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A NOTE ON SHADOW PRICES

UDK / UDC: 330.4
JEL klasifikacija / JEL classification: C61
Izvorni znanstveni rad / Original scientific paper
Primljeno / Received: 19. prosinca 2011. / December 19, 2011
Prihvaćeno za tisak / Accepted for publishing: 13. lipnja 2012. / June 13, 2012

Abstract

In this paper we examine the method of Lagrange multipliers within the context of consumer optimal choices. The action of multipliers on constraints in the utility maximization problem is represented graphically as a multiplied distance between expenditures and income in the primal, and as a multiplied distance between total and given utilities in its dual. Based on those graphical representations we offer new interpretations of the optimal values of Lagrange multipliers as well as of the consumer's ordinal properties. Comparative statics results are confirmed analytically and placed within the general framework of non-linear relationships among variables that are relevant in general equilibrium analysis.

Keywords: ***Lagrange multipliers, line of force, line of indifference, equilibrium, concavity and quasi-concavity, comparative static***

* The authors would like to thank Mr. Vladimir Sukser for technical support with the graphs.

1. INTRODUCTION

There exists considerable literature on the method of Lagrange multipliers and their interpretations (Jehle and Reny, 2001, Mas-Collel, Whinston and Green, 1995). Often, this literature neglects the economic context and the possibility of the graphical representation of the multiplier action on the constraint (Binger and Hoffman, 1998). Strict mathematical form can sometimes separate the reader from the economic context. In this paper we expand on the idea of differential calculus by using graphical representations of multiplier actions to derive their economic interpretations. We present the obtained results within the general framework of non-linear objective functions and constraints that often appear in general equilibrium problems.

The basic idea in differential calculus is to replace non-linear relationships with linear ones. The function describing a non-linear relationship is replaced by a tangent and mathematical analysis is enriched by applying linear algebra. For instance, real change in the value of a utility function gets represented by the differential, which clarifies how a consumer's utility is affected by the direction in which he travels in the commodity space.

In this paper, staying within the context of rational consumer choices, we first derive equations that describe the critical point of the Lagrange function and, using a graphical representation of the multiplier's impact on the constraint, we obtain new economic interpretations. We confirm the comparative static results by letting the consumer go in the direction of the new equilibrium. In the dual problems of the consumer's choice we rely exclusively on consumer's ordinal properties and we give new interpretation to the notion of quasi-concavity. We place our general conclusions in the area of general equilibrium analysis. In the comparative static analysis we show the possibility of interchange between the tangent surfaces while both the objective function and constraints can remain non-linear.

2. REACHING EQUILIBRIUM BY WAY OF BSTACLES

Let us denote with $\mathbf{x} = (x_1,x_2)$ quantities of goods and with $\mathbf{dx} = (dx_1,dx_2)$ changes in those quantities. Changes in quantities describe the direction in the commodity space in which the consumer makes his decisions. The change in utility that we identify with the differential is equal to

$$du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 = \nabla u\mathbf{dx}. \quad (1)$$

The change of value on the function denotes the change in utility while the change of value on the tangential plane represents the differential. In fact, it is in the close environment of the point of tangency that the function and the tangential plane are taken to be the same thereby describing the non-linear relationships with linear ones. We see that this change can be expressed as the scalar product of the gradient and the vector of changes in quantities. Let us assume that the vector of changes in quantities has a unitary length. It follows that

$$du = \|\nabla u\| \|\mathbf{dx}\| \cos\alpha,$$
$$du = \|\nabla u\| \cos\alpha. \quad (2)$$

In previous expressions α denotes the angle between the gradient and the direction in which the consumer makes his choices. The cosine of that angle and the change in utility are the greatest in the direction of the gradient. Edgeworth calls this the „line of force“ or the „line of preference“ (Edgeworth, 1881, 21-22). If the direction in which the consumer changes his choices is perpendicular to the gradient, the cosinus is zero and there is no change in utility. That perpendicular direction Edgeworth calls the „line of indifference“. The direction of indifference inspires Pareto to replace cardinal utility with subjective preferences (Pareto, 1906, 118-120). What matters in this case is only what the greatest quantity of the other good that the consumer is willing to sacrifice for an additional unit of the first good is. This relationship determines the marginal rate of substitution between goods and it shows the direction of indifference (Varian, 2002).

The other important rate in the description of equilibrium is called the marginal rate of market transformation. It shows the direction in which consumer expenditures do not change. For the consumer whose income is limited, this is the direction of the obstacle. The direction of the obstacle is perpendicular to the direction in which consumer expenditures increase the fastest. It is described by prices $\mathbf{p} = (p_1, p_2) = \nabla e.$ Consumer expenditures are given as a linear function of quantities $e(x_1, x_2) = p_1 x_1 + p_2 x_2$.

The difference between the direction of indifference and the direction of the obstacle (constraint) enables the consumer to increase his utility. When the marginal rate of substitution between goods is greater than the marginal rate of market transformation, the consumer substitutes the second good with the first one sacrificing the smaller quantity of the second good than the one he is willing to sacrifice. Because of the non-satiation the difference in the quantity of the other good makes him better off. If the marginal rate of substitution between two goods is smaller than the marginal rate of market transformation, a rational consumer follows the direction of the obstacle and substitutes the first good with the second.

In equilibrium these two magnitudes are the same and the directions of indifference and of the obstacle are also the same. Moreover, the perpendicular directions, one of the force and the other of the direction in which consumer's expenditures increase the fastest, also coincide. It follows that the gradient of utility is proportional to the vector of prices,

$$\nabla u = \lambda \mathbf{p}. \tag{3}$$

The previous equations define the critical point in the Lagrange function,

$$L = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - M), \tag{4}$$

using the model of utility maximization with the budget constraint,

$$\begin{aligned} v(p_1, p_2, M) = \max_{x_1, x_2 \geq \mathbf{0}} \quad & u(x_1, x_2) \\ & p_1 x_1 + p_2 x_2 = M. \end{aligned} \tag{5}$$

In addition to prices among the parameters of the Lagrange function there also appears the consumer's income, M. It is represented by the optimal value of the multiplier λ and the maximum utility function.

3. THE INTERPRETATION OF THE LAGRANGE MULTIPLIER AND ITS GRAPH

The Lagrange function shows the distance between utility and the multiplied difference between expenditures and income. Graphically the difference between expenditures and income is represented by a plane that contains the budget line and that intercepts the *e*-axis in its negative segment at the income level. This is shown on Figure 1.

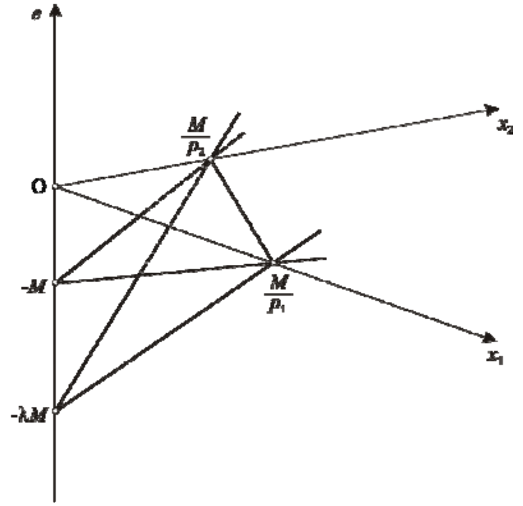

Figure 1. The difference and the multiplied difference between expenditures and income

Multiplying the difference between expenditures and income rotates the plane around the budget line until it intersects the multiplied income on the negative segment of the *e*-axis. If the utility function is concave, the consumer's equilibrium, according to which the direction of force and the direction of the fastest increase in expenditures are identical, also shows the greatest distance between utility and the multiplied difference between expenditures and income. This is shown on Figure 2.

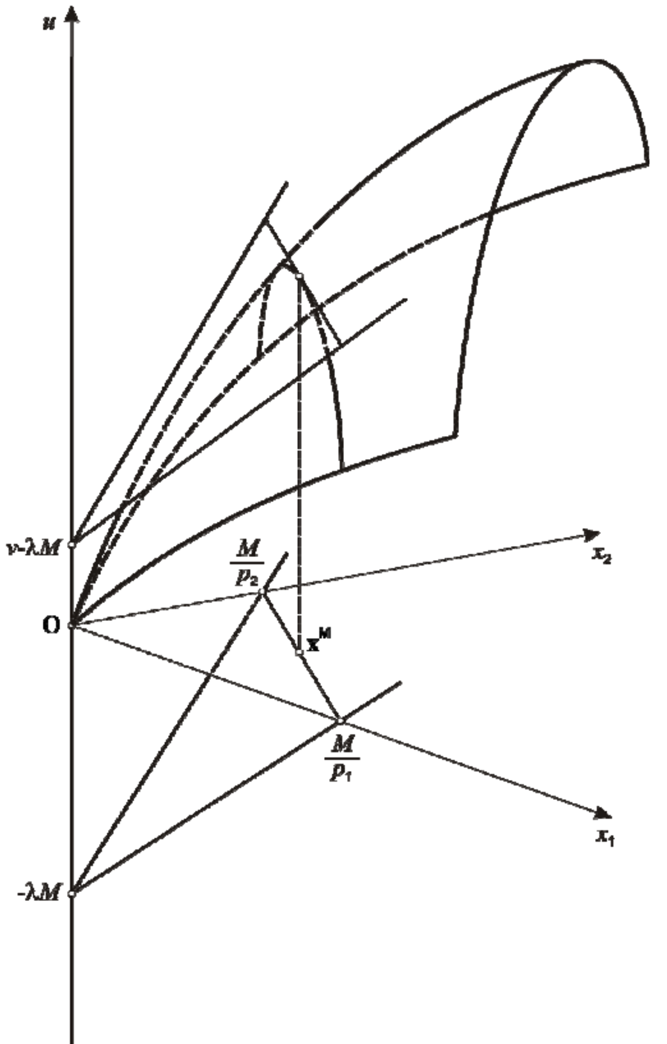

Figure 2. The distance between utility and the multiplied difference between expenditures and income

Optimal consumer choices and the optimal value of the Lagrange multiplier give the saddle point of the Lagrange function (Blume and Simon, 1994). If we translate the rotated constraint plane into the positive segment of the *e*-axis for the maximal consumer utility, it will be tangent to the utility surface above the optimal combination of goods. The utility surface in the close environment of that point can, according to Leibnitz, be identified with the obtained tangent plane.

Then the change in utility is given by the change in the *u* coordinate of the tangent plane or of the plane parallel to it. This change is also visible when the commodity space is represented by a straight line with which we cut that space perpendicularly and from the origin, as is shown on Figure 3.

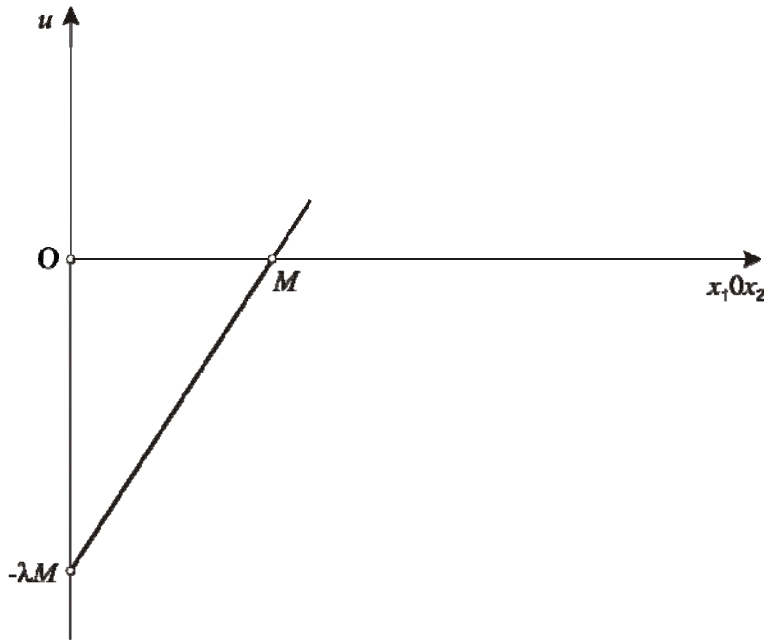

Figure 3. The change in income and the change in maximum utility

The horizontal axis represents the commodity space. The plane parallel to the tangent plane is given by the straight line intersecting the vertical axis at the multiplied income level. The intercept with the horizontal axis represents the budget constraint. When consumer's income increases from 0 to M monetary units, the intercept on the vertical axis goes from $-\lambda M$ to 0. In other words, an increase in income for *M* units is equivalent to a shift in the *u* coordinate on the vertical axis for λM. Therefore, an increase in income for a small unit increases the maximal consumer utility for λ. The economic interpretation of the optimal value of the Lagrange muliplier has been deduced from its graphical representation.

We can confirm the same result using also the description of equilibrium. In equilibrium the line of force is also the direction of the fastest increase in expenditures. The optimal value of the Lagrange multiplier is given by the quotient of the length of the utility gradient and the gradient of expenditures,

$$\|\nabla u\| = \lambda \|\nabla e\|,$$
$$\lambda = \frac{\|\nabla u\|}{\|\nabla e\|}. \tag{6}$$

When income increases the consumer goes in the direction of a new equilibrium, which, together with the directions of preference and the fastest increase in expenditures, forms an angle α. This angle enables connecting the respective gradients and changes,

$$\lambda = \frac{\|\nabla u\|\|\mathbf{dx}\|\cos\alpha}{\|\nabla e\|\|\mathbf{dx}\|\cos\alpha},\ \lambda = \frac{du}{de}. \tag{7}$$

It follows that the optimal value of the Lagrange multiplier is given as the ratio of change of utility and expenditures. This ratio represents the average change of utility or the change of utility per small unit of increase in expenditures. We call it marginal utility of money.

4. THE DUAL PROBLEM

An economic example that would illustrate the use of duality is an example of a subsidy given to a low income consumer to neutralize the impact of a tax on his welfare. The dual problem describing this situation requires solving the expenditure minimization problem for a given level of utility or

$$E(p_1, p_2, u) = \min_{x_1, x_2 \geq \mathbf{0}} \ p_1 x_1 + p_2 x_2$$
$$u(x_1, x_2) = u. \tag{8}$$

The objective function and the constraint in the dual problem change places. Now the constraint in the commodity space is represented by an indifference curve. It is natural to observe the distance between total and given utility, as Figure 4 illustrates.

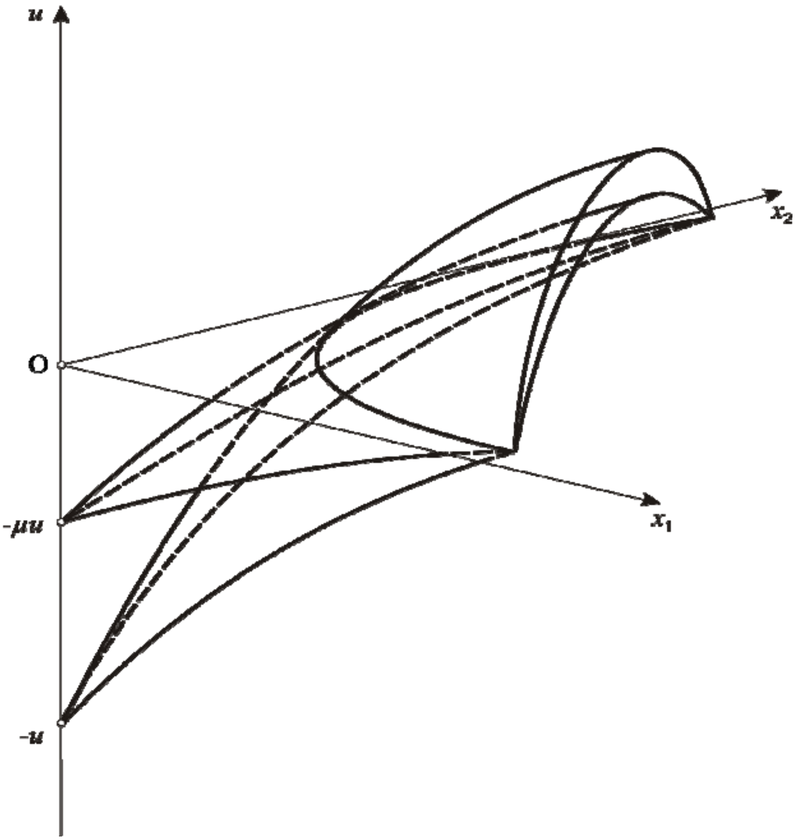

Figure 4. The difference and multiplied difference between total and given utility

Now we address the problem how to represent the multiplication of that distance graphically. Since the constraint is no longer linear we can not speak of rotation. It is essential to note that the shape of the distance does not change because positive multiplier does not affect signs of respective minors. It only stretches or shrinks along the *u*-axis until it reaches the multiplied level of utility on its negative segment. The distance between expenditures and the multiplied distance between total utility from the given utility is given by the Lagrange function.

$$L = p_1 x_1 + p_2 x_2 - \mu(u(x_1, x_2) - u). \qquad (9)$$

We are interested in that value of the Lagrange multiplier μ for which the shift in the positive direction of the multiplied distance between total and given utility for the amount of minimal expenditure is tangent to the expenditure plane above the optimal bundle. This is shown on Figure 5.

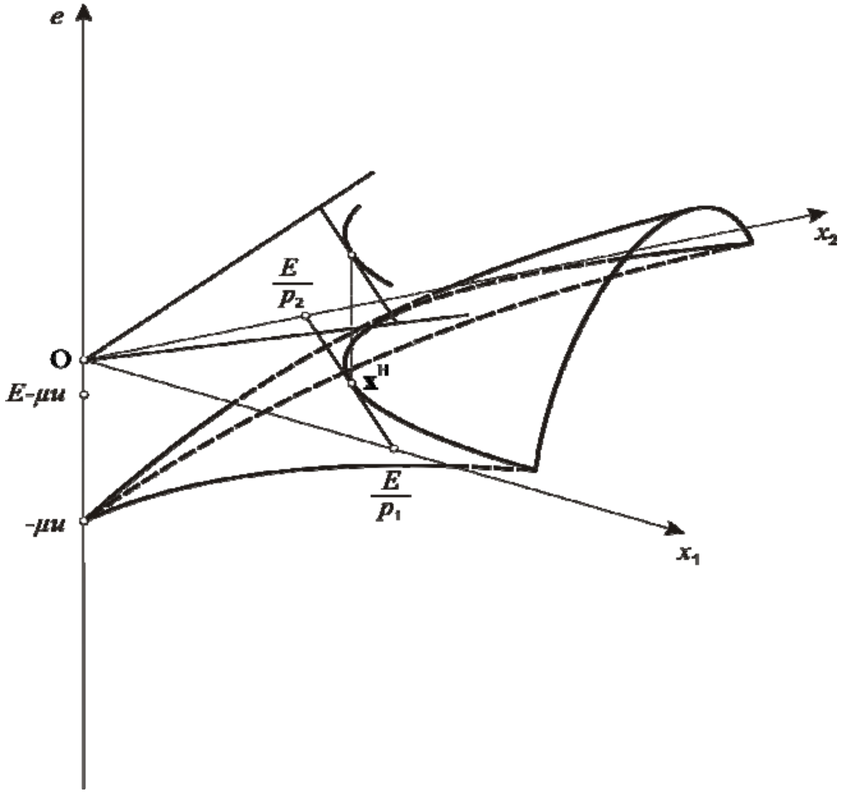

Figure 5. The distance between expenditures and multiplied distance between total and given utility

The condition for the tangetial solution gives the critical point of the Lagrange function and it establishes the relation of proportionality between the vector of prices and the gradient of the utility function.

$$\mathbf{p} = \mu \nabla u. \tag{10}$$

In a small neighborhood around that point, the level of expenditures can be regarded as the translated multiplied distance from the utility. Notice that by replacing the linear relationship with the non-linear one, we are simplifying the analysis. The change in expenditure is shown on the axis depicting changes in the mutiplied distance in the utility. This change is also visible when we represent the commodity space with a line that cuts that space vertically and passes through the origin, as Figure 6 shows.

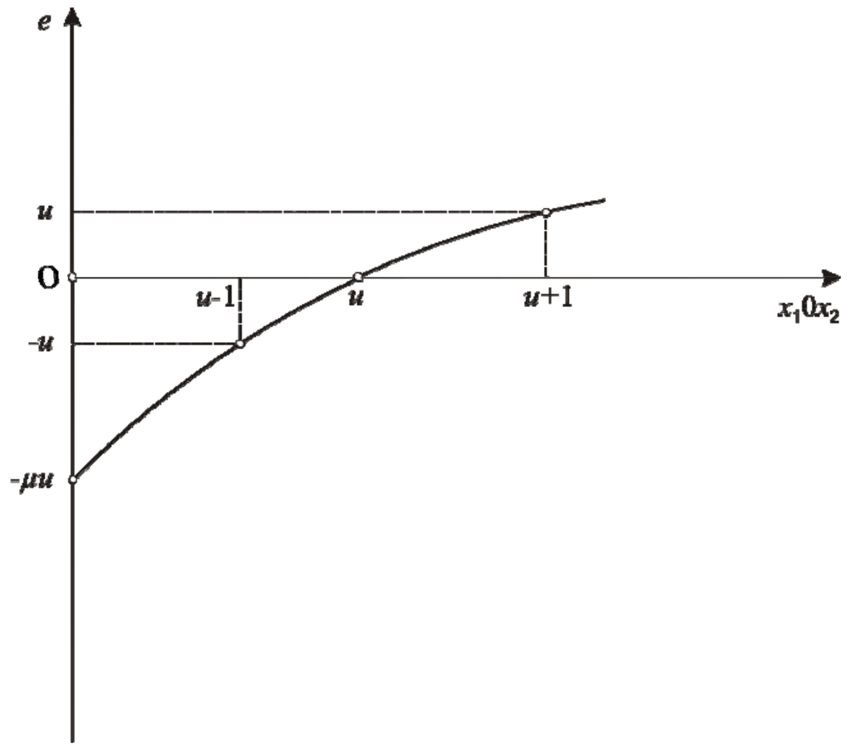

Figure 6. The change in given utility and the change in minimum expenditures

The horizontal axis represents the commodity space. The multiplied distance between total and given utility is represented by a curve that intercepts the multiplied utility level on the vertical axis. The intercept with the horizontal axis is given by the indifference curve. When the utility level for the consumer increases by *u*, the *e*-coordinate value increases by μu. Therefore, the optimal value of the Lagrange multiplier μ shows the increase in minimal expenditure per small unit of increase in utility. We can note that in the case of a concave function the same increase in utility on the horizontal axis is matched by the greater and greater distance between points giving the respective indifference curves. The graphical interpretation of the optimal value of the Lagrange multiplier is confirmed by the description of equilibrium according to which the gradients of expenditures and utility are proportional

$$\nabla e = \mu \nabla u. \tag{11}$$

The optimal value of the Lagrange multiplier gives the quotient of the lengths of gradients of expenditures and utility,

$$\|\nabla e\| = \mu \|\nabla u\|,\ \mu = \frac{\|\nabla e\|}{\|\nabla u\|}. \tag{12}$$

When utility increases, the consumer moves in the direction of a new equilibrium. This direction and the direction of the fastest increase of expenditures form an angle of magnitude α. This angle enables the connection between the respective gradients and changes,

$$\mu = \frac{\|\nabla e\| \|\mathbf{dx}\| \cos\alpha}{\|\nabla u\| \|\mathbf{dx}\| \cos\alpha},$$

$$\mu = \frac{de}{du}. \tag{13}$$

The optimal value of the Lagrange multiplier gives the ratio of the change in expenditures and the change in utility. This ratio is the average change in expenditures or the change in expenditures per a small unit of increase in utility. We call it marginal cost of utility and it is reciprocal to the marginal utility of money when the given utility level is the highest that the consumer can achieve given his disposable income and prices. In this case, in both the model of utility maximization given the budget constraint, and in the model of expenditure minimization given the level of utility, the optimal consumer choices are the same. In both optimization models, the optimal value of the Lagrange multiplier gives the change in the optimal value of the objective function per small unit of change in the constraint.

5. CONCAVITY AND QUASI-CONCAVITY

According to the ordinal theory of consumer behavior positive monotone transformations of utility do not affect optimal consumer choices. Utility surfaces can change shape but level sets remain convex and enable a unique solution. Cardinality is the property that loses importance in analysis. Therefore, the ordinal approach is more general and we prefer to use it in our analysis. It ensures that the utility function is invariant up to a positive monotone transformation. This property places a horizontal cut of the utility function or indifference curves above the budget line. Convexity of the indifference curves guarantee the decreasing marginal rates of substitution between goods. Also, the consumer likes averages at least as much as extremes. A quasi-concave utility function of such a consumer requires less rigorous assumptions and it is the starting point for the contemporary analysis of consumer behavior. Even though they do not fulfill the conditions for the saddle point of the Lagrange function, the tangency conditions that we relied upon in our description of equlibrium and in comparative static analysis remain. The convexity of indifference curves determines the concave shape of the vertical cut of the utility surface passing through the budget line as Figure 7 suggests.

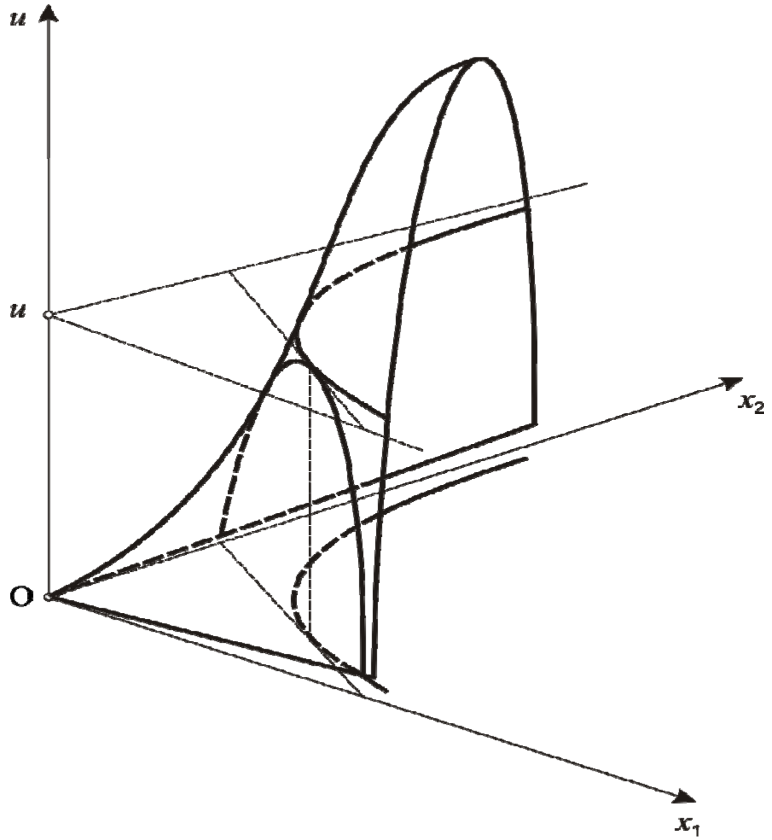

Figure 7. Quasi-concavity or concavity in the direction of indifference

To justify this statement, we start with the consumer who chooses an optimal bundle. Moving along an indifference curve above the budget line does not affect utility and unnecessarily increases expenditures. At the same time, movement along the budget line does not affect expenditures and it unnecessarily lowers utility. The vertical intersection of the utility surface passing through the budget line lies below the parallel to the budget line. In equilibrium, the direction of indifference is identical to the one of the budget line. Therefore, quasi-concavity of the utility function can be thought of as concavity in the direction of indifference. It doesn't matter whether optimal consumer choices are given by a segment, instead of just by a single point of quasi-concavity or concavity in the direction of indifference.

6. GENERAL FINDINGS

Due to their structural symmetry, comparative static results in consumer behavior theory can be applied directly in producer theory. The interpretations deduced so far are even more important when applied to production because of their cardinal character. Production is maximized with respect to the cost constraints and total economic costs are minimized for the given level of production. The non-linear objective function in the primal problem represents the non- linear constraint of the dual problem. The linearity of functions that appear in those optimization problems bears no importance on the economic interpretation of the Lagrange multipliers. A general optimization problem can consist of a non-linear objective function and

$$\begin{aligned} v(k) &= \max_{x_1,x_2} f(x_1,x_2) \\ &g(x_1,x_2) = k. \end{aligned} \tag{14}$$

Such is the problem of the Robinson Crusoe (Jehle and Reny, 2001, Mas-Collel, Whinston and Green, 1995), who is both a consumer and a producer and who maximizes utility on a non-linear production possibility frontier. Thus the problem becomes that of maximizing a non-liner function on a non-liner function.

The Lagrange function gives the distance between the objective and the multiplied difference between the constraint and the given value,

$$L = f(x_1,x_2) - \lambda(g(x_1,x_2) - k). \tag{15}$$

In interpreting the optimal value of the Lagrange multiplier, the graph of the objective function is replaced by the tangential surface representing the translated multiplied distance between the value of the constraint function and the given value as in Figure 8. This replacement is essential.

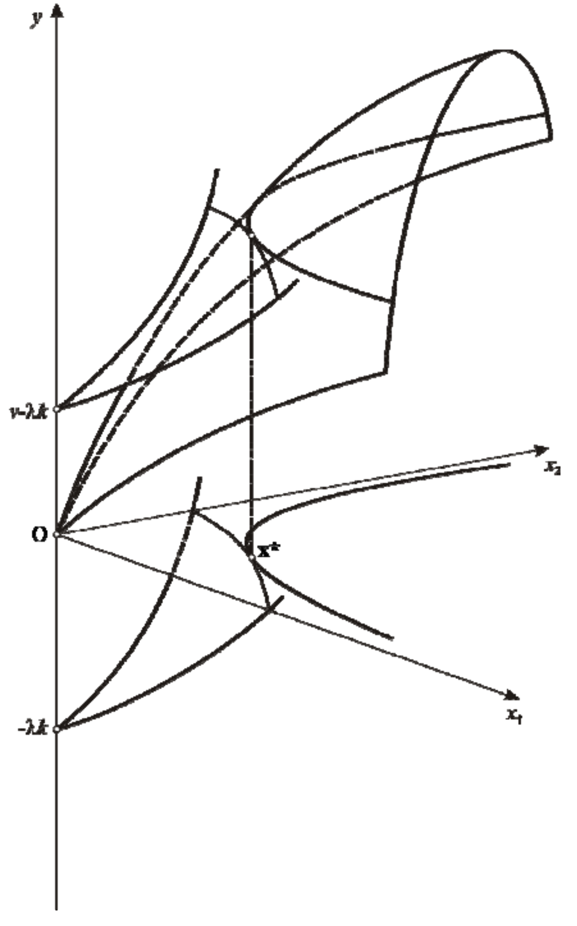

Figure 8. The distance between the objective and the multiplied difference between the constraint and the given value

When the given value of the constraint increases by *k* units, the tangency surface coordinate increases by λk. Therefore, the optimal value of the Lagrange multiplier shows how the relaxation of the constraint for a small unit affects the optimal value of the objective function. In the Robinson Crusoe problem, the relaxation of the constraint is represented by the shift in the production possibility frontier because of the technological progress or an increase in producer's resources. The optimal value of the Lagrange multiplier then shows the increase in Robinson Crusoe's welfare per small unit of the relaxation of the constraint. This finding follows analytically from the equilibrium condition involving the critical point of the Lagrange function and the optimal multiplier:

$$\nabla f = \lambda \nabla g,$$
$$\|\nabla f\| = \lambda \|\nabla g\|,$$
$$\lambda = \frac{\|\nabla f\|}{\|\nabla g\|},$$
$$\lambda = \frac{\|\nabla f\| \|\mathbf{dx}\| \cos\alpha}{\|\nabla g\| \|\mathbf{dx}\| \cos\alpha},$$
$$\lambda = \frac{df}{dg}. \tag{16}$$

The positive sign of the multiplier follows from the same monotonicity of the objective and the constraint whose relaxation takes Robinson Crusoe into the direction of a new equilibrium. The optimal value of the Lagrange multiplier is a ratio of the change of value of the objective function and the change in value of the constraint. It is the average change in the objective function showing the change in the objective per increase of a constraint by a small unit. In the Robinson Crusoe problem, we started from the production possibility frontier that is derived from maximizing the production of the second product while keeping the level of production of the first one fixed. Labor and capital are also assumed fixed. Every constraint comes with its own multiplier with its own interpretation. Prices of factors and goods are determined on the markets and their effects are described by welfare theorems. At the same time, the multipliers that belong to the constraints of limited resources describe how an increase in the fixed factor for a small unit affects the production of the second good while keeping the level of production of the first one fixed. Then the value of the factor of production is expressed by means of physical production. Since value is usually represented by prices it is appropriate to call these multipliers shadow prices.

7. CONCLUSION

The obstacle that the budget constraint represents prevents the consumer to go in the direction of preference. It leads the consumer into an equilibrium where the direction of the force and the direction of the fastest increase in the consumer expenditures are the same. The historically important economic context determines the critical point of the Lagrange function that gives the distance between utility and the multiplied difference between expenditures and income. The impact of the multiplier on the linear constraint is represented graphically as the rotation of the constraint plane and it enables the comparison between the change in income and the change in consumer's welfare. Original interpretations of the multiplier are applied to the problems dual to the theory of consumer behavior, where ordinal properties of the utility function prevail. Such is the diminishing marginal rate of substitution between goods that is characteristic of a quasi-concave utility function. Quasi-concavity of a utility function is interpreted as concavity in the direction of indifference. The impact of the dual multiplier on the difference between total utility and the given one enables generalization of the deduced conclusions within the non-linear relationships among variables that are particularly important in general equilibrium problems.

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CIJENE U SJENI

Sažetak

U ovom radu analiziramo metodu Lagrangeovih množitelja u okviru optimalnih izbora potrošača. Djelovanje množitelja na ograničenja kod problema maksimizacije korisnosti grafički je prikazano kao pomnožena udaljenost između rashoda i prihoda u primarnoj funkciji, i kao pomnožena udaljenost između ukupnih i danih korisnosti u dualnoj formi. Na temelju tih grafičkih prikaza nudimo nova tumačenja optimalnih vrijednosti Lagrangeovih množitelja, kao i potrošačkih rednih svojstava. Rezultati komparativne statike su analitički potvrđeni i postavljeni unutar općeg okvira nelinearnih veza između varijabli važnih za analizu opće ravnoteže.

Ključne riječi: Lagrangeovi množitelji, krivulja sile, krivulja indiferentnosti, ravnoteža, konkavnost i kvazi-konkavnost, komparativna statika.

JEL klasifikacija: C61