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DESCRIBING THE PROPER n–SHAPE CATEGORY BY USING NON–CONTINUOUS FUNCTIONS

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Abstract. In this paper, we describe the proper n-shape category by using non-continuous functions. Moreover, applying non-continuous homotopies, we show that the Čech expansion is a polyhedral expansion in the proper n-homotopy category.

1. Introduction

In homotopy theory, because of pathological situations (e.g., some spaces have only constant maps from spheres), spaces should be restricted to ones having homotopy type of polyhedra (simplicial complexes) or ANR's. But this restriction can be removed by approximating spaces by inverse systems of polyhedra or ANR's (called polyhedral expansions or ANR expansions). Roughly speaking, shape theory is a homotopy theory of such inverse systems approximating spaces, where a morphism between spaces X and Y is represented by a system of maps between spaces (polyhedra or ANR's) consisting of inverse systems approximating X and Y. Thus the shape category is described by using external elements. Refer to [16]. To treat non-compact but locally compact spaces, proper maps and proper homotopies are very useful. Proper shape theory corresponds to proper homotopy theory.

As an intrinsic description without external elements, after Felt's work [13], Sanjurjo [19] described a shape category of compacta by using non-continuous functions between spaces themselves, which is extended to arbitrary spaces in [7]. In [20], he also gave another intrinsic description of shape by using upper semi-continuous (u.s.c.) multi-valued functions, which is also extended to arbitrary spaces in [5] and [8]. By using the method in [20], Čerin [5] gave an intrinsic description of the proper shape, where the proper shape is defined by the proper homotopy category of topological spaces and the proper homotopy category of polyhedra (cf. [16]).

As Menger manifold theory developed, (proper) n-shape theory was introduced by using (proper) n-homotopy instead of (proper) homotopy (cf. [9], [10], [11], [1]). It is said that (proper) maps $f, g : X \to Y$ are *(proper) n-homotopic* if $f \circ h$ and $g \circ h$ are (proper) homotopic for any (proper) map $h : Z \to X$ of a space Z with $\dim Z \leqslant n$. In this paper, we describe (proper) n-shape category

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by using non-continuous functions between spaces themselves as in [13] and [19]. Our approach is also available to obtain the (proper) n-shape version of the other Sanjurjo's description using upper semi-continuous multi-valued functions in [20] and [5]. Since the definition of (proper) n-homotopy itself is external, it cannot be expected purely internal description, but our description is sufficiently internal. We mainly concern ourselves with the description of the proper n-shape category. Removing the words "proper(ly)" and "locally compact", we obtain the description of the n-shape category in the class of separable metrizable spaces or the class of compact Hausdorff spaces, that will be discussed in the last section.

In the general theory of shape [16, Ch.I, §2], the notion of expansions of spaces is fundamental. The Čech expansion for X is a typical polyhedral expansion of X in the homotopy category, which is proved by applying [16, Ch.I, §4, Lemma 1]. However, in the proper n-homotopy category the same approach is difficult. Even in the proper homotopy category, we have troubles because the path space is not locally compact and maps are not proper in the proof of [16, Ch.I, §4, Lemma 1]. In §5, we apply the notion of non-continuous n-homotopy to prove that the Čech expansion of X is a polyhedral expansion of X in the proper n-homotopy category. It should be remarked that, even in the n-homotopy category, the existence of polyhedral expansions of non-metrizable compacta are discussed in [15]. In the last section, we prove that if X is a separable metrizable space or a compact Hausdorff space, then the Čech expansion of X is a polyhedral expansion of X in the n-homotopy category.

As shape theory, proper shape theory can be described by various methods and all descriptions except the Ball-Sher's method [4] are equivalent to each others, that is, they are categorically isomorphic to each others (cf. [3]). In [18], it is shown that the Ball-Sher's category is isomorphic to a subcategory of the proper shape category. In proper n-shape theory, we have the same situation.

2. Preliminaries

Throughout the paper, maps are assumed to be continuous but functions are not. Except the last section, spaces are assumed to be **locally compact separable metrizable.**

For collections $\mathscr{A}$ and $\mathscr{B}$ of subsets of X, $\mathscr{A} \prec \mathscr{B}$ means that $\mathscr{A}$ refines $\mathscr{B}$, that is, each $A \in \mathscr{A}$ is contained in some $B \in \mathscr{B}$. By $\operatorname{cov}(X)$, we denote the set of all *star-finite* covers of a space X consisting of *relatively compact*[1] open sets, where $\operatorname{cov}(X)$ is directed by the refinement, (i.e., $\mathscr{U} \leqslant \mathscr{V}$ means $\mathscr{V} \prec \mathscr{U}$). By $\Omega(X)$, we denote the directed set of finite subcollections of $\operatorname{cov}(X)$ with the order $\subset$. For $\mathscr{U} \in \operatorname{cov}(X)$ and $A \subset X$, let $\operatorname{st}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \mid A \cap U \neq \emptyset\}$. We define $\operatorname{st}\mathscr{U} = \{\operatorname{st}(U, \mathscr{U}) \mid U \in \mathscr{U}\}$.

Let $\mathscr{V} \in \operatorname{cov}(Y)$. Two functions $f, g : X \to Y$ are $\mathscr{V}$*-close* to each other (written

[1] A subset of X is said to be *relatively compact* if it has the compact closure in X.

by $f \overset{\mathcal{V}}{=} g$) if $\{\{f(x), g(x)\} \mid x \in X\} \prec \mathcal{V}$. For proper maps $f, g : X \to Y$, $f \simeq_p g$ means that f is properly homotopic to g. It is said that f is *properly n-homotopic* to g (written by $f \overset{n}{\simeq}_p g$) if $fh \simeq_p gh$ for any proper map $h : Z \to X$ from an arbitrary space Z with $\dim Z \leqslant n$. The proper homotopy class (or the proper n-homotopy class) of a proper map $f : X \to Y$ is denoted by $[f]_p$ (or $[f]_p^n$). By $[X, Y]_p$ (or $[X, Y]_p^n$), we denote the collection of all proper homotopy classes (or the collection of all proper n-homotopy classes) of proper maps from X to Y.

By $K_{\mathcal{U}}$, we denote the nerve of $\mathcal{U} \in \mathrm{cov}(X)$ and its polyhedron. Since $\mathcal{U}$ is star-finite, $K_{\mathcal{U}}$ is a locally finite simplicial complex, hence it is locally compact and metrizable. Each $U \in \mathcal{U}$ is a vertex of $K_{\mathcal{U}}$ and $\mathcal{U}$ is the 0-skeleton $K_{\mathcal{U}}^{(0)}$ of $K_{\mathcal{U}}$. The simplex spanned by vertices $U_0, U_1, \ldots, U_k \in \mathcal{U}$ is denoted by $\langle U_0, U_1, \cdots, U_k\rangle$. Let $U \in \mathcal{U} = K_{\mathcal{U}}^{(0)}$. The star and the link at U in $K_{\mathcal{U}}$ are denoted by $\mathrm{St}(U, K_{\mathcal{U}})$ and $\mathrm{Lk}(U, K_{\mathcal{U}})$, respectively. Let $U^\circ = \mathrm{St}(U, K_{\mathcal{U}}) \setminus \mathrm{Lk}(U, K_{\mathcal{U}})$, which is called the open star at U in $K_{\mathcal{U}}$. We denote $\mathcal{U}^\circ = \{U^\circ \mid U \in \mathcal{U}\} \in \mathrm{cov}(K_{\mathcal{U}})$.

Let $\phi_{\mathcal{U}} : X \to K_{\mathcal{U}}$ be a canonical map for $\mathcal{U}$, that is, $\phi_{\mathcal{U}}^{-1}(U^\circ) \subset U$ for every $U \in \mathcal{U}$, in other words, $\phi_{\mathcal{U}}(x) \in \langle U_0, U_1, \cdots, U_k\rangle$ if $U_0, U_1, \ldots, U_k \in \mathcal{U}$ are all members containing x. (Since canonical maps for $\mathcal{U}$ are contiguous to each others, the proper homotopy class of $\phi_{\mathcal{U}}$ is uniquely determined.) Then $\phi_{\mathcal{U}}$ is a proper map. Observe

$$\phi_{\mathcal{U}}^{-1}(\mathring{\sigma}) \subset \bigcap_{U \in \sigma^{(0)}} U \quad \text{for each simplex } \sigma \text{ of } K_{\mathcal{U}},$$

where $\mathring{\sigma}$ is the interior of σ and $\sigma^{(0)}$ is the set of vertices of σ. One should remark that $\phi_{\mathcal{U}}(U) \subset \mathrm{St}(U, K_{\mathcal{U}})$ but $\phi_{\mathcal{U}}^{-1}(\mathrm{St}(U, K_{\mathcal{U}})) \not\subset \mathrm{cl}\, U$ nor $\phi_{\mathcal{U}}(U) \not\subset U^\circ$ in general. For example, let $X = [0, 1]$ and $\mathcal{U} = \{U, V\} \in \mathrm{cov}(X)$, where $U = [0, \frac{3}{4})$, $V = (\frac{1}{4}, 1]$. Define $f : X \to K_{\mathcal{U}}$ by $f([0, \frac{1}{3}]) = U$, $f([\frac{2}{3}, 1]) = V$ and

$$f(t) = (2 - 3x)U + (3x - 1)V \in \langle U, V\rangle \quad \text{for } x \in [\tfrac{1}{3}, \tfrac{2}{3}].$$

Then f is a canonical map, whence $f^{-1}(\mathrm{St}(U, K_{\mathcal{U}})) = f^{-1}(K_{\mathcal{U}}) = X \not\subset \mathrm{cl}\, U$ nor $f(U) \not\subset U^\circ$ because $f(U) \ni V$.

For simplicity of arguments, we assume that a canonical map $\phi_{\mathcal{U}} : X \to K_{\mathcal{U}}$ satisfies the condition that $\phi_{\mathcal{U}}(U) \subset U^\circ$ for each $U \in \mathcal{U}$ and $\phi_{\mathcal{U}}^{-1}(\mathring{\sigma}) \neq \emptyset$ for each simplex σ of $K_{\mathcal{U}}$. In fact, let $(k_U)_{U \in \mathcal{U}}$ be a partition of unity on X such that $k_U(x) > 0$ for each $x \in U$ (cf. [17, p.27]), and define $\phi_{\mathcal{U}}(x) = \sum_{U \in \mathcal{U}} k_U(x)U \in K_{\mathcal{U}}$ (note $\mathcal{U} = K_{\mathcal{U}}^{(0)}$).

For $\mathcal{U}' \prec \mathcal{U} \in \mathrm{cov}(X)$, let $\phi_{\mathcal{U}}^{\mathcal{U}'} : K_{\mathcal{U}'} \to K_{\mathcal{U}}$ be a simplicial map such that $U \subset \phi_{\mathcal{U}}^{\mathcal{U}'}(U)$ for each $U \in K_{\mathcal{U}'}^{(0)} = \mathcal{U}'$. Since such maps from $K_{\mathcal{U}'}$ to $K_{\mathcal{U}}$ are contiguous to each others, the proper homotopy class of $\phi_{\mathcal{U}}^{\mathcal{U}'}$ is uniquely determined. Observe that $\phi_{\mathcal{U}}^{\mathcal{U}'}$ is a proper map. For each $U' \in \mathcal{U}'$, $\phi_{\mathcal{U}}^{\mathcal{U}'}(\mathrm{St}(U', K_{\mathcal{U}'})) \subset \mathrm{St}(\phi_{\mathcal{U}}^{\mathcal{U}'}(U'))$ and $\phi_{\mathcal{U}}^{\mathcal{U}'}(U'^\circ) \subset \phi_{\mathcal{U}}^{\mathcal{U}'}(U')^\circ$.

Let K be a simplicial complex and $\mathcal{U} \in \operatorname{cov}(X)$. A *partial $\mathcal{U}$-realization* of K is a map $f : L \to X$ of a subcomplex L of K with $K^{(0)} \subset L$ such that for each simplex σ of K, $f(\sigma \cap L)$ is contained in some member of $\mathcal{U}$. In case $L = K$, f is called a *full $\mathcal{U}$-realization* of K. In order that X is an ANR, it is necessary and sufficient that each $\mathcal{U} \in \operatorname{cov}(X)$ has a refinement $\mathcal{V} \in \operatorname{cov}(X)$ such that any partial $\mathcal{V}$-realization of an arbitrary simplicial complex extends to a full $\mathcal{U}$-realization [14]. We call such a refinement $\mathcal{V}$ a *Lefschetz refinement* of $\mathcal{U}$.

3. Proper $\mathcal{V}$-continuous functions

Let $\mathcal{V} \in \operatorname{cov}(Y)$. A function $f : X \to Y$ is said to be *$\mathcal{V}$-continuous* if each $x \in X$ has a neighborhood U such that $f(U)$ is contained in some member of $\mathcal{V}$, in other words, there exists $\mathcal{U} \in \operatorname{cov}(X)$ such that $f(\mathcal{U}) \prec \mathcal{V}$, i.e., $\mathcal{U} \prec f^{-1}(\mathcal{V})$. Clearly, $f : X \to Y$ is continuous if and only if f is $\mathcal{V}$-continuous for every $\mathcal{V} \in \operatorname{cov}(Y)$.

Note that a map $f : X \to Y$ is proper if and only if for any compact set $D \subset Y$ there is a compact set $C \subset X$ such that $f(X \setminus C) \subset Y \setminus D$. A **non-continuous** function $f : X \to Y$ is similarly said to be *proper* if for any compact set $D \subset Y$ there is a compact set $C \subset X$ such that $f(X \setminus C) \subset Y \setminus D$. Then the following follows:

LEMMA 3.1. *A function $f : X \to Y$ is proper if it is $\mathcal{V}$-close to a proper function $g : X \to Y$ for some $\mathcal{V} \in \operatorname{cov}(Y)$.*

Proof. For any compact set $D \subset Y$, let D' be the closure of $\operatorname{st}(D, \mathcal{V})$ in Y. Since $\mathcal{V}$ is star-finite, D meets only finitely many members of $\mathcal{V}$, and the closure of each of them is compact, hence D' is compact. Then there is a compact set $C \subset X$ such that $g(X \setminus C) \subset Y \setminus D'$. For each $x \in X \setminus C$, $\{f(x), g(x)\}$ is contained in some $V \in \mathcal{V}$ and $g(x) \notin \operatorname{st}(D, \mathcal{V})$, hence $V \cap D = \emptyset$. Thus we have $f(X \setminus C) \subset Y \setminus D$. □

It is said that two proper $\mathcal{V}$-continuous functions $f, g : X \to Y$ are *properly $\mathcal{V}$-continuously homotopic* to each others (written by $f \overset{\mathcal{V}}{\simeq}_p g$) if there is a proper $\operatorname{st}\mathcal{V}$-continuous function $H : X \times [0, 1] \to Y$ such that $H_0 = f$, $H_1 = g$, and $H|W$ is $\mathcal{V}$-continuous for some neighborhood W of $X \times \{0, 1\}$ in $X \times [0, 1]$ (i.e., each $x \in X$ has a neighborhood U in X with $\varepsilon > 0$ such that each of $H(U \times [0, \varepsilon))$ and $H(U \times (1 - \varepsilon, 1])$ are contained in some member of $\mathcal{V}$). We call H a *proper $\mathcal{V}$-continuous homotopy* from f to g.

One might wonder why we don't define H in the above to be $\mathcal{V}$-continuous. He should remark that, given two proper $\mathcal{V}$-continuous functions $F, G : X \times [0, 1] \to Y$ such that $F_1 = G_0$, if $H : X \times [0, 1] \to Y$ is defined by $H_t = F_{2t}$ for $t \leqslant \frac{1}{2}$ and $H_t = G_{2t-1}$ for $t \geqslant \frac{1}{2}$ then H is $\operatorname{st}\mathcal{V}$-continuous (cf. the proof of Lemma 3.2 below) but, in general, it is **not** $\mathcal{V}$-continuous. For example, let $Y = [0, 1]$, $\mathcal{V} = \{[0, \frac{2}{3}), (\frac{1}{3}, 1]\} \in \operatorname{cov}(Y)$, and let $F, G : X \times [0, 1] \to Y$ be $\mathcal{V}$-continuous functions defined by $F_t(X) = 0$ for $t < 1$, $F_1(X) = \frac{1}{2}$, $G_0 = \frac{1}{2}$ and $G_t(X) = 1$ for $t > 0$. Then the above H is not $\mathcal{V}$-continuous. Thus, in order that the relation $\overset{\mathcal{V}}{\simeq}_p$

is an equivalence relation on the set of proper $\mathscr{V}$-continuous functions from X to Y, we have to define as above.

LEMMA 3.2. *For any two proper $\mathscr{V}$-continuous functions $f, g : X \to Y$, $f \overset{\mathscr{V}}{=} g$ implies $f \overset{\mathscr{V}}{\sim}_p g$.*

Proof. We define a proper function $H : X \times [0, 1] \to Y$ by $H_t = f$ for $t < \frac{1}{2}$ and $H_t = g$ for $t \geqslant \frac{1}{2}$. Then $H|X \times [0, 1/2) \cup X \times (1/2, 1]$ is $\mathscr{V}$-continuous. On the other hand, each $x \in X$ has a neighborhood U in X such that $f(U)$ is contained in $V_1 \in \mathscr{V}$ and $g(U)$ is contained in $V_2 \in \mathscr{V}$. Since $f \overset{\mathscr{V}}{=} g$, $\{f(x), g(x)\}$ is contained in some $V_0 \in \mathscr{V}$. Then $V_1 \cap V_0 \neq \emptyset$ and $V_2 \cap V_0 \neq \emptyset$, hence $H(U \times [0, 1]) = f(U) \cup g(U) \subset \mathrm{st}(V_0, \mathscr{V})$. Thus H is st $\mathscr{V}$-continuous. □

The following is obvious:

LEMMA 3.3. *Let $h : Y \to Z$ be a function such that $h(\mathscr{V}) \prec \mathscr{W}$, where $\mathscr{V} \in \mathrm{cov}(Y)$ and $\mathscr{W} \in \mathrm{cov}(Z)$. Then $f \overset{\mathscr{V}}{\sim}_p g$ implies $hf \overset{\mathscr{W}}{\sim}_p hg$ for any two $\mathscr{V}$-continuous functions $f, g : X \to Y$.* □

We also have the following:

LEMMA 3.4. *Let $\mathscr{V} \in \mathrm{cov}(Y)$ and $f, g : X \to Y$ be $\mathscr{V}$-continuous functions. Then, $f \overset{\mathscr{V}}{\sim}_p g$ implies that there exists some $\mathscr{U} \in \mathrm{cov}(X)$ such that $f(\mathscr{U}), g(\mathscr{U}) \prec \mathscr{V}$ and $fh \overset{\mathscr{V}}{\sim}_p gh$ for any proper $\mathscr{U}$-continuous function $h : Z \to X$ of an arbitrary space Z.*

Proof. We have a proper $\mathscr{V}$-continuous homotopy $H : X \times [0, 1] \to Y$ from f to g, which is st $\mathscr{V}$-continuous and $H|W$ is $\mathscr{V}$-continuous for some neighborhood W of $X \times \{0, 1\}$ in $X \times [0, 1]$. Choose $\mathscr{U} \in \mathrm{cov}(X)$ so that for each $U \in \mathscr{U}$, there are $0 < t_1 < \cdots < t_n < 1$ such that $H(U \times [0, t_1])$ and $H(U \times [t_n, 1])$ are contained in some member of $\mathscr{V}$ and each $H(U \times [t_{i-1}, t_i])$ is contained in some member of st$\mathscr{V}$. If $h : Z \to X$ is $\mathscr{U}$-continuous, $h(\mathscr{W}) \prec \mathscr{U}$ for some $\mathscr{W} \in \mathrm{cov}(Z)$, hence $H\circ(h \times \mathrm{id}) : Z \times [0, 1] \to X$ is a proper $\mathscr{V}$-continuous homotopy from fh to gh. Thus $fh \overset{\mathscr{V}}{\sim}_p gh$. □

Let $\mathscr{U} \in \mathrm{cov}(X)$. Recall $K_{\mathscr{U}}$ denotes the nerve of $\mathscr{U}$ and $\phi_{\mathscr{U}} : X \to K_{\mathscr{U}}$ a canonical map such that $\phi_{\mathscr{U}}(U) \subset U^\circ$ for each $U \in \mathscr{U}$ and $\phi_{\mathscr{U}}^{-1}(\mathring{\sigma}) \neq \emptyset$ for each simplex σ of $K_{\mathscr{U}}$. A (non-continuous) function $\psi_{\mathscr{U}} : K_{\mathscr{U}} \to X$ is called a *canonical function* for $\mathscr{U}$ if $\psi_{\mathscr{U}}(\mathring{\sigma}) \subset \bigcap_{U \in \sigma^{(0)}} U$ for each simplex σ of $K_{\mathscr{U}}$, whence $\psi_{\mathscr{U}}(U^\circ) \subset U$ for each $U \in \mathscr{U}$. It is easy to see that $\psi_{\mathscr{U}}$ is a proper $\mathscr{U}$-continuous function. Since such functions are $\mathscr{U}$-close to each others, the proper $\mathscr{U}$-continuous homotopy class of $\psi_{\mathscr{U}}$ is uniquely determined.

LEMMA 3.5. *Let $\mathscr{U}' \prec \mathscr{U} \in \mathrm{cov}(X)$. Then the following hold.*

1. $\phi_{\mathscr{U}}^{-1}(\mathscr{U}^\circ) \prec \mathscr{U}$, $\psi_{\mathscr{U}}(\mathscr{U}^\circ) \prec \mathscr{U}$;
2. $\psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{U}}{=} \mathrm{id}_X$, $\phi_{\mathscr{U}}\psi_{\mathscr{U}} \overset{\mathscr{U}^\circ}{=} \mathrm{id}_{K_{\mathscr{U}}}$;

3. $\psi_{\mathscr{U}}\phi_{\mathscr{U}}^{\mathscr{U}'} \overset{\mathscr{U}}{=} \psi_{\mathscr{U}'}$, $\phi_{\mathscr{U}} \overset{\mathscr{U}^\circ}{=} \phi_{\mathscr{U}}^{\mathscr{U}'}\phi_{\mathscr{U}'}$, $\phi_{\mathscr{U}} \simeq_p \phi_{\mathscr{U}}^{\mathscr{U}'}\phi_{\mathscr{U}'}$.

Proof. (1): Obvious. (2): Since $\psi_{\mathscr{U}}\phi_{\mathscr{U}}(U) \subset \psi_{\mathscr{U}}(U^\circ) \subset U$ for each $U \in \mathscr{U}$, we have $\psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{U}}{=} \mathrm{id}_X$. Since $\phi_{\mathscr{U}}\psi_{\mathscr{U}}(U^\circ) \subset \phi_{\mathscr{U}}(U) \subset U^\circ$ for each $U \in \mathscr{U}$, we have $\phi_{\mathscr{U}}\psi_{\mathscr{U}} \overset{\mathscr{U}^\circ}{=} \mathrm{id}_{K_{\mathscr{U}}}$. (3): For each $U \in \mathscr{U}'$,

$$\psi_{\mathscr{U}'}(U^\circ) \subset U \subset \phi_{\mathscr{U}}^{\mathscr{U}'}(U) \text{ and } \psi_{\mathscr{U}}\phi_{\mathscr{U}}^{\mathscr{U}'}(U^\circ) \subset \psi_{\mathscr{U}}\left(\phi_{\mathscr{U}}^{\mathscr{U}'}(U)^\circ\right) \subset \phi_{\mathscr{U}}^{\mathscr{U}'}(U).$$

Then $\psi_{\mathscr{U}}\phi_{\mathscr{U}}^{\mathscr{U}'} \overset{\mathscr{U}}{=} \psi_{\mathscr{U}'}$. For each $U \in \mathscr{U}'$, $U \subset \phi_{\mathscr{U}}^{\mathscr{U}'}(U) \in \mathscr{U}$, $\phi_{\mathscr{U}'}(U) \subset U^\circ$, $\phi_{\mathscr{U}}(\phi_{\mathscr{U}}^{\mathscr{U}'}(U)) \subset \phi_{\mathscr{U}}^{\mathscr{U}'}(U)^\circ$ and $\phi_{\mathscr{U}}^{\mathscr{U}'}(U^\circ) \subset \phi_{\mathscr{U}}^{\mathscr{U}'}(U)^\circ$. Then $\phi_{\mathscr{U}} \overset{\mathscr{U}^\circ}{=} \phi_{\mathscr{U}}^{\mathscr{U}'}\phi_{\mathscr{U}'}$. Moreover, $\phi_{\mathscr{U}} \simeq_p \phi_{\mathscr{U}}^{\mathscr{U}'}\phi_{\mathscr{U}'}$ because $\phi_{\mathscr{U}}$ and $\phi_{\mathscr{U}}^{\mathscr{U}'}\phi_{\mathscr{U}'}$ are contiguous. □

Let $\mathscr{V} \in \mathrm{cov}(Y)$ and $n \in \mathbb{N} \cup \{\infty\}$. Two proper $\mathscr{V}$-continuous functions $f, g : X \to Y$ are *properly $\mathscr{V}$-continuously n-homotopic* (written by $f \overset{(\mathscr{V},n)}{\sim_p} g$) if there is some $\mathscr{U} \in \mathrm{cov}(X)$ such that $f(\mathscr{U}), g(\mathscr{U}) \prec \mathscr{V}$ and $f\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)}$ ($f\psi_{\mathscr{U}} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}$ in case $n = \infty$). Then it should be remarked that $f\psi_{\mathscr{U}'}|K_{\mathscr{U}'}^{(n)} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}'}|K_{\mathscr{U}'}^{(n)}$ for every $\mathscr{U}' \prec \mathscr{U} \in \mathrm{cov}(X)$. In fact, since $\psi_{\mathscr{U}}\phi_{\mathscr{U}}^{\mathscr{U}'} \overset{\mathscr{U}}{\sim}_p \psi_{\mathscr{U}'}$ and $\phi_{\mathscr{U}}^{\mathscr{U}'}$ is continuous, it follows that

$$f\psi_{\mathscr{U}'}|K_{\mathscr{U}'}^{(n)} \overset{\mathscr{V}}{\sim}_p f\psi_{\mathscr{U}}\phi_{\mathscr{U}}^{\mathscr{U}'}|K_{\mathscr{U}'}^{(n)} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}\phi_{\mathscr{U}}^{\mathscr{U}'}|K_{\mathscr{U}'}^{(n)} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}'}|K_{\mathscr{U}'}^{(n)}.$$

Then it is easy to see that the relation $\overset{(\mathscr{V},n)}{\sim_p}$ is an equivalence relation on the set of $\mathscr{V}$-continuous proper functions from X to Y.

By the following lemma, the phrase "properly $\mathscr{V}$-continuously homotopic" can be replaced by "properly $(\mathscr{V}, \infty)$-homotopic".

LEMMA 3.6. *For $\mathscr{V}$-continuous functions $f, g : X \to Y$, $f \overset{\mathscr{V}}{\sim}_p g$ if and only if $f \overset{(\mathscr{V},\infty)}{\sim_p} g$. Hence, $f \overset{\mathscr{V}}{\sim}_p g$ implies $f \overset{(\mathscr{V},n)}{\sim_p} g$ for every $n \in \mathbb{N}$.*

Proof. First, assume $f \overset{\mathscr{V}}{\sim}_p g$. By Lemma 3.4, we have $\mathscr{U} \in \mathrm{cov}(X)$ such that $fh \overset{\mathscr{V}}{\sim}_p gh$ for any proper $\mathscr{U}$-continuous function $h : Z \to X$ of an arbitrary space Z. Then $f\psi_{\mathscr{U}} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}$, that is, $f \overset{(\mathscr{V},\infty)}{\sim_p} g$.

Conversely, assume $f \overset{(\mathscr{V},\infty)}{\sim_p} g$, that is, there exists some $\mathscr{U} \in \mathrm{cov}(X)$ such that $f(\mathscr{U}), g(\mathscr{U}) \prec \mathscr{V}$ and $f\psi_{\mathscr{U}} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}$. Since $\phi_{\mathscr{U}}$ is continuous, we have $f\psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}\phi_{\mathscr{U}}$. On the other hand, since $\psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{U}}{=} \mathrm{id}_X$, it follows that $f \overset{\mathscr{V}}{=} f\psi_{\mathscr{U}}\phi_{\mathscr{U}}$ and $g \overset{\mathscr{V}}{=} g\psi_{\mathscr{U}}\phi_{\mathscr{U}}$. Consequently, $f \overset{\mathscr{V}}{\sim}_p g$. □

For proper maps $f, g : X \to Y$, it is obvious that $f \simeq_p g$ implies $f \overset{\mathscr{V}}{\sim}_p g$ for every $\mathscr{V} \in \mathrm{cov}(Y)$. However, the n-homotopy version is non-trivial.

LEMMA 3.7. *For proper maps $f, g : X \to Y$, $f \overset{n}{\simeq}_p g$ implies $f \overset{(\mathscr{V},n)}{\sim_p} g$ for every $\mathscr{V} \in \mathrm{cov}(Y)$.*

Proof. Consider X and Y as closed sets in $Q\setminus\{\mathrm{pt}\}$. Let $\tilde{f}, \tilde{g} : Q\setminus\{\mathrm{pt}\} \to Q\setminus\{\mathrm{pt}\}$ be proper maps extending f and g, respectively. We have a closed neighborhood N of Y in $Q\setminus\{\mathrm{pt}\}$ and $\tilde{\mathscr{V}} = \{\tilde{V} \mid V \in \mathrm{cov}(Y)\} \in \mathrm{cov}(N)$ such that $\tilde{V} \cap Y = V$ for each $V \in \mathrm{cov}(Y)$ and $\tilde{V}_0 \cap \cdots \cap \tilde{V}_k \neq \emptyset$ implies $V_0 \cap \cdots \cap V_k \neq \emptyset$ for any $V_0, \cdots, V_k \in \mathscr{V}$, hence the nerve $K_{\tilde{\mathscr{V}}}$ can be identified with $K_{\mathscr{V}}$. Let $r = \psi_{\mathscr{V}} \circ \phi_{\tilde{\mathscr{V}}} : N \to Y$. Recall that each $\tilde{V} \in \tilde{\mathscr{V}}$ is relatively compact in N. Since $r(\tilde{V}) \subset \psi_{\mathscr{V}}(V^\circ) \subset V$ for each $V \in \mathscr{V}$, r is $\mathscr{V}$-continuous and $\tilde{\mathscr{V}}$-close to id_N, hence it is proper. It should be also noticed that $r|Y$ is $\mathscr{V}$-close to id_N.

By [12], there exists an n-invertible map $\xi : \mu^n \to Q$, where μ^n is the n-dimensional universal Menger compactum. Since $\dim \xi^{-1}(X) \leqslant n$, $f\xi|\xi^{-1}(X) \simeq_p g\xi|\xi^{-1}(X)$ in Y. Since $\mathrm{int}\, N$ is an ANR, $\xi^{-1}(X)$ has a neighborhood W in $\mu^n\setminus\xi^{-1}(\mathrm{pt})$ such that $\tilde{f}\xi|W \simeq_p \tilde{g}\xi|W$ in $\mathrm{int}\, N$.

Let M be an open neighborhood of X in $Q\setminus\{\mathrm{pt}\}$ and $\mathscr{U}' \in \mathrm{cov}(M)$ such that $\xi^{-1}(M) \subset W$ and $\tilde{f}(\mathrm{st}\,\mathscr{U}'), \tilde{g}(\mathrm{st}\,\mathscr{U}') \prec \tilde{\mathscr{V}}$. Note that $r\tilde{f}(\mathrm{st}\,\mathscr{U}'), r\tilde{g}(\mathrm{st}\,\mathscr{U}') \prec \mathscr{V}$. Since M is an ANR, $\mathscr{U}'$ has a Lefschetz refinement $\mathscr{U}'' \in \mathrm{cov}(M)$. Choose $\mathscr{U} \in \mathrm{cov}(X)$ so that $\mathrm{st}\,\mathscr{U} \prec \mathscr{U}''$. Then $\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(0)} : K_{\mathscr{U}}^{(0)} \to X$ is a partial $\mathscr{U}''$-realization of $K_{\mathscr{U}}$, which extends to a full $\mathscr{U}'$-realization $h : K_{\mathscr{U}} \to M$. Observe that h is $\mathrm{st}\,\mathscr{U}'$-close to $\psi_{\mathscr{U}}$, hence it is proper. Since ξ is n-invertible, we have a proper map $\tilde{h} : K_{\mathscr{U}}^{(n)} \to \xi^{-1}(M) \subset W$ such that $\xi\tilde{h} = h|K_{\mathscr{U}}^{(n)}$, whence

$$\tilde{f}h|K_{\mathscr{U}}^{(n)} = \tilde{f}\xi\tilde{h} \simeq_p \tilde{g}\xi\tilde{h} = \tilde{g}h|K_{\mathscr{U}}^{(n)} \quad \text{in } N.$$

Therefore, $r\tilde{f}h|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}}{\sim}_p r\tilde{g}h|K_{\mathscr{U}}^{(n)}$ in Y.

On the other hand, since h is $\mathrm{st}\,\mathscr{U}'$-close to $\psi_{\mathscr{U}}$ and $r\tilde{f}(\mathrm{st}\,\mathscr{U}') \prec \mathscr{V}$, we have

$$r\tilde{f}h \overset{\mathscr{V}}{=} r\tilde{f}\psi_{\mathscr{U}} = rf\psi_{\mathscr{U}} \overset{\mathscr{V}}{=} f\psi_{\mathscr{U}},$$

hence $r\tilde{f}h \overset{\mathscr{V}}{\sim}_p f\psi_{\mathscr{U}}$. Similarly $r\tilde{g}h \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}$. Consequently, $f\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}}{\sim}_p g\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)}$, that is, $f \overset{(\mathscr{V},n)}{\sim}_p g$. □

For each $\mathscr{V}$-continuous function $f : X \to Y$, we have some $\mathscr{U} \in \mathrm{cov}(X)$ such that $f(\mathscr{U}) \prec \mathscr{V}$. Then, for each $U \in \mathscr{U} = K_{\mathscr{U}}^{(0)}$, choosing $\varphi(U) \in \mathscr{V} = K_{\mathscr{V}}^{(0)}$ so that $f(U) \subset \varphi(U)$, we can obtain a simplicial map $\varphi : K_{\mathscr{U}} \to K_{\mathscr{V}}$. Such a simplicial map φ is said to be *associated* with f. For every $\mathscr{U}' \prec \mathscr{U}$, $\varphi\phi_{\mathscr{U}}^{\mathscr{U}'} : K_{\mathscr{U}'} \to K_{\mathscr{V}}$ is also associated with f. In fact, for each $U \in \mathscr{U}'$, since $U \subset \phi_{\mathscr{U}}^{\mathscr{U}'}(U) \in \mathscr{U}$, we have $f(U) \subset f(\phi_{\mathscr{U}}^{\mathscr{U}'}(U)) \subset \varphi(\phi_{\mathscr{U}}^{\mathscr{U}'}(U))$.

LEMMA 3.8. *Let $f : X \to Y$ be a $\mathscr{V}$-continuous function and $\varphi : K_{\mathscr{U}} \to K_{\mathscr{V}}$ a simplicial map associated with f. Then, $\varphi\phi_{\mathscr{U}} \overset{\mathscr{V}^\circ}{=} \phi_{\mathscr{V}}f$ and $\psi_{\mathscr{V}}\varphi\phi_{\mathscr{U}} \overset{\mathscr{V}}{=} f$. Hence, if f is proper then so is φ.*

Proof. For each $U \in \mathscr{U}$, $\varphi\phi_{\mathscr{U}}(U) \subset \varphi(U^\circ) \subset \varphi(U)^\circ$ and $\phi_{\mathscr{V}}f(U) \subset \phi_{\mathscr{V}}\varphi(U) \subset \varphi(U)^\circ$, which implies $\varphi\phi_{\mathscr{U}} \overset{\mathscr{V}^\circ}{=} \phi_{\mathscr{V}}f$. Moreover, $\psi_{\mathscr{V}}\varphi\phi_{\mathscr{U}}(U) \subset \psi_{\mathscr{V}}(\varphi(U)^\circ) \subset \varphi(U)$, hence $\psi_{\mathscr{V}}\varphi\phi_{\mathscr{U}} \overset{\mathscr{V}}{=} f$. □

In case Y is an ANR, we have the following converse of Lemma 3.7.

LEMMA 3.9. *In case Y is an ANR, there exists* $\mathcal{V}_Y \in \operatorname{cov}(Y)$ *such that* $f \overset{(\mathcal{V}_Y,n)}{\sim_p} g$ *implies* $f \overset{n}{\simeq_p} g$ *and* $f \overset{\mathcal{V}_Y}{\sim_p} g$ *implies* $f \simeq_p g$ *for any proper maps* $f, g : X \to Y$ *from an arbitrary space X.*

Proof. Since Y is an ANR, there exists $\mathcal{V}_0 \in \operatorname{cov}(Y)$ such that any two $\operatorname{st}\mathcal{V}_0$-close proper maps from an arbitrary space to Y are proerly homotopic. Choose $\mathcal{V}_1, \mathcal{V}, \mathcal{V}_Y \in \operatorname{cov}(Y)$ so that $\mathcal{V}_1 \in \operatorname{cov}(Y)$ is a star-refinement of $\mathcal{V}_0$, $\operatorname{st}\mathcal{V}$ is a Lefschetz refinement of $\mathcal{V}_1$ and $\mathcal{V}_Y \in \operatorname{cov}(Y)$ is a star-refinement of $\mathcal{V}$. Since $\psi_{\mathcal{V}}|K_{\mathcal{V}}^{(0)}$ is partial $\operatorname{st}\mathcal{V}$-realization of $K_{\mathcal{V}}$, it extends to a full $\mathcal{V}_1$-realization $q : K_{\mathcal{V}} \to Y$. Observe that $q(\mathcal{V}^\circ) \prec \operatorname{st}\mathcal{V}_1 \prec \mathcal{V}_0$ and $q\phi_{\mathcal{V}} \overset{\mathcal{V}_0}{=} \mathrm{id}_Y$, hence $q\phi_{\mathcal{V}} \simeq_p \mathrm{id}_Y$.

Let $f, g : X \to Y$ be proper maps such that $f \overset{(\mathcal{V}_Y,n)}{\sim_p} g$, that is, $f(\mathcal{U}), g(\mathcal{U}) \prec \mathcal{V}_Y$ and $f\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{V}_Y}{\sim_p} g\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$ for some $\mathcal{U} \in \operatorname{cov}(X)$. Then we have a proper $\operatorname{st}\mathcal{V}_Y$-continuous function $H : K_{\mathcal{U}}^{(n)} \times [0,1] \to Y$ such that $H_0 = f\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$, $H_1 = g\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$ and $H|W$ is $\mathcal{V}_Y$-continuous for some neighborhood W of $K_{\mathcal{U}}^{(n)} \times \{0,1\}$ in $K_{\mathcal{U}}^{(n)} \times [0,1]$. Since H is $\mathcal{V}$-continuous, there is $\mathcal{W} \in \operatorname{cov}(K_{\mathcal{U}}^{(n)} \times [0,1])$ and a simplicial map $H' : K_{\mathcal{W}} \to K_{\mathcal{V}}$ associated with H, whence $\phi_{\mathcal{V}}H \overset{\mathcal{V}^\circ}{=} H'\phi_{\mathcal{W}}$. Then we have a proper homotopy $\tilde{H} = qH'\phi_{\mathcal{W}} : K_{\mathcal{U}}^{(n)} \times [0,1] \to Y$ such that $\tilde{H} \overset{\mathcal{V}_0}{=} q\phi_{\mathcal{V}}H$, hence $\tilde{H}_0 \overset{\mathcal{V}_0}{=} q\phi_{\mathcal{V}}f\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$ and $\tilde{H}_1 \overset{\mathcal{V}_0}{=} q\phi_{\mathcal{V}}g\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$. Let $h : Z \to X$ be a proper map from a space Z with $\dim Z \leqslant n$. Then we have a proper map $h' : Z \to K_{\mathcal{U}}^{(n)}$ such that $\phi_{\mathcal{U}}h \overset{\mathcal{U}^\circ}{=} h'$. Since $f(\mathcal{U}) \prec \mathcal{V}$ and $q\phi_{\mathcal{V}} \overset{\mathcal{V}_0}{=} \mathrm{id}_Y$, it follows that $fh \overset{\mathcal{V}}{=} f\psi_{\mathcal{U}}\phi_{\mathcal{U}}h \overset{\mathcal{V}}{=} f\psi_{\mathcal{U}}h' \overset{\mathcal{V}_0}{=} q\phi_{\mathcal{V}}f\psi_{\mathcal{U}}h' = q\phi_{\mathcal{V}}H_0h' \overset{\mathcal{V}_0}{=} \tilde{H}_0h'$, hence $fh \overset{\operatorname{st}\mathcal{V}_0}{=} \tilde{H}_0h'$, which implies that $fh \simeq_p \tilde{H}_0h'$. Similarly, we have $gh \simeq_p \tilde{H}_1h'$. Consequently, $fh \simeq_p gh$.

In the above, by replacing $K_{\mathcal{U}}^{(n)}$ with X, we can show that $f \overset{\mathcal{V}_Y}{\sim_p} g$ implies $f \simeq_p g$.
□

The following is obvious:

LEMMA 3.10. *Let* $h : Y \to Z$ *be a function such that* $h(\mathcal{V}) \prec \mathcal{W}$, *where* $\mathcal{V} \in \operatorname{cov}(Y)$ *and* $\mathcal{W} \in \operatorname{cov}(Z)$. *Then,* $f \overset{(\mathcal{V},n)}{\sim_p} g$ *implies* $hf \overset{(\mathcal{W},n)}{\sim_p} hg$ *for any two* $\mathcal{V}$*-continuous functions* $f, g : X \to Y$. □

The following is the n-homotopy version of Lemma 3.4.

LEMMA 3.11. *Let* $\mathcal{V} \in \operatorname{cov}(Y)$ *and* $f, g : X \to Y$ *be* $\mathcal{V}$*-continuous functions. Then,* $f \overset{(\mathcal{V},n)}{\sim_p} g$ *implies that there is some* $\mathcal{U} \in \operatorname{cov}(X)$ *such that* $f(\mathcal{U}), g(\mathcal{U}) \prec \mathcal{V}$ *and* $fh \overset{(\mathcal{V},n)}{\sim_p} gh$ *for any proper* $\mathcal{U}$*-continuous function* $h : Z \to X$ *of an arbitrary space Z.*

Proof. There exists some $\mathcal{U} \in \operatorname{cov}(X)$ such that $f(\mathcal{U}), g(\mathcal{U}) \prec \mathcal{V}$ and $f\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{V}}{\sim}_p g\psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$. For a proper $\mathcal{U}$-continuous function $h : Z \to X$, let $\varphi : K_{\mathcal{W}} \to K_{\mathcal{U}}$ be a simplicial map associated with h. Since $\varphi(K_{\mathcal{W}}^{(n)}) \subset K_{\mathcal{U}}^{(n)}$ and φ is continuous, it follows from Lemma 3.4 that $f\psi_{\mathcal{U}}\varphi|K_{\mathcal{W}}^{(n)} \overset{\mathcal{V}}{\sim}_p g\psi_{\mathcal{U}}\varphi|K_{\mathcal{W}}^{(n)}$. On the other hand, $\psi_{\mathcal{U}}\varphi\phi_{\mathcal{W}} \overset{\mathcal{U}}{=} h$ by Lemma 3.8. Since $\phi_{\mathcal{W}}\psi_{\mathcal{W}} \overset{\mathcal{W}^\circ}{=} \mathrm{id}_{K_{\mathcal{W}}}$ and $\varphi(\mathcal{W}^\circ) \prec \mathcal{U}^\circ$, it follows that $\psi_{\mathcal{U}}\varphi \overset{\mathcal{U}}{=} \psi_{\mathcal{U}}\varphi\phi_{\mathcal{W}}\psi_{\mathcal{W}} \overset{\mathcal{U}}{=} h\psi_{\mathcal{W}}$. Then we have $fh\psi_{\mathcal{W}} \overset{\mathcal{V}}{\sim}_p f\psi_{\mathcal{U}}\varphi$ and $gh\psi_{\mathcal{W}} \overset{\mathcal{V}}{\sim}_p g\psi_{\mathcal{U}}\varphi$. Therefore, $fh\psi_{\mathcal{W}}|K_{\mathcal{W}}^{(n)} \overset{\mathcal{V}}{\sim}_p gh\psi_{\mathcal{W}}|K_{\mathcal{W}}^{(n)}$. □

In case $n = \infty$, Lemma 3.11 is none other than Lemma 3.4.

4. Proper proximate nets

A *proper proximate net* $(f_\lambda) : X \to Y$ is a net of (non-continuous) proper functions $f_\lambda : X \to Y$ indexed by a directed set $\Lambda = (\Lambda, \leqslant)$ such that, for each $\mathcal{V} \in \operatorname{cov}(Y)$, there exists $\lambda_0 \in \Lambda$ such that $f_{\lambda_0} \overset{\mathcal{V}}{\sim}_p f_\lambda$ for all $\lambda \geqslant \lambda_0$, whence f_λ is $\mathcal{V}$-continuous for all $\lambda \geqslant \lambda_0$. It is said that (f_λ) is *properly homotopic* to a proper proximate net $(g_\delta) : X \to Y$ indexed by a directed set Δ (denoted by $(f_\lambda) \simeq_p (g_\delta)$) provided, for each $\mathcal{V} \in \operatorname{cov}(Y)$, there exist $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that $f_\lambda \overset{\mathcal{V}}{\sim}_p g_\delta$ for all $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$. Clearly, the relation $\simeq_p$ among proper proximate nets from X to Y is an equivalence relation. The equivalence class of a proper proximate net $(f_\lambda) : X \to Y$ is denoted by $[(f_\lambda)]_p$ and called the proper homotopy class of (f_λ). The collection of all proper homotopy classes of proper proximate nets is denoted by $[X, Y]_{ppn}$. Every proper map $f : X \to Y$ can be considered a proper proximate net as the net consisting of only f indexed by itself.

A *proper proximate Čech net* $(f_\mathcal{V}) : X \to Y$ is a net of proper $\mathcal{V}$-continuous functions $f_\mathcal{V} : X \to Y$ indexed by the directed set $\operatorname{cov}(Y)$ such that $f_\mathcal{V} \overset{\mathcal{V}}{\sim}_p f_{\mathcal{V}'}$ for $\mathcal{V}' \prec \mathcal{V} \in \operatorname{cov}(Y)$, where $\operatorname{cov}(Y)$ has the order $[\mathcal{V} \leqslant \mathcal{V}'] \equiv [\mathcal{V}' \prec \mathcal{V}]$. Then a proper proximate Čech net is clearly a proper proximate net.

LEMMA 4.1. *Any proper proximate net is properly homotopic to a proper proximate Čech net.*

Proof. Let $(f_\lambda) : X \to Y$ be a proper proximate net. For each $\mathcal{V} \in \operatorname{cov}(Y)$, choose $\lambda(\mathcal{V}) \in \Lambda$ so that $f_\lambda \overset{\mathcal{V}}{\sim}_p f_{\lambda(\mathcal{V})}$ for all $\lambda \geqslant \lambda(\mathcal{V})$, and let $f_\mathcal{V} = f_{\lambda(\mathcal{V})}$. Then for $\mathcal{V}' \prec \mathcal{V} \in \operatorname{cov}(Y)$, we have $\lambda \geqslant \lambda(\mathcal{V}), \lambda(\mathcal{V}')$, whence

$$f_\mathcal{V} = f_{\lambda(\mathcal{V})} \overset{\mathcal{V}}{\sim}_p f_\lambda \overset{\mathcal{V}'}{\sim}_p f_{\lambda(\mathcal{V}')} = f_{\mathcal{V}'}.$$

Therefore, $(f_\mathcal{V}) : X \to Y$ is a proper proximate Čech net which is properly homotopic to (f_λ). □

LEMMA 4.2. *For any two proper proximate Čech nets* $(f_{\mathcal{V}}), (g_{\mathcal{V}}) : X \to Y$, $(f_{\mathcal{V}}) \simeq_p (g_{\mathcal{V}})$ *if and only if* $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}}$ *for each* $\mathcal{V} \in \operatorname{cov}(Y)$.

Proof. The "if" part is trivial. To see the "only if" part, assume that $(f_{\mathcal{V}}) \simeq_p (g_{\mathcal{V}})$. For each $\mathcal{V} \in \operatorname{cov}(Y)$, we have $\mathcal{V}_1, \mathcal{V}_2 \in \Omega(Y)$ such that $f_{\mathcal{V}'} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}''}$ for all $\mathcal{V}' \prec \mathcal{V}_1$ and $\mathcal{V}'' \prec \mathcal{V}_2$. By choose $\mathcal{V}_0 \prec \mathcal{V}, \mathcal{V}_1, \mathcal{V}_2$, we have

$$f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p f_{\mathcal{V}_0} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}_0} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}}$$

because $(f_{\mathcal{V}})$ and $(g_{\mathcal{V}})$ are proper proximate Čech nets. □

We shall define the composition $[X, Y]_{ppn} \times [Y, Z]_{ppn} \to [X, Z]_{ppn}$. By Lemma 4.1, we may only consider proper proximate Čech nets.

LEMMA 4.3. *Let* $(f_{\mathcal{V}}), (f'_{\mathcal{V}}) : X \to Y$ *and* $(g_{\mathcal{W}}), (g'_{\mathcal{W}}) : Y \to Z$ *be proper proximate Čech nets, and* $g, g' : \operatorname{cov}(Z) \to \operatorname{cov}(Y)$ *be functions so that* $g_{\mathcal{W}}(g(\mathcal{W})), g'_{\mathcal{W}}(g'(\mathcal{W})) \prec \mathcal{W}$ *for each* $\mathcal{W} \in \operatorname{cov}(Z)$. *Then,* (1) $(g_{\mathcal{W}} f_{g(\mathcal{W})})$ *is a proper proximate Čech net;* (2) $(f_{\mathcal{V}}) \simeq_p (f'_{\mathcal{V}})$ *and* $(g_{\mathcal{W}}) \simeq_p (g'_{\mathcal{W}})$ *imply* $(g_{\mathcal{W}} f_{g(\mathcal{W})}) \simeq_p (g'_{\mathcal{W}} f'_{g'(\mathcal{W})})$.

Proof. (1): Each $g_{\mathcal{W}} f_{g(\mathcal{W})} : X \to Z$ is obviously a proper $\mathcal{W}$-continuous function. For each $\mathcal{W}' \prec \mathcal{W} \in \operatorname{cov}(Z)$, $g_{\mathcal{W}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'}$. By Lemma 3.4, we can obtain $\mathcal{V} \prec g(\mathcal{W}), g(\mathcal{W}')$ such that $g_{\mathcal{W}}(\mathcal{V}), g_{\mathcal{W}'}(\mathcal{V}) \prec \mathcal{W}$ and $g_{\mathcal{W}} h \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'} h$ for any proper $\mathcal{V}$-continuous function $h : X \to Y$, whence $g_{\mathcal{W}} f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'} f_{\mathcal{V}}$. Since $(f_{\mathcal{V}})$ is a proper proximate Čech net, we have $f_{g(\mathcal{W})} \overset{g(\mathcal{W})}{\sim}_p f_{\mathcal{V}}$. Since $g_{\mathcal{W}}(g(\mathcal{W})) \prec \mathcal{W}$, it follows that $g_{\mathcal{W}} f_{g(\mathcal{W})} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} f_{\mathcal{V}}$. Similarly, $g_{\mathcal{W}'} f_{g(\mathcal{W}')} \overset{\mathcal{W}'}{\sim}_p g_{\mathcal{W}'} f_{\mathcal{V}}$. Consequently, we have $g_{\mathcal{W}} f_{g(\mathcal{W})} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}'} f_{g(\mathcal{W}')}$.

(2): For each $\mathcal{W} \in \operatorname{cov}(Z)$, $g_{\mathcal{W}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}}$ by Lemma 4.2. By Lemma 3.4, we can choose $\mathcal{V} \prec g(\mathcal{W}), g'(\mathcal{W})$ so that $g_{\mathcal{W}}(\mathcal{V}), g'_{\mathcal{W}}(\mathcal{V}) \prec \mathcal{W}$ and $g_{\mathcal{W}} h \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} h$ for any proper $\mathcal{V}$-continuous function $h : X \to Y$, whence $g_{\mathcal{W}} f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} f_{\mathcal{V}}$. Since $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p f'_{\mathcal{V}}$ by Lemma 4.2, we have $g'_{\mathcal{W}} f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} f'_{\mathcal{V}}$ by Lemma 3.3. Therefore $g_{\mathcal{W}} f_{\mathcal{V}} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} f'_{\mathcal{V}}$. On the other hand, since $(g_{\mathcal{W}} f_{g(\mathcal{W})})$ and $(g'_{\mathcal{W}} f'_{g'(\mathcal{W})})$ are proper proximate Čech nets, we have $g_{\mathcal{W}} f_{g(\mathcal{W})} \overset{\mathcal{W}}{\sim}_p g_{\mathcal{W}} f_{\mathcal{V}}$ and $g'_{\mathcal{W}} f'_{g'(\mathcal{W})} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} f'_{\mathcal{V}}$. Consequently, $g_{\mathcal{W}} f_{g(\mathcal{W})} \overset{\mathcal{W}}{\sim}_p g'_{\mathcal{W}} f'_{g'(\mathcal{W})}$. □

Then we can define $[(g_{\mathcal{W}})]_p \circ [(f_{\mathcal{V}})]_p = [(g_{\mathcal{W}} f_{g(\mathcal{V})})]_p$ by using a function $g : \operatorname{cov}(Z) \to \operatorname{cov}(Y)$ such that $g_{\mathcal{W}}(g(\mathcal{W})) \prec \mathcal{W}$ for every $\mathcal{W} \in \operatorname{cov}(Z)$. In case $f_{\mathcal{V}} = \mathrm{id}_Y : Y \to Y$, we have $[(f_{\mathcal{V}})]_p = [(\mathrm{id}_Y)]_p$ and $[(g_{\mathcal{W}})]_p \circ [(f_{\mathcal{V}})]_p = [(g_{\mathcal{W}})]_p$. In case $g_{\mathcal{W}} = \mathrm{id}_Y : Y \to Y$, we have $[(g_{\mathcal{W}})]_p = [(\mathrm{id}_Y)]_p$ and $[(g_{\mathcal{W}})]_p \circ [(f_{\mathcal{V}})]_p = [(f_{\mathcal{V}})]_p$ by using $g = \mathrm{id}_{\operatorname{cov}(Y)}$. It is straightforward to see that the composition is associative. The proof is left to the readers. Thus we obtain the category $\mathscr{P}_p$ of locally compact separable metrizable spaces and proper homotopy classes of proper proximate nets.

Replacing $\overset{\mathscr{V}}{\sim}_p$ by $\overset{(\mathscr{V},n)}{\sim}_p$ in the definition of a proper proximate net and a proper proximate Čech net, a *proper n-proximate net* and a *proper n-proximate Čech net* can be defined. The *proper n-homotopy* $(f_\lambda) \overset{n}{\simeq}_p (g_\delta)$ between proper n-proximate nets can be also defined by the same replacement. Similarly to Lemmas 4.1 and 4.2, we have the following:

LEMMA 4.4. *Any proper n-proximate net is properly n-homotopic to a proper n-proximate Čech net.* □

LEMMA 4.5. *For any two proper n-proximate Čech nets $(f_\mathscr{V}), (g_\mathscr{V}) : X \to Y$, $(f_\mathscr{V}) \overset{n}{\simeq}_p (g_\mathscr{V})$ if and only if $f_\mathscr{V} \overset{(\mathscr{V},n)}{\sim}_p g_\mathscr{V}$ for each $\mathscr{V} \in \operatorname{cov}(Y)$.* □

The proper n-homotopy class of a proper n-proximate net $(f_\lambda) : X \to Y$ is denoted by $[(f_\lambda)]_p^n$ and the collection of all proper n-homotopy classes of proper n-proximate nets is denoted by $[X, Y]_{ppn}^n$. All the same as the composition $[X, Y]_{ppn} \times [Y, Z]_{ppn} \to [X, Z]_{ppn}$ but using Lemma 3.11 instead of Lemma 3.4, we can defined the composition $[X, Y]_{ppn}^n \times [Y, Z]_{ppn}^n \to [X, Z]_{ppn}^n$ by using proper n-proximate Čech nets as follows: $[(g_\mathscr{W})]_p^n \circ [(f_\mathscr{V})]_p^n = [(g_\mathscr{W} f_{g(\mathscr{W})})]_p^n$, where $g : \operatorname{cov}(Z) \to \operatorname{cov}(Y)$ is a function such that $g_\mathscr{W}(g(\mathscr{W})) \prec \mathscr{W}$ for every $\mathscr{W} \in \operatorname{cov}(Z)$. Thus we obtain the category $\mathscr{P}_p^n$ of locally compact separable metrizable spaces proper n-homotopy classes of proper n-proximate nets.

5. The proper n-shape theory and the Čech expansion

Let $\mathscr{H}_p^n$ be the category whose objects are locally compact separable metrizable spaces and whose morphisms are the proper n-homotopy classes of proper maps. By $\mathscr{H}_p^n\mathrm{Pol}$, we denote the full subcategory of $\mathscr{H}_p^n$ whose objects are spaces having the proper n-homotopy classes of polyhedra. The *proper n-shape category* $\mathscr{S}_p^n$ is defined as the category whose objects are locally compact spaces and whose morphisms from X to Y are natural transformations from $[Y, -]_p^n$ to $[X, -]_p^n$ called a *proper n-shaping* (cf. [3]), where $[X, -]_p^n$ is the functor from $\mathscr{H}_p^n\mathrm{Pol}$ to the category of sets.

Let $\mathbf{p} : X \to \mathbf{X}$ be a morphism in the pro-category pro-$\mathscr{H}_p^n$ from a locally compact space X to an inverse system $\mathbf{X}$ in $\mathscr{H}_p^n\mathrm{Pol}$. We call $\mathbf{p}$ is an $\mathscr{H}_p^n\mathrm{Pol}$-*expansion* of X if it satisfies the following:

- for any inverse system $\mathbf{Y}$ in $\mathscr{H}_p^n\mathrm{Pol}$ and any morphism $\mathbf{q} : X \to \mathbf{Y}$ in pro-$\mathscr{H}_p^n$ from X to $\mathbf{Y}$, there exists a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathscr{H}_p^n$ such that $\mathbf{q} = \mathbf{fp}$.

By [16, Ch. §2, Theorem 1], $\mathbf{p} = (p_\lambda)_{\lambda\in\Lambda} : X \to \mathbf{X} = (X_\lambda, p_\lambda^{\lambda'}, \Lambda)$ is an $\mathscr{H}_p^n\mathrm{Pol}$-expansion of X if and only if the following conditions are satisfied:

1. for any locally compact polyhedron P and any proper map $f : X \to P$, there exist $\lambda \in \Lambda$ and a proper map $q : X_\lambda \to P$ such that $f \overset{n}{\simeq}_p qp_\lambda$;

2. for any locally compact polyhedron P and any two proper maps $g, h : X_\lambda \to P$ satisfying $gp_\lambda \overset{n}{\simeq}_p hp_\lambda$, there exists $\lambda' \geqslant \lambda$ such that $gp_\lambda^{\lambda'} \overset{n}{\simeq}_p hp_\lambda^{\lambda'}$.

The following is Theorem 2.1 in [18]:

THEOREM 5.1. *The category* $\mathscr{H}_p^n\mathrm{Pol}$ *is dense in* $\mathscr{H}_p^n$, *that is, every locally compact space* X *admits an* $\mathscr{H}_p^n\mathrm{Pol}$-*expansion.*

Due to [16, Ch. I, §2], this theorem guarantees the shape theory for the pair $(\mathscr{H}_p^n, \mathscr{H}_p^n\mathrm{Pol})$. By [16, Ch. I, §2, Theorem 7], the shape category defined to the pair $(\mathscr{H}_p^n, \mathscr{H}_p^n\mathrm{Pol})$ is isomorphic to the proper n-shape category $\mathscr{S}_p^n$ defined as the above.

We call $\check{C}(X) = (K_{\mathscr{U}}, \phi_{\mathscr{U}}^{\mathscr{V}}, \mathrm{cov}(X))$ the *Čech system* for X, where $\mathrm{cov}(X)$ is directed by the order $[\mathscr{U} \leqslant \mathscr{V}] \equiv [\mathscr{U} \succ \mathscr{V}]$. Then $[\check{C}(X)]_p^n = (K_{\mathscr{U}}, [\phi_{\mathscr{U}}^{\mathscr{V}}]_p^n, \mathrm{cov}(X))$ is an inverse system in $\mathscr{H}_p^n$ and $\phi_X = ([\phi_{\mathscr{U}}]_p^n)_{\mathscr{U} \in \mathrm{cov}(X)} : X \to [\check{C}(X)]_p^n$ is a morphism in pro-$\mathscr{H}_p^n$, where $\phi_{\mathscr{U}} : X \to K_{\mathscr{U}}$ is a canonical map. As mentioned in Introduction, the same approach as in [16] cannot be available to show the following theorem. We apply the results in §3 to prove the following theorem.

THEOREM 5.2. *For every locally compact space* X, *the morphism* ϕ_X *in* pro-$\mathscr{H}_p^n$ *is an* $\mathscr{H}_p^n\mathrm{Pol}$-*expansion of* X, *which is called the Čech expansion of* X.

Proof. (1): Let $f : X \to P$ be a proper map from X to a locally compact polyhedron P. Since P is a locally compact ANR, there exists $\mathscr{V} \in \mathrm{cov}(P)$ such that any two $\mathscr{V}$-close proper maps from an arbitrary space to P are proper homotopic. Choose $\mathscr{V}_1, \mathscr{V}_2 \in \mathrm{cov}(P)$ so that $\mathscr{V}_1$ is a star-refinement of $\mathscr{V}$ and $\mathrm{st}\,\mathscr{V}_2$ is a Lefschetz refinement of $\mathscr{V}_1$. Let $\mathscr{U} \in \mathrm{cov}(X)$ be a refinement of $f^{-1}(\mathscr{V}_2)$. Since $f\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(0)}$ is a partial $\mathrm{st}\,\mathscr{V}_2$-realization of $K_{\mathscr{U}}$, it extends to a full $\mathscr{V}_1$-realization $q : K_{\mathscr{U}} \to P$. Since $f \overset{\mathscr{V}_1}{=} f\psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{V}_1}{=} q\phi_{\mathscr{U}}$, we have $f \simeq_p q\phi_{\mathscr{U}}$, hence $f \overset{n}{\simeq}_p q\phi_{\mathscr{U}}$.

(2): Let $g, h : K_{\mathscr{U}} \to P$ be two proper maps from $K_{\mathscr{U}}$ to a locally compact polyhedron P such that $g\phi_{\mathscr{U}} \overset{n}{\simeq}_p h\phi_{\mathscr{U}}$. Since P is a locally compact ANR, there exists $\mathscr{U}_P \in \mathrm{cov}(P)$ satisfying Lemma 3.9. Let $K'_{\mathscr{U}}$ be a subdivision of $K_{\mathscr{U}}$ such that the star-covering $\mathscr{K} = \{\mathrm{St}(v, K'_{\mathscr{U}}) \mid v \in (K'_{\mathscr{U}})^{(0)}\}$ refines both $g^{-1}(\mathscr{U}_P)$ and $h^{-1}(\mathscr{U}_P)$. Let $i : K'_{\mathscr{U}} \to K_{\mathscr{U}}$ be a simplicial approximation of the identity. Then $gi \simeq_p g$ and $hi \simeq_p h$. Let $\mathscr{U}' \in \mathrm{cov}(X)$ be a refinement of $\phi_{\mathscr{U}}^{-1}(\mathscr{K})$. Then there exists a simplicial map $p : K_{\mathscr{U}'} \to K'_{\mathscr{U}}$ such that $\phi_{\mathscr{U}}^{\mathscr{U}'} \simeq_p ip$. By the assumption, there exists $\mathscr{V} \geqslant \mathscr{U}'$ such that $g\phi_{\mathscr{U}}\psi_{\mathscr{V}} \overset{(\mathscr{V},n)}{\sim_p} h\phi_{\mathscr{U}}\psi_{\mathscr{V}}$ by Lemma 3.7. Since $\phi_{\mathscr{U}} \simeq_p \phi_{\mathscr{U}}^{\mathscr{U}'}\phi_{\mathscr{U}'}$, $gip\phi_{\mathscr{U}'}\psi_{\mathscr{V}} \overset{(\mathscr{U},n)}{\sim_p} hip\phi_{\mathscr{U}'}\psi_{\mathscr{V}}$. Since $\phi_{\mathscr{U}'}^{\mathscr{V}} \overset{\mathscr{U}'^{\circ}}{=} \phi_{\mathscr{U}'}\psi_{\mathscr{V}}$, $gip\phi_{\mathscr{U}'}^{\mathscr{V}} \overset{\mathscr{U}}{=} gip\phi_{\mathscr{U}'}\psi_{\mathscr{V}}$ and $hip\phi_{\mathscr{U}'}^{\mathscr{V}} \overset{\mathscr{U}}{=} hip\phi_{\mathscr{U}'}\psi_{\mathscr{V}}$. Thus, $g\phi_{\mathscr{U}}^{\mathscr{V}} \simeq_p^n gip\phi_{\mathscr{U}'}^{\mathscr{V}} \overset{(\mathscr{V},n)}{\sim_p} hip\phi_{\mathscr{U}'}^{\mathscr{V}} \simeq_p^n h\phi_{\mathscr{U}}^{\mathscr{V}}$, which implies that $g\phi_{\mathscr{U}}^{\mathscr{V}} \simeq_p^n h\phi_{\mathscr{U}}^{\mathscr{V}}$, i.e., $[g]_p^n[\phi_{\mathscr{U}}^{\mathscr{V}}]_p^n = [h]_p^n[\phi_{\mathscr{U}}^{\mathscr{V}}]_p^n$. □

By the above theorem, we can describe $\mathscr{S}_p^n$ by using Čech systems. A *proper n-morphism* $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$ from $\check{C}(X)$ to $\check{C}(Y)$ consists of a function $f : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ and proper maps $f_{\mathscr{V}} : K_{f(\mathscr{V})} \to K_{\mathscr{V}}$, $\mathscr{V} \in \mathrm{cov}(Y)$, satisfying the condition: for each $\mathscr{V}' \prec \mathscr{V} \in \operatorname{cov}(Y)$ there is some $\mathscr{U} \in \operatorname{cov}(X)$ such that $f_{\mathscr{V}}\phi^{\mathscr{U}}_{f(\mathscr{V})} \overset{n}{\simeq}_p \phi^{\mathscr{V}'}_{\mathscr{V}} f_{\mathscr{V}'}\phi^{\mathscr{U}}_{f(\mathscr{V}')}$, which is equivalent to the condition: $f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{n}{\simeq}_p \phi^{\mathscr{V}'}_{\mathscr{V}} f_{\mathscr{V}'}\phi_{f(\mathscr{V}')}$. The *composition* of proper n-morphisms $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$ and $(g_{\mathscr{W}}, g) : \check{C}(Y) \to \check{C}(Z)$ is defined by $(g_{\mathscr{W}} f_{g(\mathscr{W})}, fg) : \check{C}(X) \to \check{C}(Z)$. Two proper n-morphisms $(f_{\mathscr{V}}, f), (f'_{\mathscr{V}}, f') : \check{C}(X) \to \check{C}(Y)$ are *properly n-homotopic* to each others provided for each $\mathscr{V} \in \operatorname{cov}(Y)$ there is some $\mathscr{U} \in \operatorname{cov}(X)$ such that $\mathscr{U} \prec f(\mathscr{V}), f'(\mathscr{V})$ and $f_{\mathscr{V}}\phi^{\mathscr{U}}_{f(\mathscr{V})} \overset{n}{\simeq}_p f'_{\mathscr{V}}\phi^{\mathscr{U}}_{f'(\mathscr{V})}$, which is equivalent to the condition: $f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{n}{\simeq}_p f'_{\mathscr{V}}\phi_{f'(\mathscr{V})}$. It should be noted that every proper n-morphism $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$ is properly n-homotopic to a proper n-morphism $(f'_{\mathscr{V}}, f') : \check{C}(X) \to \check{C}(Y)$ such that each $f'_{\mathscr{V}} : K_{f'(\mathscr{V})} \to K_{\mathscr{V}}$ is simplicial. In fact, for each $f_{\mathscr{V}} : K_{f(\mathscr{V})} \to K_{\mathscr{V}}$, we have $\mathscr{U} \prec f(\mathscr{V})$ (say $f'(\mathscr{V}) = \mathscr{U}$) and a simplicial map $f'_{\mathscr{V}} : K_{\mathscr{U}} \to K_{\mathscr{V}}$ such that $f'_{\mathscr{V}}\phi_{\mathscr{U}}$ is contiguous (hence properly homotopic) to $f_{\mathscr{V}}\phi_{f(\mathscr{V})}$. By $[(f_{\mathscr{V}}, f)]^n_p$, we denote the proper n-homotopy class of a proper n-morphism $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$, which is a proper n-shape morphism from X to Y. Thus we have the proper n-shape category $\mathscr{S}^n_p$.

6. A categorical isomorphism

In this section, we construct a categorical isomorphism $\eta : \mathscr{S}^n_p \to \mathscr{P}^n_p$. We start to show the following:

LEMMA 6.1. *Let $(f_{\mathscr{V}}, f), (f'_{\mathscr{V}}, f') : \check{C}(X) \to \check{C}(Y)$ be proper n-morphisms. Then,*

1. *$(\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})})$ is a proper n-proximate Čech net from X to Y;*
2. *$(\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})}) \overset{n}{\simeq}_p (\psi_{\mathscr{V}} f'_{\mathscr{V}}\phi_{f'(\mathscr{V})})$ if and only if $(f_{\mathscr{V}}, f) \overset{n}{\simeq}_p (f'_{\mathscr{V}}, f')$.*

Proof. (1): First note that each $\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})} : X \to Y$ is a proper $\mathscr{V}$-continuous function. For $\mathscr{V}' \prec \mathscr{V}$, since $f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{n}{\simeq}_p \phi^{\mathscr{V}'}_{\mathscr{V}} f_{\mathscr{V}'}\phi_{f(\mathscr{V}')}$, it follows from Lemmas 3.7 and 3.10 that

$$\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{(\mathscr{V},n)}{\sim_p} \psi_{\mathscr{V}}\phi^{\mathscr{V}'}_{\mathscr{V}} f_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \overset{\mathscr{V}}{=} \psi_{\mathscr{V}'} f_{\mathscr{V}'}\phi_{f(\mathscr{V}')}.$$

Then $(\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})})$ is a proper n-proximate Čech net from X to Y.

(2): First, assume that $(f_{\mathscr{V}}, f) \overset{n}{\simeq}_p (f'_{\mathscr{V}}, f')$, that is, $f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{n}{\simeq}_p f'_{\mathscr{V}}\phi_{f'(\mathscr{V})}$, for each $\mathscr{V} \in \operatorname{cov}(Y)$. It follows from Lemmas 3.7 and 3.10 that $\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{(\mathscr{V},n)}{\sim_p} \psi_{\mathscr{V}} f'_{\mathscr{V}}\phi_{f'(\mathscr{V})}$ for each $\mathscr{V} \in \operatorname{cov}(Y)$, hence $(\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})}) \overset{n}{\simeq}_p (\psi_{\mathscr{V}} f'_{\mathscr{V}}\phi_{f'(\mathscr{V})})$.

Conversely, assume that $(\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})}) \overset{n}{\simeq}_p (\psi_{\mathscr{V}} f'_{\mathscr{V}}\phi_{f'(\mathscr{V})})$, that is, $\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{(\mathscr{V},n)}{\sim_p} \psi_{\mathscr{V}} f'_{\mathscr{V}}\phi_{f'(\mathscr{V})}$ for each $\mathscr{V} \in \operatorname{cov}(Y)$. By Lemma 3.9, we have $\mathscr{W} \in \operatorname{cov}(K_{\mathscr{V}})$ such that $\mathscr{W} \prec \mathscr{V}^{\circ}$ and $g \overset{(\mathscr{W},n)}{\sim_p} g'$ implies $g \overset{n}{\simeq}_p g$ for any proper maps $g, g' : X \to K_{\mathscr{V}}$. Let K' be a subdivision of $K_{\mathscr{V}}$ such that the star of each vertex of K' is contained in some member of $\mathscr{W}$. Then we have $\mathscr{V}' \in \operatorname{cov}(Y)$ and a simplicial isomorphism $h : K_{\mathscr{V}'} \to K'$ such that $\mathscr{V}' \prec \mathscr{V}$ and $h \simeq_p \phi^{\mathscr{V}}_{\mathscr{V}'}$. Since $\psi_{\mathscr{V}'} f_{\mathscr{V}'} \phi_{f(\mathscr{V}')} \overset{(\mathscr{V}',n)}{\sim_p} \psi_{\mathscr{V}'} f'_{\mathscr{V}'} \phi_{f'(\mathscr{V}')}$ and $h\phi_{\mathscr{V}'}(\mathscr{V}') \prec h(\mathscr{V}'^{\circ}) \prec \mathscr{W}$, we have

$$hf_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \overset{\mathscr{W}}{=} h\phi_{\mathscr{V}'}\psi_{\mathscr{V}'}f_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \overset{(\mathscr{V},n)}{\sim_p} h\phi_{\mathscr{V}'}\psi_{\mathscr{V}'}f'_{\mathscr{V}'}\phi_{f'(\mathscr{V}')} \overset{\mathscr{W}}{=} hf'_{\mathscr{V}'}\phi_{f'(\mathscr{V}')},$$

hence $hf_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \overset{(\mathscr{W},n)}{\sim_p} hf'_{\mathscr{V}'}\phi_{f'(\mathscr{V}')}$, which implies $hf_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \overset{n}{\simeq}_p hf'_{\mathscr{V}'}\phi_{f'(\mathscr{V}')}$. Then it follows that

$$\begin{aligned} f_{\mathscr{V}}\phi_{f(\mathscr{V})} \overset{n}{\simeq}_p \phi^{\mathscr{V}'}_{\mathscr{V}} f_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \simeq_p hf_{\mathscr{V}'}\phi_{f(\mathscr{V}')} \overset{n}{\simeq}_p hf'_{\mathscr{V}'}\phi_{f'(\mathscr{V}')} \\ \simeq_p \phi^{\mathscr{V}'}_{\mathscr{V}} f'_{\mathscr{V}'}\phi_{f'(\mathscr{V}')} \overset{n}{\simeq}_p f'_{\mathscr{V}}\phi_{f'(\mathscr{V}')}, \end{aligned}$$

which implies that $f_{\mathscr{V}}\phi_{f(\mathscr{V})} \simeq_p f'_{\mathscr{V}}\phi_{f'(\mathscr{V})}$. Therefore, $(f_{\mathscr{V}}, f) \overset{n}{\simeq}_p (f'_{\mathscr{V}}, f')$. □

By the above lemma, we can define $\eta : \mathscr{S}^n_p \to \mathscr{P}^n_p$ by $\eta(X) = X$ for each object X of $\mathscr{S}^n_p$ and $\eta([(f_{\mathscr{V}}, f)]^n_p) = [(\psi_{\mathscr{V}} f_{\mathscr{V}} \phi_{f(\mathscr{V})})]^n_p$ for each proper n-morphism $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$.

LEMMA 6.2. *$\eta : \mathscr{S}^n_p \to \mathscr{P}^n_p$ is a functor.*

Proof. In case $f = \mathrm{id}_{\operatorname{cov}(X)}$ and $f_{\mathscr{U}} = \mathrm{id}_{K_{\mathscr{U}}}$ for each $\mathscr{U} \in \operatorname{cov}(X)$, since $\psi_{\mathscr{U}} f_{\mathscr{U}} \phi_{f(\mathscr{U})} = \psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{U}}{=} \mathrm{id}_X$, we have $\eta([(f_{\mathscr{V}}, f)]^n_p) = [(\mathrm{id}_X)]^n_p \in [X, Y]^n_{ppn}$.

Let $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$ and $(g_{\mathscr{W}}, g) : \check{C}(Y) \to \check{C}(Z)$ be proper n-morphisms. It can be assumed that each $g_{\mathscr{W}}$ is simplicial. We have to show that

$$\eta([(g_{\mathscr{W}}, g)]^n_p[(f_{\mathscr{V}}, f)]^n_p) = \eta([(g_{\mathscr{V}}, f)]^n_p)\eta([(f_{\mathscr{V}}, f)]^n_p).$$

Observe that

$$\begin{aligned} \eta([(g_{\mathscr{W}}, g)]^n_p[(f_{\mathscr{V}}, f)]^n_p) &= \eta([(g_{\mathscr{W}} f_{g(\mathscr{W})}, fg)]^n_p) = [(\psi_{\mathscr{W}} g_{\mathscr{W}} f_{g(\mathscr{W})}\phi_{fg(\mathscr{W})})]^n_p; \\ \eta([(g_{\mathscr{W}}, g)]^n_p)\eta([(f_{\mathscr{V}}, f)]^n_p) &= [(\psi_{\mathscr{W}} g_{\mathscr{W}}\phi_{g(\mathscr{W})})]^n_p[(\psi_{\mathscr{V}} f_{\mathscr{V}}\phi_{f(\mathscr{V})})]^n_p \\ &= [(\psi_{\mathscr{W}} g_{\mathscr{W}}\phi_{g(\mathscr{W})}\psi_{g'(\mathscr{W})} f_{g'(\mathscr{W})}\phi_{f(g'(\mathscr{W}))})]^n_p, \end{aligned}$$

where $g' : \operatorname{cov}(Z) \to \operatorname{cov}(Y)$ is a function such that $\psi_{\mathscr{W}} g_{\mathscr{W}}\phi_{g(\mathscr{W})}(g'(\mathscr{W})) \prec \mathscr{W}$ for each $\mathscr{W} \in \operatorname{cov}(Z)$. We may assume that $g'(\mathscr{W}) \prec g(\mathscr{W})$ for each $\mathscr{W} \in \operatorname{cov}(Z)$. Since each $V \in g'(\mathscr{W})$ is contained in $\phi^{g'(\mathscr{W})}_{g(\mathscr{W})}(V) \in g(\mathscr{W})$ and $\phi^{g'(\mathscr{W})}_{g(\mathscr{W})} : K_{g'(\mathscr{W})} \to K_{g(\mathscr{W})}$ is simplicial, we have

$$\phi_{g(\mathscr{W})}\psi_{g'(\mathscr{W})}(V^{\circ}) \subset \phi_{g(\mathscr{W})}(V) \subset \phi_{g(\mathscr{W})}(\phi^{g'(\mathscr{W})}_{g(\mathscr{W})}(V)) \subset \phi^{g'(\mathscr{W})}_{g(\mathscr{W})}(V)^{\circ} = \phi^{g'(\mathscr{W})}_{g(\mathscr{W})}(V^{\circ}),$$

which implies that $\phi_{g(\mathscr{W})}\psi_{g'(\mathscr{W})} \overset{g(\mathscr{W})^{\circ}}{=} \phi^{g'(\mathscr{W})}_{g(\mathscr{W})}$. Since $g_{\mathscr{W}}$ is simplicial, $\psi_{\mathscr{W}} g_{\mathscr{W}}(g(\mathscr{W})^{\circ}) \prec \psi_{\mathscr{W}}(\mathscr{W}^{\circ}) \prec \mathscr{W}$, hence $\psi_{\mathscr{W}} g_{\mathscr{W}}\phi_{g(\mathscr{W})}\psi_{g'(\mathscr{W})} \overset{\mathscr{W}}{=} \psi_{\mathscr{W}} g_{\mathscr{W}}\phi^{g'(\mathscr{W})}_{g(\mathscr{W})}$. Thus we have

$$\psi_{\mathscr{W}} g_{\mathscr{W}}\phi_{g(\mathscr{W})}\psi_{g'(\mathscr{W})} f_{g'(\mathscr{W})}\phi_{f(g'(\mathscr{W}))} \overset{\mathscr{W}}{=} \psi_{\mathscr{W}} g_{\mathscr{W}}\phi^{g'(\mathscr{W})}_{g(\mathscr{W})} f_{g'(\mathscr{W})}\phi_{f(g'(\mathscr{W}))}.$$

On the other hand, $\phi_{g(\mathscr{W})}^{g'(\mathscr{W})} f_{g'(\mathscr{W})}\phi_{f(g'(\mathscr{W}))} \overset{n}{\simeq}_p f_{g(\mathscr{W})}\phi_{f(g(\mathscr{W}))}$, which implies that

$$\psi_{\mathscr{W}} g_{\mathscr{W}} \phi_{g(\mathscr{W})}^{g'(\mathscr{W})} f_{g'(\mathscr{W})}\phi_{f(g'(\mathscr{W}))} \overset{(\mathscr{W},n)}{\sim_p} \psi_{\mathscr{W}} g_{\mathscr{W}} f_{g(\mathscr{W})}\phi_{f(g(\mathscr{W}))}$$

by Lemma 3.10. Consequently,

$$\psi_{\mathscr{W}} g_{\mathscr{W}} \phi_{g(\mathscr{W})}\psi_{g'(\mathscr{W})} f_{g'(\mathscr{W})}\phi_{f(g'(\mathscr{W}))} \overset{(\mathscr{W},n)}{\sim_p} \psi_{\mathscr{W}} g_{\mathscr{W}} f_{g(\mathscr{W})}\phi_{f(g(\mathscr{W}))}.$$

This completes the proof. □

THEOREM 6.3. *The functor $\eta : \mathscr{S}_p^n \to \mathscr{P}_p^n$ is a categorical isomorphism.*

Proof. By Lemma 6.1(2), η gives an injection between morphisms of $\mathscr{S}_p^n$ and ones of $\mathscr{P}_p^n$. Then it is left to show that η gives a surjection between morphisms of $\mathscr{S}_p^n$ and ones of $\mathscr{P}_p^n$.

To this end, let $(f_{\mathscr{V}}) : X \to Y$ be a proper n-proximate Čech net. By Lemma 3.9, we have $\mathscr{W}_{\mathscr{V}} \in \operatorname{cov}(K_{\mathscr{V}})$, $\mathscr{V} \in \operatorname{cov}(Y)$, such that $\mathscr{W}_{\mathscr{V}} \prec \mathscr{V}^\circ$ and $g \overset{(\mathscr{W},n)}{\sim_p} g'$ implies $g \overset{n}{\simeq}_p g$ for any proper maps $g, g' : X \to K_{\mathscr{V}}$. Let $K'_{\mathscr{V}}$ be a subdivision of $K_{\mathscr{V}}$ such that the star at each vertex of $K'_{\mathscr{V}}$ is contained in some member of $\mathscr{W}_{\mathscr{V}}$. There exists $\tilde{\mathscr{V}} \in \operatorname{cov}(Y)$ with a simplicial isomorphism $h_{\mathscr{V}} : K_{\tilde{\mathscr{V}}} \to K'_{\mathscr{V}}$ such that $\tilde{\mathscr{V}} \prec \mathscr{V}$ and $h_{\mathscr{V}} \simeq_p \phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}$. We have $\varphi : \operatorname{cov}(Y) \to \operatorname{cov}(X)$ such that $f_{\tilde{\mathscr{V}}}(\varphi(\mathscr{V})) \prec \tilde{\mathscr{V}}$. For each $\mathscr{V} \in \operatorname{cov}(Y)$, let $\varphi_{\mathscr{V}} : K_{\varphi(\mathscr{V})} \to K_{\tilde{\mathscr{V}}}$ be a simplicial map associated with $f_{\tilde{\mathscr{V}}}$. Then $(\phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}, \varphi) : \check{C}(X) \to \check{C}(Y)$ is a proper n-morphism. In fact, for $\mathscr{V}' \prec \mathscr{V}$, choose $\mathscr{V}_0 \prec \tilde{\mathscr{V}}', \tilde{\mathscr{V}}$. Let $\varphi_0 : K_{\mathscr{U}} \to K_{\mathscr{V}_0}$ be a simplicial map associated with $\phi_{\mathscr{V}_0} f_{\mathscr{V}_0}$. Since $h_{\mathscr{V}}(\tilde{\mathscr{V}}^\circ) \prec \mathscr{W}_{\mathscr{V}}$ and $f_{\tilde{\mathscr{V}}} \overset{(\tilde{\mathscr{V}},n)}{\sim_p} f_{\mathscr{V}_0}$,

$$\begin{aligned}
\phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})} \simeq_p h_{\mathscr{V}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})} &\overset{\mathscr{W}_{\mathscr{V}}}{=} h_{\mathscr{V}}\phi_{\tilde{\mathscr{V}}} f_{\tilde{\mathscr{V}}} \overset{(\mathscr{W}_{\mathscr{V}},n)}{\sim_p} h_{\mathscr{V}}\phi_{\tilde{\mathscr{V}}} f_{\mathscr{V}_0} \\
&\overset{\mathscr{W}_{\mathscr{V}}}{=} h_{\mathscr{V}}\phi_{\tilde{\mathscr{V}}}^{\mathscr{V}_0}\phi_{\mathscr{V}_0} f_{\mathscr{V}_0} \overset{\mathscr{W}_{\mathscr{V}}}{=} h_{\mathscr{V}}\phi_{\tilde{\mathscr{V}}}^{\mathscr{V}_0}\varphi_0\phi_{\mathscr{U}} \simeq_p \phi_{\mathscr{V}}^{\mathscr{V}_0}\varphi_0\phi_{\mathscr{U}},
\end{aligned}$$

hence $\phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})} \overset{n}{\simeq}_p \phi_{\mathscr{V}}^{\mathscr{V}_0}\varphi_0\phi_{\mathscr{U}}$. Similarly, we have $\phi_{\mathscr{V}'}^{\tilde{\mathscr{V}}'}\varphi_{\mathscr{V}'}\phi_{\varphi(\mathscr{V}')} \overset{n}{\simeq}_p \phi_{\mathscr{V}'}^{\mathscr{V}_0}\varphi_0\phi_{\mathscr{U}}$, hence

$$\phi_{\mathscr{V}}^{\mathscr{V}'}\phi_{\mathscr{V}'}^{\tilde{\mathscr{V}}'}\varphi_{\mathscr{V}'}\phi_{\varphi(\mathscr{V}')} \overset{n}{\simeq}_p \phi_{\mathscr{V}}^{\mathscr{V}'}\phi_{\mathscr{V}'}^{\mathscr{V}_0}\varphi_0\phi_{\mathscr{U}} \overset{n}{\simeq}_p \phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})}.$$

Since $\psi_{\mathscr{V}}\phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})} \overset{\mathscr{V}}{=} \psi_{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})} \overset{\tilde{\mathscr{V}}}{=} f_{\tilde{\mathscr{V}}} \overset{(\mathscr{V},n)}{\sim_p} f_{\mathscr{V}}$ for each $\mathscr{V} \in \operatorname{cov}(Y)$, we have

$$\eta([(\phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}, \varphi)]_p^n) = [(\psi_{\mathscr{V}}\phi_{\mathscr{V}}^{\tilde{\mathscr{V}}}\varphi_{\mathscr{V}}\phi_{\varphi(\mathscr{V})})]_p^n = [(f_{\mathscr{V}})]_p^n.$$

This completes the proof. □

7. Approach by using multi-valued maps

In this section, we consider proper upper multi-valued functions instead of non-continuous functions.

Let $F : X \to Y$ be a multi-valued function. For $A \subset X$, we denote $F(A) = \bigcup_{x \in A} F(x) \subset Y$. A multi-valued function $F : X \to Y$ is called *upper semi-continuous* (u.s.c.) if for each neighborhood U of $F(x)$ there exists a neighborhood V of x in X such that $F(V) \subset U$. For simplicity, F is here called a *$\mathfrak{K}$-function* if F is u.s.c. and $F(x) \neq \emptyset$ is compact for each $x \in X$.

LEMMA 7.1. *Let $F : X \to Y$ and $G : Y \to Z$ be $\mathfrak{K}$-functions. Then GF is also a $\mathfrak{K}$-function.*

Proof. Let $x \in X$ and let $\mathscr{W}$ be an open cover of $GF(x)$ in Z, i.e., $GF(x) = G(F(x)) \subset \bigcup \mathscr{W}$. For each $y \in F(x)$, since $G(y)$ is compact, $G(y)$ is covered by finite $W_1^y, \cdots, W_{k(y)}^y \in \mathscr{W}$. Since G is u.s.c., y has an open neighborhood V_y in Y such that $G(V_y) \subset \bigcup_{i=1}^{k(y)} W_i^y$. Because of compactness of $F(x)$, $F(x) \subset \bigcup_{j=1}^{l} V_{y_j}$ for some $y_1, \cdots, y_l \in F(x)$, whence

$$GF(x) \subset G(\textstyle\bigcup_{j=1}^{l} V_{y_j}) \subset \displaystyle\bigcup_{j=1}^{l} G(V_{y_j}) \subset \bigcup_{j=1}^{l} \bigcup_{i=1}^{k(y_j)} W_i^{y_j}.$$

Thus $GF(x)$ is compact. Moreover, since F is u.s.c., we have an open neighborhood U of x in X such that $F(U) \subset \bigcup_{j=1}^{l} V_{y_j}$. Then $GF(U) \subset \bigcup \mathscr{W}$. By the arbitrariness of $\mathscr{W}$, this implies that GF is u.s.c. □

A $\mathfrak{K}$-function $F : X \to Y$ is *proper* if for any compact set $D \subset Y$ there is a compact set $C \subset X$ such that $F(X \setminus C) \subset Y \setminus D$. For $\mathscr{V} \in \operatorname{cov}(Y)$, F is said to be *$\mathscr{V}$-small* if there exists $\mathscr{U} \in \operatorname{cov}(X)$ such that $F(\mathscr{U}) = \{F(U) \mid U \in \mathscr{U}\} \prec \mathscr{V}$. Two multi-valued functions $F, G : X \to Y$ are *$\mathscr{V}$-close* if $F(x) \subset \operatorname{st}(G(x), \mathscr{V})$ and $G(x) \subset \operatorname{st}(F(x), \mathscr{V})$ for each $x \in X$. Lemma 3.1 is valid for $\mathfrak{K}$-functions.

LEMMA 7.2. *A $\mathfrak{K}$-function $F : X \to Y$ is proper if it is $\mathscr{V}$-close to a proper $\mathfrak{K}$-function $G : X \to Y$ for some $\mathscr{V} \in \operatorname{cov}(Y)$.*

Proof. For any compact set $D \subset Y$, let D' be the closure of $\operatorname{st}(D, \mathscr{V})$ in Y. Then D' is compact. Choose a compact set $C \subset X$ so that $G(X \setminus C) \subset Y \setminus D'$. For each $x \in X \setminus C$, $G(x) \cap \operatorname{st}(D, \mathscr{V}) = \emptyset$, hence $\operatorname{st}(G(x), \mathscr{V}) \cap D = \emptyset$. Since $F(x) \subset \operatorname{st}(G(x), \mathscr{V})$, we have $F(x) \cap D = \emptyset$. Therefore, $F(X \setminus C) \subset Y \setminus D$. □

Two $\mathscr{V}$-small proper $\mathfrak{K}$-functions $F, G : X \to Y$ are *properly $\mathscr{V}$-homotopic* to each others (written by $F \overset{\mathscr{V}}{\sim}_p G$) if there is a $\mathscr{V}$-small proper $\mathfrak{K}$-function $H : X \times [0, 1] \to Y$ such that $H_0 = F$ and $H_1 = G$, where $H_t(x) = H(x, t)$. It is easy to see that $\overset{\mathscr{V}}{\sim}_p$ is an equivalence relation among $\mathscr{V}$-small proper u.s.c. $\mathfrak{K}$-functions because of upper semi-continuity.[2] We call H a *proper $\mathscr{V}$-small $\mathfrak{K}$-homotopy* from F to G.

Corresponding to Lemmas 3.2, 3.3 and 3.4, we have the following:

[2]If the upper semi-continuity is not assumed, then we have to define similarly to §3, i.e., in the above, H is st $\mathscr{V}$-small and $H|W$ is $\mathscr{V}$-small for some neighborhood W of $X \times \{0, 1\}$ in $X \times [0, 1]$.

LEMMA 7.3. *For any two proper $\mathscr{V}$-small $\mathfrak{K}$-functions $F, G : X \to Y$, $F \stackrel{\mathscr{V}}{=} G$ implies $F \stackrel{\mathscr{V}}{\sim}_p G$.*

Proof. Define $H : X \times [0,1] \to Y$ by $H_0 = F$ and $H_t = G$ for $t > 0$. Since $F \stackrel{\mathscr{V}}{=} G$, for each $x \in X$ there exists $V \in \mathscr{V}$ such that $F(x) \cup G(x) \subset V$. Since F and G are u.s.c., there exist open neighborhoods U_1 and U_2 of x in X such that $F(U_1) \subset V$ and $G(U_2) \subset V$. Then $H((U_1 \cap U_2) \times [0,1]) \subset V$, which implies that H is a proper $\mathscr{V}$-small $\mathfrak{K}$-homotopy from F to G. □

LEMMA 7.4. *Let $\mathscr{W} \in \operatorname{cov}(Z)$ and $G, G' : Y \to Z$ be proper $\mathscr{W}$-small $\mathfrak{K}$-functions with $G \stackrel{\mathscr{W}}{\sim}_p G'$. Then there exists $\mathscr{V} \in \operatorname{cov}(Y)$ such that $GF \stackrel{\mathscr{W}}{\sim}_p G'F'$ for any proper $\mathscr{V}$-small $\mathfrak{K}$-functions $F, F' : X \to Y$ of an arbitrary space X with $F \stackrel{\mathscr{V}}{\sim}_p F'$.*

Proof. Let $H' : Y \times [0,1] \to Z$ be a proper $\mathscr{W}$-small $\mathfrak{K}$-homotopy from G to G'. Since $[0,1]$ is compact, we can choose $\mathscr{V} \in \operatorname{cov}(Y)$ so that for each $V \in \mathscr{V}$, there are $0 = t_0^V < t_1^V < \cdots < t_{n(V)}^V = 1$ such that each $H'(V \times [t_{i-1}^V, t_i^V])$ is contained in some member of $\mathscr{W}$. Let $H'' : X \times [0,1] \to Y$ be a proper $\mathscr{V}$-small $\mathfrak{K}$-homotopy from F and F'. We define $\tilde{H}'' : X \times [0,1] \to Y \times [0,1]$ by $\tilde{H}''(x,t) = H''(x) \times \{t\}$. Then $\tilde{H}''$ is clearly a proper $\mathfrak{K}$-function. Thus we have a proper $\mathfrak{K}$-function $H = H'\tilde{H}'' : X \times [0,1] \to Z$ such that $H_t = H_t'H_t''$ for each $t \in [0,1]$, hence $H_0 = GF$ and $H_1 = G'F'$. Choose $\mathscr{U} \in \operatorname{cov}(X)$ so that, for each $U \in \mathscr{U}$, there are $0 = s_0^U < s_1^U < \cdots < s_{m(U)}^U = 1$ such that each $H''(U \times [s_{j-1}^U, s_j^U])$ is contained in some $V \in \mathscr{V}$. Then it can be assumed that each $[s_{j-1}^U, s_j^U]$ is contained in some $[t_{i-1}^V, t_i^V]$, whence each $H(U \times [s_{i-1}^U, s_i^U])$ is contained in in some member of $\mathscr{W}$. Thus H is $\mathscr{W}$-small. We can conclude $GF \stackrel{\mathscr{W}}{\sim}_p G'F'$. □

For $\mathscr{U} \in \operatorname{cov}(X)$, let $\{F_U \mid U \in \mathscr{U}\}$ be a closed shrinking of $\mathscr{U}$ (i.e., $\bigcup_{U \in \mathscr{U}} F_U = X$, F_U is closed in X and $F_U \subset U$ for each $U \in \mathscr{U}$) such that, for each $U_1, \cdots, U_n \in \mathscr{U}$, $\bigcap_{i=1}^n F_U \neq \emptyset$ if and only if $\bigcap_{i=1}^n U \neq \emptyset$. (Recall we assume that each $U \in \mathscr{U}$ is not an empty set.) We define a multi-valued function $\Psi_{\mathscr{U}} : K_{\mathscr{U}} \to X$ by $\Psi_{\mathscr{U}}(x) = \bigcap_{U \in \sigma^{(0)}} F_U \subset \bigcap_{U \in \sigma^{(0)}} U$ where σ is the carrier of x in $K_{\mathscr{U}}$ (i.e., σ is the simplex of $K_{\mathscr{U}}$ with $x \in \mathring{\sigma}$).

LEMMA 7.5. *$\Psi_{\mathscr{U}} : K_{\mathscr{U}} \to X$ is a proper $\mathscr{U}$-small $\mathfrak{K}$-function. In fact, $\Psi_{\mathscr{U}}(U^\circ) \subset U$ for each $U \in \mathscr{U}$, hence $\Psi_{\mathscr{U}}(\mathscr{U}^\circ) \prec \mathscr{U}$.*

Proof. Consider the graph $\mathrm{Gr}\Psi_{\mathscr{U}}$ of $\Psi_{\mathscr{U}}$:

$$\mathrm{Gr}\Psi_{\mathscr{U}} = \bigcup \{\mathring{\sigma} \times \textstyle\bigcap_{U \in \sigma^{(0)}} F_U \mid \sigma \text{ is a simplex of } K_{\mathscr{U}}\}.$$

It is easy to see that $\mathrm{Gr}\Psi_{\mathscr{U}}$ is closed in $K_{\mathscr{U}} \times X$, which implies that $\Psi_{\mathscr{U}}$ is u.s.c. Hence Ψ is a $\mathfrak{K}$-function.

For each compactum $D \subset X$, let $\mathscr{U}_D = \{U \in \mathscr{U} \mid U \cap D \neq \emptyset\}$. Since $\mathscr{U}_D$ is finite, $C = \bigcup_{U \in \mathscr{U}_D} \mathrm{St}(U, K_{\mathscr{U}})$ is compact. For each $x \in K_{\mathscr{U}} \setminus C$, let

σ is the carrier of x in $K_{\mathcal{U}}$. Since $U \cap D = \emptyset$ for each $U \in \sigma^{(0)}$, we have $\Psi_{\mathcal{U}}(x) = \bigcap_{U \in \sigma^{(0)}} F_U \subset \bigcap_{U \in \sigma^{(0)}} U \subset X \setminus D$, which implies that $\Psi_{\mathcal{U}}$ is proper.

Let $U \in \mathcal{U}$. For each simplex σ in $K_{\mathcal{U}}$ with $U \in \sigma^{(0)}$, $\Psi_{\mathcal{U}}(\mathring{\sigma}) \subset \bigcap_{V \in \sigma^{(0)}} F_V \subset F_U \subset U$. Since $U^\circ = \bigcup\{\mathring{\sigma} \mid U \in \sigma^{(0)}\}$, we have $\Psi_{\mathcal{U}}(U^\circ) \subset U$. $\square$

LEMMA 7.6. *For each $\mathcal{U} \in \operatorname{cov}(X)$, $\Psi_{\mathcal{U}}\phi_{\mathcal{U}} \overset{\mathcal{U}}{=} \mathrm{id}_X$ and $\phi_{\mathcal{U}}\Psi_{\mathcal{U}} \overset{\mathcal{U}^\circ}{=} \mathrm{id}_{K_{\mathcal{U}}}$. If $\mathcal{U}' \prec \mathcal{U}$ then $\Psi_{\mathcal{U}}\phi_{\mathcal{U}}^{\mathcal{U}'} \overset{\mathcal{U}}{=} \Psi_{\mathcal{U}'}$.*

Proof. Since $\Psi_{\mathcal{U}}\phi_{\mathcal{U}}(U) \subset \Psi_{\mathcal{U}}(U^\circ) \subset U$ for each $U \in \mathcal{U}$, we have $\Psi_{\mathcal{U}}\phi_{\mathcal{U}} \overset{\mathcal{U}}{=} \mathrm{id}_X$. Since $\phi_{\mathcal{U}}\Psi_{\mathcal{U}}(U^\circ) \subset \phi_{\mathcal{U}}(U) \subset U^\circ$ for each $U \in \mathcal{U}$, we have $\phi_{\mathcal{U}}\Psi_{\mathcal{U}} \overset{\mathcal{U}^\circ}{=} \mathrm{id}_{K_{\mathcal{U}}}$. For each $U \in \mathcal{U}'$,

$$\Psi_{\mathcal{U}'}(U^\circ) \subset U \subset \phi_{\mathcal{U}}^{\mathcal{U}'}(U) \text{ and } \Psi_{\mathcal{U}}\phi_{\mathcal{U}}^{\mathcal{U}'}(U^\circ) \subset \Psi_{\mathcal{U}}\big(\phi_{\mathcal{U}}^{\mathcal{U}'}(U)^\circ\big) \subset \phi_{\mathcal{U}}^{\mathcal{U}'}(U).$$

Then $\Psi_{\mathcal{U}}\phi_{\mathcal{U}}^{\mathcal{U}'} \overset{\mathcal{U}}{=} \Psi_{\mathcal{U}'}$. $\square$

Let $\mathcal{V} \in \operatorname{cov}(Y)$ and $n \in \mathbb{N} \cup \{\infty\}$. Two proper $\mathcal{V}$-small $\mathfrak{K}$-functions $F, G : X \to Y$ are *properly $\mathcal{V}$-small n-homotopic* (written by $F \overset{(\mathcal{V},n)}{\sim_p} G$) if there is some $\mathcal{U} \in \operatorname{cov}(X)$ such that $F(\mathcal{U}), G(\mathcal{U}) \prec \mathcal{V}$ and $F\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$ ($F\Psi_{\mathcal{U}} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}$ in case $n = \infty$). Then $F\Psi_{\mathcal{U}'}|K_{\mathcal{U}'}^{(n)} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}'}|K_{\mathcal{U}'}^{(n)}$ for every $\mathcal{U}' \prec \mathcal{U} \in \operatorname{cov}(X)$. In fact, since $\Psi_{\mathcal{U}}\phi_{\mathcal{U}}^{\mathcal{U}'} \overset{\mathcal{U}}{\sim}_p \Psi_{\mathcal{U}'}$ and $\phi_{\mathcal{U}}^{\mathcal{U}'}$ is continuous, it follows that

$$F\Psi_{\mathcal{U}'}|K_{\mathcal{U}'}^{(n)} \overset{\mathcal{V}}{\sim}_p F\Psi_{\mathcal{U}}\phi_{\mathcal{U}}^{\mathcal{U}'}|K_{\mathcal{U}'}^{(n)} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}\phi_{\mathcal{U}}^{\mathcal{U}'}|K_{\mathcal{U}'}^{(n)} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}'}|K_{\mathcal{U}'}^{(n)}.$$

It is easy to see that the relation $\overset{(\mathcal{V},n)}{\sim_p}$ is an equivalence relation on the set of proper $\mathcal{V}$-small $\mathfrak{K}$-functions from X to Y.

Corresponding to Lemma 3.6, we have the following:

LEMMA 7.7. *For any proper $\mathcal{V}$-small $\mathfrak{K}$-functions $F, G : X \to Y$, $F \overset{\mathcal{V}}{\sim}_p G$ if and only if $F \overset{(\mathcal{V},\infty)}{\sim_p} G$. Hence, $F \overset{\mathcal{V}}{\sim}_p G$ implies $F \overset{(\mathcal{V},n)}{\sim_p} G$ for every $n \in \mathbb{N}$.*

Proof. First, assume $F \overset{\mathcal{V}}{\sim}_p G$. By Lemma 7.4, we have $\mathcal{U} \in \operatorname{cov}(X)$ such that $F\Psi_{\mathcal{U}} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}$, that is, $F \overset{(\mathcal{V},\infty)}{\sim_p} G$. Conversely, assume $F \overset{(\mathcal{V},\infty)}{\sim_p} G$. Choose $\mathcal{U} \in \operatorname{cov}(X)$ so that then $F(\mathcal{U}), G(\mathcal{U}) \prec \mathcal{V}$ and $F\Psi_{\mathcal{U}} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}$. Then $F\Psi_{\mathcal{U}}\phi_{\mathcal{U}} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}\phi_{\mathcal{U}}$ by Lemma 7.4. Since $\Psi_{\mathcal{U}}\phi_{\mathcal{U}} \overset{\mathcal{U}}{\sim}_p \mathrm{id}_X$ by Lemmas 7.6 and 7.3, it follows that $F \overset{\mathcal{U}}{\sim}_p F\Psi_{\mathcal{V}}\phi_{\mathcal{U}} \overset{\mathcal{V}}{\sim}_p G\Psi_{\mathcal{U}}\phi_{\mathcal{V}} \overset{\mathcal{U}}{\sim}_p G$. $\square$

For each $\mathcal{V}$-small $\mathfrak{K}$-function $F : X \to Y$, we define similarly simplicial map $\varphi : K_{\mathcal{U}} \to K_{\mathcal{V}}$ associated with F, that is, $f(U) \subset \varphi(U) \in \mathcal{V} = K_{\mathcal{U}}^{(0)}$. Similarly to Lemma 3.8, we have

LEMMA 7.8. *Let $F : X \to Y$ be a $\mathcal{V}$-small $\mathfrak{K}$-function and $\varphi : K_{\mathcal{U}} \to K_{\mathcal{V}}$ a simplicial map associated with F. Then, $\varphi\phi_{\mathcal{U}} \overset{\mathcal{V}^\circ}{=} \phi_{\mathcal{V}} F$ and $\Psi_{\mathcal{V}}\varphi\phi_{\mathcal{U}} \overset{\mathcal{V}}{=} F$. Hence, if F is proper then so is φ.* □

Corresponding to Lemmas 3.10 and 3.11, we have the following:

LEMMA 7.9. *Let $G, G' : Y \to Z$ be $\mathcal{W}$-small $\mathfrak{K}$-functions with $G \overset{(\mathcal{V},n)}{\sim_p} G'$, where $\mathcal{W} \in \mathrm{cov}(Z)$. Then there exists $\mathcal{V} \in \mathrm{cov}(Y)$ such that $GF \overset{(\mathcal{V},n)}{\sim_p} G'F'$ for any proper $\mathcal{V}$-small $\mathfrak{K}$-functions $F, F' : X \to Y$ of an arbitrary space X with $F \overset{(\mathcal{V},n)}{\sim_p} F'$.*

Proof. Choose $\mathcal{V} \in \mathrm{cov}(Y)$ so that $F(\mathcal{V}), G(\mathcal{V}) \prec \mathcal{W}$ and $G\Psi_{\mathcal{V}}|K_{\mathcal{V}}^{(n)} \overset{\mathcal{W}}{\sim_p} G'\Psi_{\mathcal{V}}|K_{\mathcal{V}}^{(n)}$. For two proper $\mathcal{V}$-small $\mathfrak{K}$-functions $F, F' : X \to Y$, suppose that $F(\mathcal{U}), F'(\mathcal{U}) \prec \mathcal{V}$ and $F\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{V}}{\sim_p} F'\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$. Let $\varphi : K_{\mathcal{U}} \to K_{\mathcal{V}}$ be a simplicial map associated with F. Since $\varphi(K_{\mathcal{U}}^{(n)}) \subset K_{\mathcal{V}}^{(n)}$, it follows from Lemmas 7.4 and 7.3 that

$$GF\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{W}}{\sim_p} G\Psi_{\mathcal{V}}\varphi\phi_{\mathcal{U}}\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{W}}{\sim_p} G'\Psi_{\mathcal{V}}\varphi\phi_{\mathcal{U}}\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}$$
$$\overset{\mathcal{W}}{\sim_p} G'F\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)} \overset{\mathcal{W}}{\sim_p} G'F'\Psi_{\mathcal{U}}|K_{\mathcal{U}}^{(n)}.$$

The lemma is proved. □

A proper Čech $(n$-$)$multi-net $(F_{\mathcal{V}}) : X \to Y$ is a net of proper $\mathcal{V}$-small $\mathfrak{K}$-functions $F_{\mathcal{V}} : X \to Y$ indexed by the directed set $\mathrm{cov}(Y)$ such that $F_{\mathcal{V}} \overset{\mathcal{V}}{\sim_p} F_{\mathcal{V}'}$ $(F_{\mathcal{V}} \overset{(\mathcal{V},n)}{\sim_p} F_{\mathcal{V}'})$ for $\mathcal{V}' \prec \mathcal{V} \in \mathrm{cov}(Y)$. Two proper Čech $(n$-$)$multi-nets $(F_{\mathcal{V}}), (G_{\mathcal{V}}) : X \to Y$ are *properly $(n$-$)$homotopic* if $F_{\mathcal{V}} \overset{\mathcal{V}}{\sim_p} G_{\mathcal{V}}$ $(F_{\mathcal{V}} \overset{(\mathcal{V},n)}{\sim_p} G_{\mathcal{V}})$ for each $\mathcal{V} \in \mathrm{cov}(Y)$. By $[(F_{\mathcal{V}})]_p$ $([(F_{\mathcal{V}})]_p^n)$, we denote the proper $(n$-$)$homotopy classes of proper Čech $(n$-$)$multi-nets. Similarly to Section 4, we can discuss by using $(n$-$)$multi-nets indexed by arbitrary directed sets, but for simplicity, we use only Čech $(n$-$)$multi-nets which are indexed by open covers.

LEMMA 7.10. *Let $(F_{\mathcal{V}}), (F'_{\mathcal{V}}) : X \to Y$ and $(G_{\mathcal{W}}), (G'_{\mathcal{W}}) : Y \to Z$ be proper Čech multi-nets and $g, g' : \mathrm{cov}(Z) \to \mathrm{cov}(Y)$ functions such that $G_{\mathcal{W}}(g(\mathcal{W})), G'_{\mathcal{W}}(g'(\mathcal{W})) \prec \mathcal{W}$ for each $\mathcal{W} \in \mathrm{cov}(Z)$. Then, (1) $(G_{\mathcal{W}}F_{g(\mathcal{W})})$ is a proper Čech multi-net; (2) $(F_{\mathcal{V}}) \simeq_p (F'_{\mathcal{V}})$ and $(G_{\mathcal{W}}) \simeq_p (G'_{\mathcal{W}})$ imply $(G_{\mathcal{W}}F_{g(\mathcal{W})}) \simeq_p (G'_{\mathcal{W}}F'_{g'(\mathcal{W})})$.*

Proof. (1): By Lemma 7.1, each $G_{\mathcal{W}}F_{g(\mathcal{W})}$ is a proper $\mathfrak{K}$-function. For each $\mathcal{W}' \prec \mathcal{W} \in \mathrm{cov}(Z)$, $G_{\mathcal{W}} \overset{\mathcal{W}}{\sim_p} G_{\mathcal{W}'}$. Since $F_{g(\mathcal{W})} \overset{g(\mathcal{W})}{\sim_p} F_{\mathcal{V}} \overset{g(\mathcal{W}')}{\sim_p} F_{g(\mathcal{W}')}$ for each $\mathcal{V} \prec g(\mathcal{W}), g(\mathcal{W}')$, it follows from by Lemma 7.4 that $G_{\mathcal{W}}F_{g(\mathcal{W})} \overset{\mathcal{W}}{\sim_p} G_{\mathcal{W}'}F_{\mathcal{V}} \overset{\mathcal{W}}{\sim_p} G_{\mathcal{W}'}F_{g(\mathcal{W})}$.

(2): For each $\mathcal{W} \in \mathrm{cov}(Z)$, $G_{\mathcal{W}} \overset{\mathcal{W}}{\sim_p} G'_{\mathcal{W}}$. Since $F_{g(\mathcal{W})} \overset{g(\mathcal{W})}{\sim_p} F_{\mathcal{V}} \overset{g(\mathcal{W}')}{\sim_p} F'_{\mathcal{V}} \overset{g(\mathcal{W}')}{\sim_p} F'_{g(\mathcal{W}')}$, it follows from Lemma 7.4 that $G_{\mathcal{W}}F_{g(\mathcal{W})} \overset{\mathcal{W}}{\sim_p} G_{\mathcal{W}}F_{\mathcal{V}} \overset{\mathcal{W}}{\sim_p} G'_{\mathcal{W}}F'_{\mathcal{V}} \overset{\mathcal{W}}{\sim_p}$

$G'_{\mathscr{W}} F'_{g(\mathscr{W}')}$. $\square$

For any proper Čech multi-net $(G_{\mathscr{W}}) : Y \to Z$ and $\mathscr{V} \in \mathrm{cov}(Z)$, since each $G_{\mathscr{W}}$ is u.s.c., there exists a function $g : \mathrm{cov}(Z) \to \mathrm{cov}(Y)$ such that $G_{\mathscr{W}}(g(\mathscr{W})) \prec \mathscr{W}$. By Lemma 7.10, we can define $[(G_{\mathscr{W}})]_p \circ [(F_{\mathscr{V}})]_p = [(G_{\mathscr{W}} F_{g(\mathscr{V})})]_p$. Thus we obtain the category $\mathscr{M}_p$ of locally compact separable metrizable space and proper homotopy classes of proper Čech multi-nets.

Replacing $\overset{\mathscr{V}}{\sim}_p$ by $\overset{(\mathscr{V},n)}{\sim}_p$, we can define $\mathscr{M}_p^n$ of locally compact separable metrizable space and proper n-homotopy classes of proper Čech n-multi-nets.

We can also define $\eta' : \mathscr{S}_p^n \to \mathscr{M}_p^n$ similar to η by $\eta'(X) = X$ for each object X of $\mathscr{S}_p^n$ and $\eta'([(f_{\mathscr{V}}, f)]_p^n) = [(\Psi_{\mathscr{V}} f_{\mathscr{V}} \phi_{f(\mathscr{V})})]_p^n$ for each proper n-morphism $(f_{\mathscr{V}}, f) : \check{C}(X) \to \check{C}(Y)$. By the same arguments of Section 5, we have the following

THEOREM 7.11. *$\eta' : \mathscr{S}_p^n \to \mathscr{M}_p^n$ is a categorical isomorphism.* $\square$

8. Remarks on the n-shape theory

In this section, *spaces are not assumed to be locally compact.*

Let $\mathscr{H}^n$ be the n-homotopy category whose objects are spaces in a suitable class (here is considered the class of separable metrizable spaces or the class of compact Hausdorff spaces). By $\mathscr{H}^n$Pol, we denote the full subcategory of $\mathscr{H}^n$ whose objects are spaces having the n-homotopy classes of polyhedra. The *n-shape category $\mathscr{S}^n$* is defined as the category whose objects are same as $\mathscr{H}^n$ and whose morphisms from X to Y are natural transformations from $[Y, -]^n$ to $[X, -]^n$ called a *n-shaping*, where $[X, -]^n$ is the functor from $\mathscr{H}^n$Pol to the category of sets. On the other hand, if the category $\mathscr{H}^n$Pol is dense in $\mathscr{H}^n$ (i.e., every space in the considered class has an $\mathscr{H}^n$Pol-expansion), then it follows from [16, Ch.I, §2, Theorem 7] that the n-shape category $\mathscr{S}^n$ defined as above is isomorphic to the shape theory for the pair $(\mathscr{H}^n, \mathscr{H}^n\mathrm{Pol})$ discussed in [16, Ch.I, §2]. Thus the following question is important:

Does every space X in the considered class have an $\mathscr{H}^n$Pol-expansion?
Or, is the Čech expansion of X an $\mathscr{H}^n$Pol-expansion?

Here the Čech expansion of X means the morphism in pro-$\mathscr{H}^n$ below:

$$\phi_X = ([\phi_{\mathscr{U}}]^n)_{\mathscr{U} \in \mathrm{cov}(X)} : X \to [\check{C}(X)]^n = (K_{\mathscr{U}}, [\phi_{\mathscr{U}}^{\mathscr{V}}]^n, \mathrm{cov}(X)).$$

In case spaces are **separable metrizable**, removing the words "proper(ly)" and "locally compact", we can obtain the same results on the n-shape category of separable metrizable spaces as the previous sections. But now Lemmas 3.1 and 7.2 are nonsense. Spaces cannot be considered any longer as closed sets in $Q \setminus \{\mathrm{pt}\}$. Hence we have to make some changes in the proof of the corresponding result to Lemma 3.7. In the proof of Lemma 3.7, replace $Q \setminus \{\mathrm{pt}\}$ by AR's M_X and M_Y which contain X and Y as closed sets, respectively. Now we consider $X \subset M_X \subset Q$.

Then, since $\xi^{-1}(X)$ is a closed set in $\xi^{-1}(M_X)$ and $\xi|\xi^{-1}(M_X) : \xi^{-1}(M_X) \to M_X$ is n-invertible by the definition of n-invertibility, we can obtain the result by the same arguments. Among the same results as previous sections, we can answer affirmatively to the above question in case X is separable metrizable.

THEOREM 8.1. *For any separable metrizable space X, the Čech expansion of X is an $\mathscr{H}^n$Pol-expansion, that is, the morphism ϕ_X in pro-$\mathscr{H}^n$ is an $\mathscr{H}^n$Pol-expansion of X.*

In case spaces are **compact Hausdorff**, let $\operatorname{cov}(X)$ be the collection of all finite covers of X. Now, spaces are non-metrizable, but it can be assumed that a canonical map $\phi_{\mathscr{U}} : X \to K_{\mathscr{U}}$ satisfies the condition that $\phi_{\mathscr{U}}(U) \subset U^\circ$ for each $U \in \mathscr{U}$ and $\phi_{\mathscr{U}}^{-1}(\mathring{\sigma}) \neq \emptyset$ for each simplex σ of $K_{\mathscr{U}}$. In fact, let $(k_U)_{U\in\mathscr{U}}$ be a partition of unity on X subordinated by $\mathscr{U}$ and let $V_U = k_U^{-1}((0,1])$ for each $U \in \mathscr{U}$. Then we have $\mathscr{V} = \{V_U \mid U \in \mathscr{U}, V_U \neq \emptyset\} \prec \mathscr{U}$. We define $\phi_{\mathscr{V}} : X \to K_{\mathscr{V}}$ by $\phi_{\mathscr{V}}(x) = \sum_{U\in\mathscr{U}} k_U(x)V_U \in K_{\mathscr{V}}$. Then $\phi_{\mathscr{V}} : X \to K_{\mathscr{V}}$ is a canonical map satisfying the above condition. Therefore, the subcollection of $\operatorname{cov}(X)$ satisfying the above condition is cofinal in $\operatorname{cov}(X)$.

Similarly to the above, we can obtain the same results on the n-shape category of compact Hausdorff spaces as the previous sections. However, by the same reason as the above, we need to prove the corresponding result to Lemma 3.7. By [12, Theorem 9], for any cardinal τ and $n \in \mathbb{N}$, there exists a compact Hausdorff space D_n^τ with a map $\xi_n^\tau : D_n^\tau \to I^\tau$ onto the Tychonoff cube I^τ such that

1. the weight of D_n^τ is τ and $\dim D_n^\tau = n$;
2. $D_n^\tau \in AE(n-1) \cap AE(n-2, n) \cap LC^{n-1} \cap C^{n-1}$;
3. D_n^τ is universal in the class of spaces of the weight τ and the dimension $\leqslant n$;
4. ξ_n^τ is $(n-1)$-soft, $(n-2,n)$-soft and polyhedral n-soft;
5. ξ_n^τ is universal in the class of maps between spaces of the weight τ and the dimension $\leqslant n$,

where the dimension is the Lebesgue covering dimension. For the definition of each term above, refer to [12]. We can apply this map $\xi_n^\tau : D_n^\tau \to I^\tau$ to prove the following:

LEMMA 8.2. *Let X and Y be compact Hausdorff. For maps $f, g : X \to Y$, $f \overset{n}{\simeq} g$ implies $f \overset{(\mathscr{V},n)}{\sim} g$ for every $\mathscr{V} \in \operatorname{cov}(Y)$.*

Proof. Embed X into the Tychonoff cube I^τ, where τ is the weight of X. Let $\xi_n^\tau : D_n^\tau \to I^\tau$ be the map above. For each $Z \subset I^\tau$, we denote $\tilde{Z} = (\xi_n^\tau)^{-1}(Z) \subset D_n^\tau$. Since $\dim \tilde{X} = n$ and $f \overset{n}{\simeq} g$, it follows that $f\xi_n^\tau|\tilde{X} \simeq g\xi_n^\tau|\tilde{X}$, hence $\phi_{\mathscr{V}} f\xi_n^\tau|\tilde{X} \simeq \phi_{\mathscr{V}} g\xi_n^\tau|\tilde{X}$. Since $K_{\mathscr{V}}$ is a neighborhood extensor for normal spaces, the maps $\phi_{\mathscr{V}} f$ and $\phi_{\mathscr{V}} g$ extend to maps $f', g' : N \to K_{\mathscr{V}}$, respectively, where N is an open neighborhood of X in I^τ. Replacing N with a smaller one if necessary, we can assume that $f'\xi_n^\tau|\tilde{N} \simeq g'\xi_n^\tau|\tilde{N}$. Choose $\mathscr{W} \in \operatorname{cov}(N)$ so that $f'(\operatorname{st}\mathscr{W}), g'(\operatorname{st}\mathscr{W}) \prec \mathscr{V}^\circ$. Since N is LC^{n-1}, $\mathscr{W}$ has an open refinement $\mathscr{W}'$ such that any partial $\mathscr{W}'$-realization of an arbitrary n-dimensional simplicial complex extends to a full $\mathscr{W}$-realization. Choose $\mathscr{U} \in \operatorname{cov}(X)$ so that $\operatorname{st}\mathscr{U} \prec \mathscr{W}'$. Then $\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(0)}$ is a partial $\mathscr{W}'$-realization of $K_{\mathscr{U}}^{(n)}$, hence it extends to a full $\mathscr{W}$-realization $h : K_{\mathscr{U}}^{(n)} \to N$. Since ξ_n^τ is polyhedral n-soft, we have $\tilde{h} : K_{\mathscr{U}}^{(n)} \to D_n^\tau$ such that $\xi_n^\tau \tilde{h} = h$, whence $\tilde{h}(K_{\mathscr{U}}^{(n)}) \subset \tilde{N}$. Then $f'\xi_n^\tau\tilde{h} \simeq g'\xi_n^\tau\tilde{h}$. On the other hand, $h\phi_{\mathscr{U}} \overset{\operatorname{st}\mathscr{W}}{=} \psi_{\mathscr{U}}\phi_{\mathscr{U}} \overset{\mathscr{U}}{=} \mathrm{id}_X$, hence $h\phi_{\mathscr{U}} \overset{\operatorname{st}\mathscr{W}}{\sim} \mathrm{id}_X$. Then $f'\xi_n^\tau\tilde{h}\phi_{\mathscr{U}} = f'h\phi_{\mathscr{U}} \overset{\mathscr{V}^\circ}{\sim} f'$, and similarly $g'\xi_n^\tau\tilde{h}\phi_{\mathscr{U}} \overset{\mathscr{V}^\circ}{\sim} g'$. Therefore, $f' \overset{\mathscr{V}^\circ}{\sim} g'$, which implies

$$\phi_{\mathscr{V}} f\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} = f'\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}^\circ}{\sim} g'\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} = \phi_{\mathscr{V}} g\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)}.$$

Since $\psi_{\mathscr{V}}\phi_{\mathscr{V}} \overset{\mathscr{V}}{=} \mathrm{id}_Y$ and $\psi_{\mathscr{V}}(\mathscr{V}^\circ) \prec \mathscr{V}$, it follows that

$$f\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}}{=} \psi_{\mathscr{V}}\phi_{\mathscr{V}} f\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}}{\sim} \psi_{\mathscr{V}}\phi_{\mathscr{V}} g\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)} \overset{\mathscr{V}}{=} g\psi_{\mathscr{U}}|K_{\mathscr{U}}^{(n)}.$$

This means that $f \overset{(\mathscr{V},n)}{\sim} g$. □

Among the same results as previous sections, we have the affirmative answer to the above question in case X is compact Hausdorff.

THEOREM 8.3. *For any compact Hausdorff space X, the Čech expansion of X is an $\mathscr{H}^n$Pol-expansion, that is, the morphism ϕ_X in pro-$\mathscr{H}^n$ is an $\mathscr{H}^n$Pol-expansion of X.*

We shall finish the paper with pointing out that, if Lemma 3.7 could be extended to more general spaces than the above, then we could weaken the restriction for spaces.

Theorem 8.1 is valid for non-separable metrizable spaces. To see this, it suffices to prove Lemma 3.7 for non-separable metrizable spaces. This can be shown as follows: For an arbitrary metrizable space X, there exists a convex set K in a Banach space E such that X can be embedded in K as a closed set and E has the same weight as X. Then, E is an n-invertible image of n-dimensional completely metrizable universal space P with the same weight as E by Theorem 3.8 of the following paper:

A. Chigagidze and V. Valov, Universal maps and surjective characterization of completely metrizable LC^n-spaces, Proc. Amer. Math. Soc. **109** (1990), 1125–1133.

Hence, K is an n-invertible image of a subspace of P. Thus, we can similarly prove Lemma 3.7 for non-separable metrizable spaces.

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