : X \to Y$ are *properly n-homotopic* to each other if $f \circ h$ and $g \circ h$ are properly homotopic to each other for any proper map $h : Z \to X$ of an arbitrary locally compact space Z with $\dim Z \leqslant n$. In case $n = \infty$, proper ∞-homotopy is just proper homotopy. The proper n-homotopy class of f is denoted by $[f]_p^n$. Let $[X, Y]_p^n$ denote the set of all proper n-homotopy classes of proper maps

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[1] A polyhedron is the underlying space of a locally finite simplicial complex.

[2] In [3], the category $\mathscr{S}_p$ is denoted by $\mathscr{S}_p^1$ (but the separability of $X \in \mathrm{Ob}\mathscr{S}_p^1$ is not assumed) and the category $\mathscr{S}_{\mathrm{BS}}$ is denoted by $\mathscr{S}_p^0$.

from X to Y. By replacing the functor $\pi_X = [X, -]_p$ by $\pi_X^n = [X, -]_p^n$, we can define a proper n-shaping and obtain the proper n-shape category $\mathcal{S}_p^n$. On the other hand, by using proper n-homotopy instead of proper homotopy in the definition of Ball-Sher's proper shape in [4], the notion of proper n-shape was introduced in [1]. Here we call this category the proper n-shape category of *Ball-Sher type* and denote it by $\mathcal{S}_{\mathrm{BS}}^n$.

In this paper, we show that the category $\mathcal{S}_{\mathrm{BS}}^n$ is isomorphic to a subcategory of $\mathcal{S}_p^n$ by a functor which is the identity on the class of objects. Thus, we can regard $\mathcal{S}_{\mathrm{BS}}^n \subset \mathcal{S}_p^n$. In case $n = \infty$, the category $\mathcal{S}_{\mathrm{BS}}$ is isomorphic to a subcategory of $\mathcal{S}_p$. This result is also obtained in [13] independently. It remains as an open problem whether $\mathcal{S}_p$ and $\mathcal{S}_{\mathrm{BS}}$ are isomorphic to the each other. However, describing $\mathcal{S}_p^n$ by using proper n-approximative maps, we make clear where is the problem.

Each locally compact space of dimension $\leqslant n+1$ can be embedded in $\mu^{n+1} \setminus \{\mathrm{pt}\}$ as a closed set, where μ^{n+1} is the $(n+1)$-dimensional universal Menger compactum. By replacing $Q \setminus \{0\}$ ($Q = [-1, 1]^\omega$ is the Hilbert cube) by $\mu^{n+1} \setminus \{\mathrm{pt}\}$, another proper n-shape category of Ball-Sher type was obtained in [2]. Here this category is denoted by $\overline{\mathcal{S}}_{\mathrm{BS}}^n(n+1)$. Let $\mathcal{S}_{\mathrm{BS}}^n(n+1)$ (resp. $\mathcal{S}_p^n(n+1)$) be the full subcategory of $\mathcal{S}_{\mathrm{BS}}^n$ (resp. $\mathcal{S}_p^n$) whose objects are spaces of dimension $\leqslant n+1$. Akaike [2] showed that the category $\mathcal{S}_{\mathrm{BS}}^n(n+1)$ is isomorphic to a subcategory of $\overline{\mathcal{S}}_{\mathrm{BS}}^n(n+1)$. In this paper, we also show that $\overline{\mathcal{S}}_{\mathrm{BS}}^n(n+1)$ is isomorphic to a subcategory $\mathcal{S}_p^n(n+1)$. Thus, in the case when dim $\leqslant n+1$, we can regard $\mathcal{S}_{\mathrm{BS}}^n(n+1) \subset \overline{\mathcal{S}}_{\mathrm{BS}}^n(n+1) \subset \mathcal{S}_p^n(n+1)$.

2. The proper n-shape theory

Let $\mathcal{H}_p^n$ be the category whose objects are locally compact spaces and whose morphisms are the proper n-homotopy classes of proper maps. By $\mathcal{H}_p^n\mathrm{Pol}$, we denote the full subcategory of $\mathcal{H}_p^n$ whose objects are spaces having the proper n-homotopy classes of polyhedra. The *proper n-shape category* $\mathcal{S}_p^n$ is defined as the category whose objects are locally compact spaces and whose morphisms from X to Y are natural transformations from $\pi_Y^n = [Y, -]_p^n$ to $\pi_X^n = [X, -]_p^n$ called *proper n-shapings* (cf. [3]), where $\pi_X^n = [X, -]_p^n$ is the functor from $\mathcal{H}_p^n\mathrm{Pol}$ to the category of sets.

In the general theory of shape [12, Ch.I, §2], the notion of expansions of spaces is fundamental. Let $\mathbf{p} : X \to \mathbf{X}$ be a morphism in the pro-category pro-$\mathcal{H}_p^n$ from a locally compact space X to an inverse system $\mathbf{X}$ in $\mathcal{H}_p^n\mathrm{Pol}$. We call $\mathbf{p}$ an $\mathcal{H}_p^n\mathrm{Pol}$-*expansion* of X if it satisfies the following:

> for any inverse system $\mathbf{Y}$ in $\mathcal{H}_p^n\mathrm{Pol}$ and any morphism $\mathbf{q} : X \to \mathbf{Y}$ in pro-$\mathcal{H}_p^n$ from X to $\mathbf{Y}$, there exists a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{H}_p^n$ such that $\mathbf{q} = \mathbf{fp}$.

By [12, Ch. §2, Theorem 1], $\mathbf{p} = (p_\lambda)_{\lambda\in\Lambda} : X \to \mathbf{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$, is an $\mathcal{H}_p^n\mathrm{Pol}$-expansion of X if and only if the following conditions are satisfied:

(1) for any polyhedron P and any proper map $f : X \to P$, there exist $\lambda \in \Lambda$ and a proper map $q : X_\lambda \to P$ such that $f \overset{n}{\simeq}_p qp_\lambda$;

(2) for any polyhedron P and any two proper maps $g, h : X_\lambda \to P$ satisfying $gp_\lambda \overset{n}{\simeq}_p hp_\lambda$, there exists $\lambda' \geqslant \lambda$ such that $gp_{\lambda,\lambda'} \overset{n}{\simeq}_p hp_{\lambda,\lambda'}$.

Due to [12, Ch.I, §2], the following theorem guarantees the existence of the shape theory for the pair $(\mathscr{H}_p^n, \mathscr{H}_p^n\mathrm{Pol})$. By [12, Ch.I, §S, Theorem 7], the shape category defined by the pair $(\mathscr{H}_p^n, \mathscr{H}_p^n\mathrm{Pol})$ is isomorphic to the proper n-shape category $\mathscr{S}_p^n$ defined as above.

THEOREM 2.1. *Every locally compact space X admits an $\mathscr{H}_p^n\mathrm{Pol}$-expansion, namely the category $\mathscr{H}_p^n\mathrm{Pol}$ is dense in $\mathscr{H}_p^n$.*

Proof. Since X can be embedded in $Q \setminus \{0\}$ as a closed set, we can regard X as a closed set in $Q \setminus \{0\}$. Each neighborhood of X in $Q \setminus \{0\}$ contains some closed neighborhood of X which is an ANR ([3]). By Nbd(X), we denote the collection of all closed ANR neighborhoods of X in $Q \setminus \{0\}$, which is directed by the order $[U \leqslant V] \equiv [V \subset U]$. Then we have the inverse system $\mathbf{U} = (U, [i_{U,V}]_p^n, \mathrm{Nbd}(X))$ in $\mathscr{H}_p^n$ and the morphism $\mathbf{i}_X = ([i_U]_p^n)_{U \in \mathrm{Nbd}(X)} : X \to \mathbf{U}$ in pro-$\mathscr{H}_p^n$, where $i_{U,V} : V \to U$ and $i_U : X \to U$ are inclusions. We show that $\mathbf{i}_X : X \to \mathbf{U}$ is an $\mathscr{H}_p^n\mathrm{Pol}$-expansion of X.

To see (1), let $f : X \to P$ be a proper map from X to a polyhedron P. By [4, Lemma 3.2], f extends to a proper map $\tilde{f} : U \to P$ of some $U \in \mathrm{Nbd}(X)$. Then $f \overset{n}{\simeq}_p \tilde{f} i_U$ since $f = \tilde{f} i_U$. The condition (2) is a direct consequence of the following lemma. □

LEMMA 2.2. *Let X be a locally compact space, A a closed set in X, Y a locally compact ANR and let $f, g : X \to Y$ be proper maps. If $f|A \overset{n}{\simeq}_p g|A$ then $f|U \overset{n}{\simeq}_p g|U$ for some neighborhood U of A in X.*

Proof. For every locally compact space X, there exists an n-invertible[3] proper map $\alpha : Z \to X$ of an n-dimensional locally compact space Z [10]. Then the lemma can be proved similarly to [8, Proposition 1.7]. □

3. Proper n-fundamental nets and proper n-approximative maps

For simplicity, we restrict objects of the category $\mathscr{S}_{\mathrm{BS}}^n$ to closed sets X in $Q \setminus \{0\}$. As in the proof of Theorem 2.1, let Nbd(X) denote the directed set of all closed ANR neighborhoods of X in $Q \setminus \{0\}$.[4]

Let X and Y be closed sets in $Q \setminus \{0\}$. A *proper n-fundamental net* from X to Y is a net of maps $f_\lambda : Q \setminus \{0\} \to Q \setminus \{0\}$ indexed by a directed set $\Lambda = (\Lambda, \leqslant)$ satisfying the condition:

[3]A proper map $\alpha : Z \to X$ is *n-invertible* if any proper map $h : W \to X$ of an n-dimensional locally compact space W lifts to a proper map $\tilde{h} : W \to Z$, i.e., $h = \alpha\tilde{h}$.

[4]In this section, each $N \in \mathrm{Nbd}(X)$ need not be an ANR since we can use the fact that int$N$ is an ANR if necessary. However, this restriction is convenient and will be necessary in the next section.

for each $N \in \mathrm{Nbd}(Y)$, there is some $M \in \mathrm{Nbd}(X)$ and $\lambda_0 \in \Lambda$ such that $f_\lambda|M \overset{n}{\simeq}_p f_{\lambda_0}|M$ in N for all $\lambda \geqslant \lambda_0$.[5]

Two proper n-fundamental nets $(f_\lambda)_{\lambda\in\Lambda}$ and $(g_\delta)_{\delta\in\Delta}$ from X to Y are *properly n-homotopic* to each other (denoted by $(f_\lambda) \overset{n}{\simeq}_p (g_\delta)$) provided

for each $N \in \mathrm{Nbd}(Y)$, there exist $M \in \mathrm{Nbd}(X)$, $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that $f_\lambda|M \overset{n}{\simeq}_p g_\delta|M$ in N for all $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$.

The relation $\overset{n}{\simeq}_p$ is an equivalence relation among proper n-fundamental nets. The composition of two proper n-fundamental nets $(f_\lambda)_{\lambda\in\Lambda}$ from X to Y and $(g_\delta)_{\delta\in\Delta}$ from Y to Z is defined as follows:

$$(g_\delta)_{\delta\in\Delta}(f_\lambda)_{\lambda\in\Lambda} = (g_\delta f_\lambda)_{(\lambda,\delta)\in\Lambda\times\Delta},$$

where $\Lambda \times \Delta$ has the order $[(\lambda,\delta) \leqslant (\lambda',\delta')] \equiv [\lambda \leqslant \lambda',\ \delta \leqslant \delta']$. Then we have the category of closed sets in $Q\setminus\{0\}$ and proper n-fundamental nets, which is denoted by $\mathscr{F}^n_p$. The *proper n-shape category of Ball-Sher type* $\mathscr{S}^n_{\mathrm{BS}}$ is defined as the proper n-homotopy category of $\mathscr{F}^n_p$, that is, $\mathrm{Ob}\mathscr{S}^n_{\mathrm{BS}} = \mathrm{Ob}\mathscr{F}^n_p$ and $\mathrm{Mor}\mathscr{S}^n_{\mathrm{BS}} = \mathrm{Mor}\mathscr{F}^n_p / \overset{n}{\simeq}_p$.

A *proper n-approximative map* of X towards Y is a net of proper maps $f_\lambda : X \to Q\setminus\{0\}$ indexed by a directed set $\Lambda = (\Lambda, \leqslant)$ satisfying the condition:

for each $N \in \mathrm{Nbd}(Y)$, there is some $\lambda_0 \in \Lambda$ such that $f_\lambda \overset{n}{\simeq}_p f_{\lambda_0}$ in N for all $\lambda \geqslant \lambda_0$.

A net of proper maps $f_N : X \to Q\setminus\{0\}$ indexed by the directed set $\mathrm{Nbd}(Y)$ is a proper n-approximative map of X towards Y if $f_{N'} \overset{n}{\simeq}_p f_N$ in N for all $N' \subset N \in \mathrm{Nbd}(Y)$. Such a net is called a *mutational proper n-approximative map.*

Two proper n-approximative maps $(f_\lambda)_{\lambda\in\Lambda}$ and $(g_\delta)_{\delta\in\Delta}$ of X towards Y are *properly n-homotopic* to each others (denoted by $(f_\lambda) \overset{n}{\simeq}_p (g_\delta)$) provided

for each $N \in \mathrm{Nbd}(Y)$, there exist $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that $f_\lambda \overset{n}{\simeq}_p g_\delta$ in N for all $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$.

The relation $\overset{n}{\simeq}_p$ is an equivalence relation among proper n-approximative maps. The equivalence class of (f_λ) is called the *proper n-homotopy class* of (f_λ) and is denoted by $[(f_\lambda)]^n_p$. For two mutational proper n-approximative maps (f_N) and (g_N) of X towards Y,

$(f_N) \overset{n}{\simeq}_p (g_N)$ if and only if $f_N \overset{n}{\simeq}_p g_N$ in N for every $N \in \mathrm{Nbd}(Y)$.

LEMMA 3.1. *Every proper n-approximative map $(f_\lambda)_{\lambda\in\Lambda}$ of X towards Y is properly n-homotopic to a mutational proper n-approximative map.*

Proof. For each $N \in \mathrm{Nbd}(Y)$, choose $\lambda_N \in \Lambda$ so that $f_\lambda \overset{n}{\simeq}_p f_{\lambda_N}$ in N for all $\lambda \geqslant \lambda_N$, and let $f_N = f_{\lambda_N}$. For each $N' \subset N \in \mathrm{Nbd}(Y)$, by choosing $\lambda_0 \geqslant \lambda_N, \lambda_{N'}$, we have $f_{\lambda_{N'}} \overset{n}{\simeq}_p f_{\lambda_0} \overset{n}{\simeq}_p f_{\lambda_N}$ in N. Thus, we have the result. □

[5]One should note that each f_λ need not be proper but the restrictions of almost all f_λ on some neighborhoods of X are proper. Since each f_λ is not required to be proper, we can replace $Q\setminus\{0\}$ by any locally compact AR's containing X and Y as closed sets to define proper shape.

For a proper n-fundamental net (f_λ) from X to Y, $(f_\lambda|X)$ is clearly a proper n-approximative map of X towards Y.

LEMMA 3.2. *For two proper n-fundamental nets (f_λ) and (g_δ) from X to Y, the following statements are equivalent*

- $(f_\lambda) \overset{n}{\simeq}_p (g_\delta)$ *(as proper n-fundamental nets);*
- $(f_\lambda|X) \overset{n}{\simeq}_p (g_\delta|X)$ *(as proper n-approximative maps).*

Proof. The implication $(f_\lambda) \overset{n}{\simeq}_p (g_\delta) \Rightarrow (f_\lambda|X) \overset{n}{\simeq}_p (g_\delta|X)$ is clear. Assume that $(f_\lambda|X) \overset{n}{\simeq}_p (g_\delta|X)$. Then, for each $N \in \mathrm{Nbd}(Y)$, we have $\lambda_1 \in \Lambda$ and $\delta_1 \in \Delta$ such that $f_\lambda|X \overset{n}{\simeq}_p g_\delta|X$ in N for all $\lambda \geqslant \lambda_1$ and $\delta \geqslant \delta_1$. On the other hand, we have $M, M' \in \mathrm{Nbd}(X)$, $\lambda_2 \in \Lambda$ and $\delta_2 \in \Delta$ so that $f_\lambda|M \overset{n}{\simeq}_p f_{\lambda_2}|M$ in N for all $\lambda \geqslant \lambda_2$ and $g_\delta|M' \overset{n}{\simeq}_p g_{\delta_2}|M'$ in N for all $\delta \geqslant \delta_2$. Choose $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ so that $\lambda_0 \geqslant \lambda_1, \lambda_2$ and $\delta_0 \geqslant \delta_1, \delta_2$. By Lemma 2.2, we have $M_0 \in \mathrm{Nbd}(X)$ such that $M_0 \subset M \cap M'$ and $f_{\lambda_0}|M_0 \overset{n}{\simeq}_p g_{\delta_0}|M_0$ in N. Then,

$$f_\lambda|M_0 \overset{n}{\simeq}_p f_{\lambda_2}|M_0 \overset{n}{\simeq}_p f_{\lambda_0}|M_0 \overset{n}{\simeq}_p g_{\delta_0}|M_0 \overset{n}{\simeq}_p g_{\delta_2}|M_0 \overset{n}{\simeq}_p g_\delta|M_0 \quad \text{in } N$$

for all $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$. □

Composition of proper n-approximative maps cannot be defined, but composition of proper n-homotopy classes of proper n-approximative maps can be. To define composition, we show the following:

LEMMA 3.3. (1) *Let (f_N) be a mutational proper n-approximative map of X towards Y and let (g_M) be one of Y towards Z. For each $M \in \mathrm{Nbd}(Z)$, choose $N_M \in \mathrm{Nbd}(Y)$ so that g_M extends to a proper map $\tilde{g}_M : N_M \to M$. Then $(\tilde{g}_M f_{N_M})$ is a mutational proper n-approximative map of X towards Z.*

(2) *Let (f'_N) be a mutational proper n-approximative map of X towards Y and let (g'_M) be one of Y towards Z such that $(f_N) \overset{n}{\simeq}_p (f'_N)$ and $(g_M) \overset{n}{\simeq}_p (g'_M)$. For each $M \in \mathrm{Nbd}(Z)$, choose $N'_M \in \mathrm{Nbd}(Y)$ so that g'_M extends to a proper map $\tilde{g}'_M : N'_M \to M$. Then $(\tilde{g}_M f_{N_M}) \overset{n}{\simeq}_p (\tilde{g}'_M f'_{N'_M})$.*

Proof. (1): For $M' \subset M \in \mathrm{Nbd}(Z)$, $\tilde{g}_{M'}|Y = g_{M'} \overset{n}{\simeq}_p g_M = \tilde{g}_M|Y$ in M. Then we can choose $N \in \mathrm{Nbd}(Y)$ so that $N \subset N_M \cap N_{M'}$ and $\tilde{g}_{M'}|N \overset{n}{\simeq}_p \tilde{g}_M|N$ in M (Lemma 2.2). Since $f_{N_{M'}} \overset{n}{\simeq}_p f_N$ in $N_{M'}$ and $f_{N_M} \overset{n}{\simeq}_p f_N$ in N_M, it follows that

$$\tilde{g}_{M'} f_{N_{M'}} \overset{n}{\simeq}_p \tilde{g}_{M'} f_N = (\tilde{g}_{M'}|N) f_N \overset{n}{\simeq}_p (\tilde{g}_M|N) f_N \overset{n}{\simeq}_p \tilde{g}_M f_N \overset{n}{\simeq}_p \tilde{g}_M f_{N_M} \quad \text{in } M.$$

(2): For each $M \in \mathrm{Nbd}(Z)$, $\tilde{g}_M|Y = g_M \overset{n}{\simeq}_p g'_M = \tilde{g}'_M|Y$ in M. Then we can choose $N \in \mathrm{Nbd}(Y)$ so that $N_M \cap N'_M$ and $\tilde{g}_M|N \overset{n}{\simeq}_p \tilde{g}'_M|N$ in M (Lemma 2.2). Since $f_N \overset{n}{\simeq}_p f'_N$ in N, $f_N \overset{n}{\simeq}_p f_{N_M}$ in N_M and $f'_N \overset{n}{\simeq}_p f_{N'_M}$ in N'_M, we have

$$\tilde{g}_M f_{N_M} \overset{n}{\simeq}_p \tilde{g}_M f_N \overset{n}{\simeq}_p \tilde{g}_M f'_N \overset{n}{\simeq}_p \tilde{g}'_M f'_N \overset{n}{\simeq}_p \tilde{g}'_M f'_{N'_M} \quad \text{in } M.$$

Hence, $(\tilde{g}_M f_{N_M}) \overset{n}{\simeq}_p (\tilde{g}'_M f'_{N'_M})$. □

By this lemma, we can define $[(g_M)]^n_p[(f_N)]^n_p = [(\tilde{g}_M f_{N_M})]^n_p$ for two mutational proper n-approximative maps (f_N) of X towards Y and (g_M) of Y towards Z. By Lemma 3.1, the composition of proper n-homotopy classes of proper n-approximative maps can be defined. Thus we obtain the category of closed sets in $Q \setminus \{0\}$ with the proper n-homotopy classes of proper n-approximative maps, which is denoted by $\mathscr{A}^n_p$.

We have a *natural* functor $R : \mathscr{F}^n_p \to \mathscr{A}^n_p$ defined by $R(X) = X$ for all $X \in \mathrm{Ob}\mathscr{F}^n_p$ and $R((f_\lambda)) = [(f_\lambda|X)]^n_p$ for all $(f_\lambda) \in \mathrm{Mor}\mathscr{F}^n_p$. By Lemma 3.2, R induces a categorical embedding $\tilde{R} : \mathscr{S}^n_{\mathrm{BS}} \to \mathscr{A}^n_p$, that is, $\tilde{R}(X) = X$ for all $X \in \mathrm{Ob}\mathscr{S}^n_{\mathrm{BS}}$, and $\tilde{R}([(f_\lambda)]^n_p) = [(f_\lambda|X)]^n_p$ for all $[(f_\lambda)]^n_p \in \mathrm{Mor}\mathscr{S}^n_{\mathrm{BS}}$. Thus we have the following:

THEOREM 3.4. *The category $\mathscr{S}^n_{\mathrm{BS}}$ is isomorphic to a subcategory of $\mathscr{A}^n_p$ by a natural functor.* □

It is unknown whether $\tilde{R} : \mathrm{Mor}\mathscr{S}^n_{\mathrm{BS}} \to \mathrm{Mor}\mathscr{A}^n_p$ is surjective or not. The answer would be positive if every proper approximative map (f_λ) of X towards Y would extend to some proper fundamental net $(\tilde{f}_\lambda)$ from X to Y (i.e., $\tilde{f}_\lambda|X = f_\lambda$). However, the following example shows that this is not true even in the case when X is compact:

EXAMPLE 3.5. *There exists a proper approximative map (f_n) of a compactum X towards Y which cannot be extended to a proper fundamental net (sequence) $(\tilde{f}_n)$ from X to Y.*

Proof. We define

$$X = \{2^{-k} \mid k \in N\} \cup \{0\} \subset \mathbb{R},$$
$$Y = \mathbb{R} \times \{2^{-k} \mid k \in N\} \cup \bigcup_{n\in\mathbb{N}} (2n-1, 2n) \times \{0\} \subset \mathbb{R}^2 \quad \text{and}$$
$$E = \mathbb{R} \times (0, 1] \cup \bigcup_{n\in\mathbb{N}} (2n-1, 2n) \times \{0\} \subset \mathbb{R}^2.$$

Then X and Y are closed sets in locally compact AR's $[0, 1]$ and E, respectively. We may replace $Q \setminus \{0\}$ by $[0, 1]$ and E. For each $n \in \mathbb{N}$, let $f_n : X \to Y$ be the map defined by $f_n(x) = (2n - \frac{1}{2}, x)$. To see that (f_n) is a proper approximative map of X towards Y in E, let V be a neighborhood of Y in E. Choose $k_1 < k_2 < k_3 \cdots \in \mathbb{N}$ so that $\bigcup_{n\in\mathbb{N}}[2n - \frac{2}{3}, 2n - \frac{1}{3}] \times [0, 2^{-k_n}] \subset V$. For each $n \in \mathbb{N}$, we define $g_n : X \to Y$ (resp. $g'_n : X \to Y$) by $g_n(x) = (2n - \frac{1}{2}, x)$ (resp. $g'_n(x) = (1 + \frac{1}{2}, x)$) if $x > 2^{-k_n}$, and $g_n(x) = (2n - \frac{1}{2}, 2^{-k_n})$ (resp. $g'_n(x) = (1 + \frac{1}{2}, 2^{-k_n})$) if $x \leqslant 2^{-k_n}$. As is easily observed, $f_n \simeq_p g_n \simeq_p g'_n \simeq_p f_1$ in V for each $n \in \mathbb{N}$. Hence (f_n) is a proper approximative map of X towards Y in E.

Assume that (f_n) extends to a proper fundamental net (sequence) $(\tilde{f}_n)$ from X to Y in $([0, 1], E)$. Now, choose $k_1 < k_2 < k_3 \cdots \in \mathbb{N}$ so that $\tilde{f}_n([0, 2^{-k_n}]) \subset$

$[2n-\frac{2}{3}, 2n-\frac{1}{3}] \times [0,1]$. Let

$$V_0 = \mathbb{R} \times \bigcup_{k\in\mathbb{N}} [\tfrac{5}{3}2^{-k-1}, \tfrac{4}{3}2^{-k}] \cup \bigcup_{n\in\mathbb{N}} \mathrm{cl}_E(2n-1, 2n) \times [0, \tfrac{4}{3}2^{-k_n-1}].$$

Then V_0 is a closed neighborhood of Y in E. We have a closed neighborhood U_0 of X in $[0,1]$ and $n_0 \in \mathbb{N}$ such that $\tilde{f}_n|U_0 \simeq_p \tilde{f}_{n_0}|U_0$ in V_0 for all $n > n_0$. In particular, $\tilde{f}_n(U_0) \subset V_0$ for all $n > n_0$. Choose $n \in \mathbb{N}$ so that $[0, 2^{-k_n}] \subset U_0$. Then

$$\begin{aligned}\tilde{f}_n([0,2^{-k_n}]) &\subset V_0 \cap [2n-\tfrac{2}{3}, 2n-\tfrac{1}{3}] \times [0,1] \\ &= [2n-\tfrac{2}{3}, 2n-\tfrac{1}{3}] \times \left([0, \tfrac{4}{3}2^{-k_n-1}] \cup \bigcup_{k=1}^{k_n} [\tfrac{5}{3}2^{-k-1}, \tfrac{4}{3}2^{-k}]\right).\end{aligned}$$

On the other hand,

$$\tilde{f}_n(0) \in [2n-\tfrac{2}{3}, 2n-\tfrac{1}{3}] \times [0, \tfrac{4}{3}2^{-k_n-1}] \quad \text{and}$$
$$\tilde{f}_n(2^{-k_n}) \in [2n-\tfrac{2}{3}, 2n-\tfrac{1}{3}] \times [\tfrac{5}{3}2^{-k_n-1}, \tfrac{4}{3}2^{-k_n}].$$

Since $\tilde{f}_n([0,2^{-k_n}])$ is connected, this is a contradiction. □

In the above, one should remark that $(f_n)_{n\in\mathbb{N}} \simeq_p f_1$ (as proper approximative maps). Let $\tilde{f}_1 : [0,1] \to E$ be an extension of f_1. Then $\tilde{f}_1$ is a proper fundamental net from X to Y in $([0,1], E)$ indexed by a singleton.

4. A categorical isomorphism between $\mathscr{S}_p^n$ and $\mathscr{A}_p^n$

In this section, we show that the categories $\mathscr{S}_p^n$ and $\mathscr{A}_p^n$ are isomorphic to each other.

THEOREM 4.1. *There exists a categorical isomorphism $A_p^n : \mathscr{S}_p^n \to \mathscr{A}_p^n$ which is the identity on the class of objects, hence the categories $\mathscr{S}_p^n$ and $\mathscr{A}_p^n$ are isomorphic to each other.*

Proof. Without loss of generality, the category $\mathscr{S}_p^n$ is restricted to closed sets in $Q \setminus \{0\}$. (cf. [12, Appendix 2]).

We will define a functor $A_p^n : \mathscr{S}_p^n \to \mathscr{A}_p^n$ as follows: $A_p^n(X) = X$ for all $X \in \mathrm{Ob}\mathscr{S}_p^n$. Let $F : X \to Y$ be a proper n-shaping (i.e., $F \in \mathrm{Mor}\mathscr{S}_p^n$). For each $N \in \mathrm{Nbd}(Y)$, choose $f_N \in F_N([i_N^Y]_p^n)$, where $i_N^Y : Y \subset N$ is the inclusion. Then (f_N) is a mutational proper n-approximative map of X towards Y. In fact, for $N' \subset N \in \mathrm{Nbd}(Y)$,

$$\begin{aligned}[i_{N,N'}]_p^n[f_{N'}]_p^n &= \pi_X^n([i_{N,N'}]_p^n)(F_{N'}([i_{N'}^Y]_p^n)) \\ &= F_N(\pi_Y^n([i_{N,N'}]_p^n)([i_{N'}^Y]_p^n)) \\ &= F_N([i_{N,N'}]_p^n[i_{N'}^Y]_p^n) = F_N([i_N^Y]_p^n) = [f_N]_p^n,\end{aligned}$$

that is, $f_{N'} \overset{n}{\simeq}_p f_N$ in N. We call (f_N) a proper n-approximative map *associated* with F. If (f'_N) is another proper n-approximative map associated with F, then $(f_N) \overset{n}{\simeq}_p (f'_N)$ because $f_N \overset{n}{\simeq}_p f'_N$ in N for all $N \in \mathrm{Nbd}(Y)$. Then we define $A^n_p(F) = [(f_N)]^n_p$, where (f_N) is a proper n-approximative map associated with F.

Let $G : Y \to Z$ be a proper n-shaping and (g_M) a proper n-approximative map associated with G. The proper n-shaping $GF : X \to Z$ is defined as the functor $F \circ G : \pi^n_Z \to \pi^n_X$. For each $M \in \mathrm{Nbd}(Z)$, $g_M \in G_M([i^Z_M]^n_p)$ extends to a proper map $\tilde{g}_M : N_M \to M$ from some $N_M \in \mathrm{Nbd}(Y)$. Then $[(\tilde{g}_M f_{N_M})]^n_p = [(g_M)]^n_p[(f_N)]^n_p$. On the other hand,

$$\begin{aligned}
[\tilde{g}_M f_{N_M}]^n_p &= [\tilde{g}_M]^n_p[f_{N_M}]^n_p = \pi^n_X([\tilde{g}_M]^n_p)(F_{N_M}([i^Y_{N_M}]^n_p)) \\
&= F_M(\pi^n_Y([\tilde{g}_M]^n_p)([i^Y_{N_M}]^n_p)) = F_M([\tilde{g}_M]^n_p[i^Y_{N_M}]^n_p) \\
&= F_M([\tilde{g}_M i^Y_{N_M}]^n_p) = F_M([g_M]^n_p) \\
&= F_M(G_M([i^Z_M]^n_p)) = (GF)_M([i^Z_M]^n_p),
\end{aligned}$$

that is, $\tilde{g}_M f_{N_M} \in (GF)_M([i^Z_M]^n_p)$. Therefore, $(\tilde{g}_M f_{N_M})$ is a proper n-approximative map associated with the proper n-shaping GF. Thus we have

$$A^n_p(GF) = [(\tilde{g}_M f_{N_M})]^n_p = [(g_M)]^n_p[(f_N)]^n_p = A^n_p(G)A^n_p(F).$$

We show that the function $A^n_p : \mathrm{Mor}\mathscr{S}^n_p \to \mathrm{Mor}\mathscr{A}^n_p$ is bijective. Let (f_N) be a mutational proper n-approximative map of X towards Y. For each polyhedron P, any proper map $h : Y \to P$ extends to a proper map $\tilde{h} : N_h \to P$ for some $N_h \in \mathrm{Nbd}(Y)$. Then we have $\tilde{h} f_{N_h} : X \to P$. Let P' be another polyhedron, let $k : P \to P'$, $h' : Y \to P'$ and $\tilde{h}' : N_{h'} \to P'$ be proper maps such that $N_{h'} \in \mathrm{Nbd}(Y)$, $h' \overset{n}{\simeq}_p kh$ and $\tilde{h}'|Y = h$, and let (g_N) be another mutational proper n-approximative map of X towards Y such that $(f_N) \overset{n}{\simeq}_p (g_N)$, whence $f_N \overset{n}{\simeq}_p g_N$ in N for all $N \in \mathrm{Nbd}(Y)$. Then, by choosing $N \in \mathrm{Nbd}(Y)$ so that $N \subset N_h \cap N_{h'}$ and $\tilde{h}'|N \overset{n}{\simeq}_p k\tilde{h}|N$, we have

$$k\tilde{h} f_{N_h} \overset{n}{\simeq}_p k\tilde{h} f_N = (k\tilde{h}|N) f_N \overset{n}{\simeq}_p (\tilde{h}'|N) f_N = \tilde{h}' f_N \overset{n}{\simeq}_p \tilde{h}' g_N \overset{n}{\simeq}_p \tilde{h}' g_{N_{h'}},$$

that is, $[k]^n_p[\tilde{h} f_{N_h}]^n_p = [\tilde{h}' g_{N_{h'}}]^n_p$. In the case when $P' = P$ and $k = \mathrm{id}_P$, this shows that $[\tilde{h} f_{N_h}]^n_p$ depends only on $[(f_N)]^n_p$ and $[h]^n_p$. By replacing (g_N) by (f_N), the above shows that $\pi_Y([k]^n_p)([h]^n_p) = [kh]^n_p = [h']^n_p$ implies $\pi_X([k]^n_p)([\tilde{h} f_{N_h}]^n_p) = [\tilde{h}' f_{N_{h'}}]^n_p$. Therefore, we can define the natural transformation $S^n_p([(f_N)]^n_p) = F : \pi_Y \to \pi_X$ (i.e., the proper n-shaping $F : X \to Y$) by $F_P([h]^n_p) = [\tilde{h} f_{N_h}]^n_p$ for each $[h]^n_p \in [Y, P]^n_p = \pi^n_Y(P)$, where $\tilde{h} : N_h \to P$ is an extension of h over some $N_h \in \mathrm{Nbd}(Y)$. Thus, we have a function $S^n_p : \mathrm{Mor}\mathscr{A}^n_p \to \mathrm{Mor}\mathscr{S}^n_p$.

In the above, $F_N([i^Y_N]^n_p) = [f_N]^n_p$ for all $N \in \mathrm{Nbd}(Y)$, because id_N is an extension of i^Y_N. This means that (f_N) is associated with the proper n-shaping F, that is, $A^n_p(S^n_p([(f_N)]^n_p)) = A^n_p(F) = [(f_N)]^n_p$. Therefore $A^n_p \circ S^n_p = \mathrm{id}$. To see that $S^n_p \circ A^n_p = \mathrm{id}$, it suffices to show that $F = G$ in the case when (f_N) is associated with a proper n-shaping $G : X \to Y$. In fact, for each polyhedron P and each proper map $h : Y \to P$, since

$$[h]_p^n = [\bar{h}|Y]_p^n = [\bar{h}]_p^n[i_{N_h}^Y]_p^n = \pi_Y^n([\bar{h}]_p^n)([i_{N_h}^Y]_p^n)$$

and $f_{N_h} \in G_{N_h}([i_{N_h}^Y]_p^n)$, we have

$$G_P([h]_p^n) = \pi_X([\bar{h}]_p^n)(G_{N_h}([i_{N_h}^Y]_p^n)) = [\bar{h}]_p^n[f_{N_h}]_p^n = [\bar{h}f_{N_h}]_p^n = F_P([h]_p^n).$$

Hence $G_P = F_P$ for each polyhedron P. This completes the proof. □

Combining Theorems 3.4 and 4.1, we have

COROLLARY 4.2. *The category $\mathcal{S}_{\mathrm{BS}}^n$ is isomorphic to a subcategory of $\mathcal{S}_p^n$ by a functor which is the identity on the class of objects.* □

5. Proper n-shapes of $(n+1)$-dimensional spaces

For a category $\mathcal{C}$ of spaces, let $\mathcal{C}(k)$ denote the subcategory of $\mathcal{C}$ whose objects are spaces of dimension $\leqslant k$. It follows from Corollary 4.2 that the category $\mathcal{S}_{\mathrm{BS}}^n(k)$ is isomorphic to a subcategory of $\mathcal{S}_p^n(k)$.

In the case when $\dim X \leqslant n+1$, any proper map $f : X \to P$ of X to a polyhedron P is properly homotopic to a proper map $f' : X \to P^{(n+1)}$ to the $(n+1)$-skeleton[6] $P^{(n+1)}$ of P, and if two proper maps $f, g : X \to P^{(n+1)}$ are properly n-homotopic in P then they are properly n-homotopic in $P^{(n+1)}$. Then, objects of $\mathcal{H}_p^n\mathrm{Pol}(n+1)$ are spaces having the proper n-homotopy type of polyhedra of dimension $\leqslant n+1$. Each LC^n locally compact space of dimension $\leqslant n+1$ is properly n-homotopic to some polyhedron P with $\dim P \leqslant n+1$ [7, Proposition 1.5] (cf. [9, Proposition 4.1.10]). Each locally compact space of dimension $\leqslant n+1$ can be embedded in $\mu^{n+1} \setminus \{\mathrm{pt}\}$ as a closed set, where μ^{n+1} is the $(n+1)$-dimensional universal Menger compactum. Each neighborhood of a closed set X in $Q \setminus \{0\}$ contains some closed neighborhood of X which is LC^n. Similarly to Theorem 2.1, we have the following

THEOREM 5.1. *The category $\mathcal{H}_p^n\mathrm{Pol}(n+1)$ is dense in $\mathcal{H}_p^n(n+1)$.*

Then the category $\mathcal{S}_p^n(n+1)$ is none other but the shape category defined by the pair $(\mathcal{H}_p^n(n+1), \mathcal{H}_p^n\mathrm{Pol}(n+1))$.

By replacing $Q \setminus \{0\}$ by $\mu^{n+1} \setminus \{\mathrm{pt}\}$ and letting $\mathrm{Nbd}(X)$ be the directed set of all closed LC^n neighborhoods of X in $\mu^{n+1} \setminus \{\mathrm{pt}\}$, we can define the proper n-shape category of Ball-Sher type whose objects are locally compact spaces of dim $\leqslant n+1$. Here this category is denoted by $\overline{\mathcal{S}}^n_{\mathrm{BS}}(n+1)$. The following is due to Akaike [2]:

THEOREM 5.2. *The category $\mathcal{S}_{\mathrm{BS}}^n(n+1)$ is isomorphic to a subcategory of $\overline{\mathcal{S}}^n_{\mathrm{BS}}(n+1)$.*

[6]The k-skeleton of a polyhedron P is the underlying space of the k-skeleton of the simplicial complex triangulating P.

In the above, it is unknown whether $\mathscr{S}^n_{\mathrm{BS}}(n+1)$ is isomorphic to $\overline{\mathscr{S}}^n_{\mathrm{BS}}(n+1)$ itself. Note that every n-dimensional locally compact space can be embedded in an $(n+1)$-dimensional locally compact AR as a closed set ([11]). Then it is easy to prove that $\overline{\mathscr{S}}^n_{\mathrm{BS}}(n)$ is naturally isomorphic to $\mathscr{S}^n_{\mathrm{BS}}(n)$.

Since Lemma 2.2 is valid in the case when $\dim X \leqslant n+1$ and Y is LC^n, we can define the category $\overline{\mathscr{A}}^n_p(n+1)$ as in §2. Similarly to Theorem 3.4, we can show the following

THEOREM 5.3. *The category* $\overline{\mathscr{S}}^n_{\mathrm{BS}}(n+1)$ *is isomorphic to a subcategory of* $\overline{\mathscr{A}}^n_p(n+1)$.

In the above, it is unknown whether $\overline{\mathscr{S}}^n_{\mathrm{BS}}(n+1)$ is isomorphic to $\overline{\mathscr{A}}^n_p(n+1)$ itself. By the above remark on $\mathscr{S}^n_p(n+1)$, the proof of Theorem 4.1 is valid for $\overline{\mathscr{A}}^n_p(n+1)$ and $\mathscr{S}^n_p(n+1)$. Hence we have

THEOREM 5.4. *The category* $\overline{\mathscr{A}}^n_p(n+1)$ *is isomorphic to* $\mathscr{S}^n_p(n+1)$.

Summarizing the above, we have the following relationship:

$$\mathscr{S}^n_{\mathrm{BS}}(n+1) \subset \overline{\mathscr{S}}^n_{\mathrm{BS}}(n+1) \subset \overline{\mathscr{A}}^n_p(n+1) = \mathscr{S}^n_p(n+1).$$

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Institute of Mathematics
University of Tsukuba
Tsukuba 305-8571
Japan
e-mail: sakaiktr@@sakura.cc.tsukuba.ac.jp