

A HYBRID ALGORITHM FOR COMPUTATION OF RECTANGULAR CONDUCTOR INTERNAL IMPEDANCE

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A hybrid algorithm for the computation of rectangular conductor per unit length internal impedance is developed. Up to a switching frequency, the internal impedance is computed using a two-dimensional analytical formula developed by Giacoletto. For these frequencies, 150 terms of Giacoletto infinite sum give a sufficiently good approximation. For higher frequencies, it is shown that the rectangular conductor can be quite accurately approximated by a cylindrical conductor. The proposed hybrid algorithm, for higher frequencies, features improved accuracy and speed when compared to the Giacoletto formula truncated after a small number of terms.

Keywords: cylindrical conductor; exponential approximation; internal impedance; rectangular conductor

Hibridni algoritam za proračun unutarnje impedancije pravokutnog vodiča

Izvorni znanstveni članak

U radu je razvijen hibridni algoritam za proračun unutarnje impedancije po jedinici duljine pravokutnog vodiča. Do izabrane granične frekvencije vrijednost unutarnje impedancije računa se primjenom dvodimenzionalne analitičke Giacolettove formule. Za te frekvencije, Giacolettova beskonačna suma može se dovoljno dobro aproksimirati sa prvih 150 članova. Pokazano je da se za veće vrijednosti frekvencije pravokutni vodič može dovoljno točno aproksimirati pomoću cilindričnog vodiča. Predloženi hibridni algoritam, za više frekvencije, daje točnije rezultate u kratkom računskom vremenu u odnosu na Giacolettovu formulu aproksimiranu s manjim brojem članova beskonačne sume.

Ključne riječi: cilindrični vodič, eksponencijalna aproksimacija, pravokutni vodič, unutarnja impedancija

1

Introduction

The accurate and fast computation of the per unit length (pul) internal impedance of rectangular conductors is required in electromagnetic analysis of various engineering and scientific problems where high current frequencies are involved [1, 2]. This pul internal impedance of a rectangular conductor can also be used in more complicated conductor structures where the shape of the conductor is approximated by a number of rectangular conductors [3].

Methods for computing the rectangular conductor pul internal impedance are based on different theoretical assumptions. The most often used numerical methods are those that employ a subdivision of the rectangular conductor into smaller segments [4÷8]. More accurate methods are analytical one-dimensional (1D) and two-dimensional (2D) algorithms. The 2D analytical algorithms [9,10] are the most accurate ones but they are characterized by infinite sums in the expression for pul internal impedance. The Rong formula [9] has an infinite double sum in the expression for pul internal impedance, which significantly prolongs the computational time needed to acquire accurate results. On the other hand, the Giacoletto formula [10] only has a single infinite sum and produces results of similar accuracy faster. An analysis of the impact of the term numbers in the Giacoletto algorithm on accuracy and computational speed will be investigated.

In this paper, it will be shown that for high frequencies when the skin effect is well developed, the rectangular conductor can be accurately approximated by a cylindrical conductor. Then the formulas developed for the pul internal impedance of solid cylindrical conductor can be used [11, 12]. A hybrid algorithm which combines two formulas is developed. In the first part the Giacoletto formula with a truncated infinite sum is used, whereas in the second part the cylindrical conductor formula is used [11]. The hybrid algorithm developed yields highly accurate results in a shorter computational time.

2

Analytical formulas for internal impedance of rectangular conductors

A rectangular conductor of height $2 \cdot a$ and width $2 \cdot b$ is observed (Fig. 1). The height-width ratio will be denoted $d = a/b$. The conductor length tends to infinity.



Figure 1 Cross-section of a rectangular conductor

The pul internal impedance of the rectangular conductor can be computed by various formulas based on different theoretical aspects which include 1D formulas [10] or 2D formulas such as the Rong formula [9] and Giacoletto formula [10].

The 1D pul internal impedance formula can be written as [10]:

$$\bar{Z} = R_{DC} \cdot (\bar{\gamma} \cdot b) \cdot \coth(\bar{\gamma} \cdot b). \quad (1)$$

The direct current (DC) resistance of the rectangular conductor is given by:

$$R_{DC} = \frac{1}{4 \cdot \sigma \cdot a \cdot b}, \quad (2)$$

and the propagation constant $\bar{\gamma}$ is described by:

$$\bar{\gamma} = \sqrt{\omega \cdot \mu \cdot \sigma} \cdot e^{j \cdot \frac{\pi}{4}}, \tag{3}$$

where σ represents the electrical conductivity of the conductor material, $\mu = \mu_0 \cdot \mu_r$ is the permeability of the conductor material, μ_0 is the vacuum permeability, μ_r is the relative permeability of the conductor material, $\omega = 2 \cdot \pi \cdot f$ is the angular frequency and f is the frequency of the time-harmonic current.

The Rong 2D formula [9] for pul internal impedance computation of a rectangular conductor can be written as:

$$\bar{Z} = R_{DC} \cdot 16 \cdot a^2 \cdot b^2 \cdot \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{c_m^2 + c_n^2}{c_m^2 \cdot c_n^2} \cdot \frac{1}{c_m^2 + c_n^2 + j \cdot \omega \cdot \mu \cdot \sigma} \right)^{-1} \tag{4}$$

where:

$$c_m = \frac{\left(m - \frac{1}{2}\right) \cdot \pi}{a}; \quad c_n = \frac{\left(n - \frac{1}{2}\right) \cdot \pi}{b}. \tag{5}$$

The Giacoletto 2D formula [10] for pul internal impedance computation of a rectangular conductor can be expressed in the following form:

$$\bar{Z} = R_{DC} \cdot \frac{\pi^2}{8} \cdot \left\{ \sum_{m=1}^{\infty} \frac{1}{(2 \cdot m - 1)^2} \cdot \left[\frac{\tanh(\bar{u})}{\bar{u}} + \frac{\tanh(\bar{v})}{\bar{v}} \right] \right\}^{-1}, \tag{6}$$

where:

$$\bar{u} = b \cdot \sqrt{\left(\frac{(2 \cdot m - 1) \cdot \pi}{2 \cdot a}\right)^2 + \bar{\gamma}^2}; \quad \bar{v} = a \cdot \sqrt{\left(\frac{(2 \cdot m - 1) \cdot \pi}{2 \cdot b}\right)^2 + \bar{\gamma}^2}. \tag{7}$$

The formula proposed by Giacoletto (6) is the most efficient and accurate analytical formula for computation of the pul internal impedance of a rectangular conductor. Rong formula (4) employs an infinite double sum, which requires considerable more terms in the internal impedance expression to obtain a similar accuracy degree as the Giacoletto formula. This requires more computer resources and consequently a much longer processing time. On the other hand, the Giacoletto formula contains only a single

infinite sum. One of the purposes of this paper is to determine the optimal number of terms of the Giacoletto infinite sum to produce results of a desired accuracy degree. This will be addressed in the following subsection.

2.1 The influence of the Giacoletto infinite sum truncation on accuracy and speed

The number of used sum terms in the Giacoletto formula determines both accuracy and speed of the pul internal impedance computation. According to (6), the pul internal impedance can be computed using the following expression:

$$\bar{Z} = R_{DC} \cdot \frac{\pi^2}{8} \cdot \left\{ \sum_{m=1}^N \frac{1}{(2 \cdot m - 1)^2} \cdot \left[\frac{\tanh(\bar{u})}{\bar{u}} + \frac{\tanh(\bar{v})}{\bar{v}} \right] \right\}^{-1}, \tag{8}$$

where N is a total number of infinite sum terms taken into account.

Skin effect internal impedance ratios will be computed according to the next equation:

$$\bar{f}_z = f_z \cdot e^{j \cdot \varphi_z} = \bar{f}_z(k \cdot b) = \frac{\bar{Z}}{R_{DC}}, \tag{9}$$

where the complex wave number is defined by the following expression:

$$\bar{k} = k \cdot e^{-j \cdot \frac{\pi}{4}} = \sqrt{2 \cdot \pi \cdot f \cdot \mu \cdot \sigma} \cdot e^{-j \cdot \frac{\pi}{4}}. \tag{10}$$

The argument magnitude $k \cdot b$ is connected with the conductor skin depth δ by the following equation:

$$k \cdot b = \sqrt{2} \cdot \frac{b}{\delta}. \tag{11}$$

The pul internal impedance of a steel rectangular conductor 32×4 mm ($\sigma = 6,7$ MS/m and $\mu_r = 100$) will be computed for 1000 frequency samples, i.e. $k \cdot b$ samples. To assure highly accurate results for comparison purposes, the

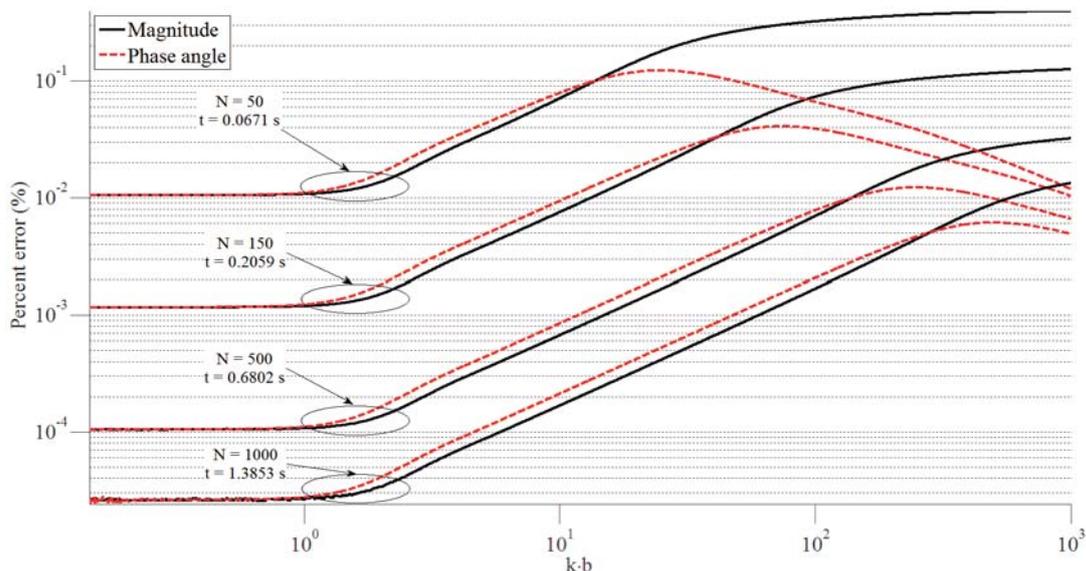


Figure 2 Percent errors of the Giacoletto formula (8) with different N relative to the exact value ($N = 5 \times 10^4$)

Giacoletto formula (8) with $N = 5 \times 10^4$ is used. The computational time for this case is $t = 80,3046$ s. The Giacioletto formula (8) with $N = 5 \times 10^4$ is, naturally, an overstatement, but nevertheless useful as these results can be used as exact.

The accuracy and computational speed analysis is performed for $N \in \{50; 150; 500; 1,000\}$ with the same compiling method as for $N = 5 \times 10^4$ sum terms. The percent errors of the Giacioletto formula approximation with a different N relative to the exact value ($N = 5 \times 10^4$) for a height-width ratio $d = 8$ are presented in Fig. 2. The magnitude percent errors and the phase angle percent errors of the skin effect internal impedance ratios are presented separately.

Even with only $N = 50$, the accuracy of the Giacioletto formula (8) is considerable – up to 0,35 % for very large $k \cdot b$ (Fig. 2). The computational time for this case is just a fraction of the time required to compute the exact value ($t = 0,0671$ s). Naturally, increasing N yields more accurate results, but a greater computational time is required.

After extensive numerical testing, the optimal N was chosen to be 150 due to a short computational time required ($t = 0,2059$ s), which also yields sufficiently accurate results (Appendix A). This choice is directly connected to the fact that, for large parameters, rectangular conductor can be substituted with cylindrical conductor to improve accuracy and speed.

3 Rectangular conductor substitution with cylindrical conductor

For large values of the function argument $\bar{k} \cdot b$, it is useful to substitute the rectangular conductor with a solid cylindrical conductor for the purpose of computing the pul internal impedance. This procedure is based on the equal circumference requirement, which involves a calculation of an equivalent radius of the solid cylindrical conductor in such a way that its circumference is identical as the circumference of the rectangular conductor. The radius of this equivalent cylindrical conductor r_e can be calculated from the following equation:

$$r_e = \frac{2 \cdot (a + b)}{\pi} \tag{12}$$

The pul internal impedance of this equivalent solid cylindrical conductor for large parameters can be computed using the following formula developed in [11], which is based on Hankel asymptotic approximations of Bessel functions:

$$\bar{Z} = \frac{\bar{k}}{2 \cdot \pi \cdot \sigma \cdot r_e} \cdot \frac{\bar{P}_0(\bar{k} \cdot r_e) + j \cdot \bar{Q}_0(\bar{k} \cdot r_e)}{\bar{Q}_1(\bar{k} \cdot r_e) - j \cdot \bar{P}_1(\bar{k} \cdot r_e)} \tag{13}$$

where:

$$\bar{P}_0(\bar{k} \cdot r_e) = 1 - \frac{0,0703125}{(\bar{k} \cdot r_e)^2} + \frac{0,1121521}{(\bar{k} \cdot r_e)^4} - \frac{0,572501421}{(\bar{k} \cdot r_e)^6} \tag{14}$$

$$\bar{P}_1(\bar{k} \cdot r_e) = 1 + \frac{0,1171875}{(\bar{k} \cdot r_e)^2} - \frac{0,144195557}{(\bar{k} \cdot r_e)^4} + \frac{0,676592588}{(\bar{k} \cdot r_e)^6} \tag{15}$$

$$\bar{Q}_0(\bar{k} \cdot r_e) = -\frac{0,125}{\bar{k} \cdot r_e} + \frac{0,0732421875}{(\bar{k} \cdot r_e)^3} - \frac{0,227108002}{(\bar{k} \cdot r_e)^5} + \frac{1,72772750258}{(\bar{k} \cdot r_e)^7} \tag{16}$$

$$\bar{Q}_1(\bar{k} \cdot r_e) = \frac{0,375}{\bar{k} \cdot r_e} - \frac{0,1025390625}{(\bar{k} \cdot r_e)^3} + \frac{0,277576447}{(\bar{k} \cdot r_e)^5} - \frac{1,99353173375}{(\bar{k} \cdot r_e)^7} \tag{17}$$

This formula gives highly accurate results for solid cylindrical conductor internal impedance for large function arguments [12].

3.1 Calculation of argument magnitude $k \cdot b$ switching value S

The results computed by the equivalent cylindrical conductor internal impedance formula (13) exhibit more

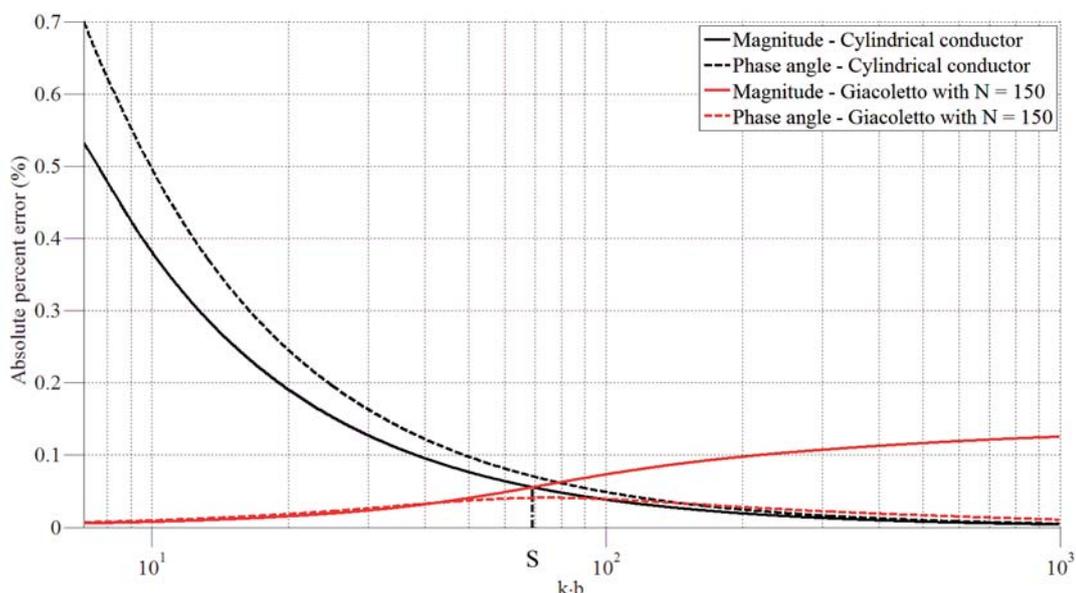


Figure 3 Absolute percent errors of the Giacioletto formula (8) with $N = 150$ and the cylindrical conductor formula (13) relative to the exact value

accurate results than the Giacoletto truncated formula (8) only for large values of argument magnitude $k \cdot b$. Let the value of the argument magnitude $k \cdot b$ at which the Giacoletto formula (8) with $N = 150$ and the cylindrical conductor formula (13) give the same magnitude absolute percent errors be called the switching value and signed as S (Fig. 3). The value S differs for various values of the height-width ratio d .

As an illustrative example, a comparison of absolute percent errors of the Giacoletto formula (8) with $N = 150$ and the cylindrical conductor formula (13), for magnitude and phase angle, relative to the exact value is presented in Fig. 3 for $d=8$.

It is evident from Fig. 3 that the phase angle absolute percent errors of the Giacoletto formula (8) with $N = 150$ and the cylindrical conductor formula (13) coincide for $k \cdot b \geq 100$. However, the magnitude absolute percent errors of the cylindrical conductor formula (13) are lower than the Giacoletto formula (8) with $N = 150$ for $k \cdot b \geq 69,056$. The same applies for the Giacoletto formula (8) with different N .

Therefore, it can be stated that for $k \cdot b \geq S$ one can accurately approximate the rectangular conductor with a cylindrical conductor for pul internal impedance computation purposes. The only issue here is the selection of the switching value S . The problem will be solved using the least squares method by determining an expression that connects S with the height-width ratio d which will be valid for $1 \leq d \leq 8$. Tab. 1 contains 15 values of S for different values of d .

According to (10), switching frequency depends on switching value S of argument magnitude $k \cdot b$, half-width of the rectangular conductor b , conductor conductivity σ and conductor permeability μ . For example, the switching value for $k \cdot b$ is $S = 69,056$ for a steel rectangular conductor 32×4 mm ($d = 8$) with $\sigma = 6,7$ MS/m and $\mu_r = 100$. After introducing this value of $k \cdot b$ into (10), switching frequency of 225,36 kHz is obtained.

The approximation function is chosen to be:

$$S = a_1 \cdot e^{-\frac{d}{\beta_1}} + a_2 \cdot e^{-\frac{d}{\beta_2}}, \tag{18}$$

where β_1 and β_2 are the prescribed parameters of the

Table 1 The values of S for different values of d

Key no. i	d_i	S_i
1	1,0	393,000
2	1,5	308,000
3	2,0	249,000
4	2,5	206,900
5	3,0	176,000
6	3,5	153,000
7	4,0	135,000
8	4,5	120,800
9	5,0	109,200
10	5,5	99,500
11	6,0	91,500
12	6,5	84,600
13	7,0	78,750
14	7,5	73,570
15	8,0	69,056

exponential approximation, whereas a_1 and a_2 represent the unknown coefficients which are computed using the least squares method. The point collocation method is used and it is required that the exponential approximation must match the 15 sampled values given in Tab. 1. The following system of linear equations needs to be solved to find the unknown coefficients [13]:

$$\sum_{n=1}^2 a_n \cdot E_{in} = S_i; i = 1, 2, \dots, 15, \tag{19}$$

of the system of linear equations are deduced from (18) and (19):

$$E_{in} = e^{-\frac{d_i}{\beta_n}}. \tag{20}$$

The system of linear equations can be written in matrix form:

$$[E] \cdot \{a\} = \{S\}, \tag{21}$$

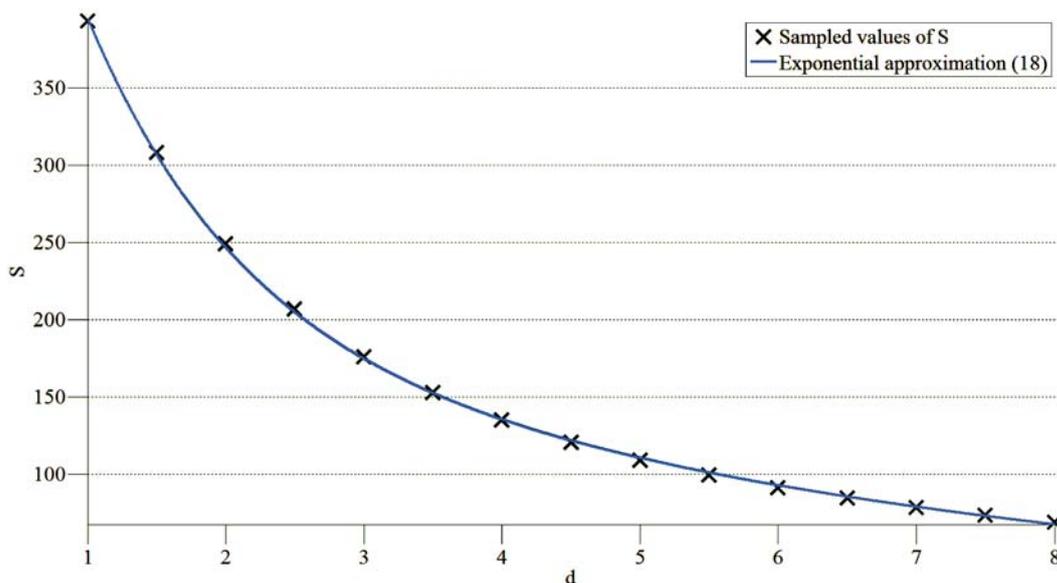


Figure 4 Exponential approximation of the switching value S

where $[E]$ is a (15×2) matrix whose elements can be obtained from (20) and $\{S\}$ is the vector whose elements are given in Tab. 1. Using the least squares method, the unknown vector $\{a\}$ can be computed using the following matrix equation [13]:

$$\{a\} = ([E]^T \cdot [E]^{-1}) \cdot [E]^T \cdot \{S\}. \quad (22)$$

By introducing the computed coefficients into (18), the exponential approximation is completed (Fig. 4):

$$S = 505,67 \cdot e^{-\frac{d}{1,1}} + 222,12 \cdot e^{-\frac{d}{6,7}}. \quad (23)$$

4

The hybrid algorithm-accuracy and speed

The hybrid algorithm developed computes the pul internal impedance by a combination of the Giacoletto formula (8) with $N = 150$ and the cylindrical conductor formula (13) in such a way that:

$$\bar{Z} = \begin{cases} \text{equation (8)} ; k \cdot b < S \\ \text{equation (13)} ; k \cdot b \geq S \end{cases} \quad (24)$$

According to Fig. 3, substituting the rectangular conductor with an equivalent cylindrical conductor produces more accurate results of the skin effect internal impedance ratio magnitudes for large argument magnitudes $k \cdot b$. The skin effect internal impedance ratio phase angles are in this case slightly less accurate.

In addition to the increased accuracy of the skin effect internal impedance ratio magnitudes, the proposed hybrid algorithm has an additional advantage: the time needed for computing the cylindrical conductor formula (13) is shorter than for the Giacoletto formula (8) with $N = 150$. This implies that the hybrid algorithm will be considerably faster for higher values of d due to the fact that the cylindrical conductor internal impedance formula is used for smaller values of $k \cdot b$. The times required for computing the hybrid algorithm are presented in Tab. 2 for various values of d . The pul internal impedance of a steel rectangular conductor ($\sigma = 6,7 \text{ MS/m}$ and $\mu_r = 100$) is computed for 1000 frequency samples. As a reminder, the time needed to compute the Giacoletto formula (8) with $N = 150$ is $t = 0,2059 \text{ s}$.

Table 2 Computational time of the hybrid algorithm

d	t/s
1	0,1778
2	0,1576
3	0,1529
4	0,1498
5	0,1466
6	0,1404
7	0,1388
8	0,1342

For higher values of d , the computational speed increases up to 53,43 %. This implies that the hybrid algorithm can be used with the Giacoletto formula (8) with more than $N = 150$ since the computational speed will be

improved. Naturally, the upper limit of the term numbers is when the Giacoletto formula (8) starts to produce more accurate results than the cylindrical conductor formula.

5

Conclusion

Using existing formulas for computing per unit length internal impedance of rectangular and solid cylindrical conductor, an efficient hybrid algorithm for rectangular conductors is developed. This hybrid algorithm combines the Giacoletto truncated formula with 150 sum terms to compute the per unit length internal impedance up to the switching value of the function argument. After this value, the rectangular conductor is substituted with a cylindrical conductor. Using the least squares method, an expression for computation of the function argument switching value in relation to rectangular conductor height-width ratio is obtained. The hybrid algorithm developed is highly accurate in a short execution time.

6

References

- [1] Sarajčev, P.; Vujević, S. Grounding Grid Analysis: Historical Background and Classification of Methods. //International Review of Electrical Engineering (IREE), 4, 4(2009), 670-683.
- [2] Zhihua, Z.; Weiming, M. Frequency characteristics of ground impedance for a flat steel plane. // IEEE Transactions on Electromagnetic Compatibility, 45, 2(2003), 468-474.
- [3] Demeester, T.; De Zutter D. Internal impedance of composite conductors with arbitrary cross section. // IEEE Transactions on Electromagnetic Compatibility, 51, 1(2009), 101-107.
- [4] Silvester, P. AC resistance and reactance of isolated rectangular conductors. // IEEE Transactions on Power Apparatus and Systems, Vol. PAS-86, 6(1967), 770-774.
- [5] Silvester, P. The accurate calculation of skin effect in conductors of complicated shape. // IEEE Transactions on Power Apparatus and Systems, Vol. PAS-87, 3(1968), 735-742.
- [6] Chari, M. V. K.; Csendes Z. J. Finite element analysis of the skin effect in current carrying conductors. // IEEE Transactions on Magnetics, Vol. MAG-13, 2(1977), 1125-1127.
- [7] Tsuk, M. J.; Kong, J. A. A hybrid method for the calculation of the resistance and inductance of transmission lines with arbitrary cross sections. // IEEE Transactions on Microwave Theory and Techniques, 39, 8(1991), 1338-1347.
- [8] Antonini, G.; Orlandi A.; Clayton, R.P. Internal impedance of conductors of rectangular Cross Section. // IEEE Transactions on Microwave Theory and Techniques, 47, (1999), 979-985.
- [9] Rong, A.; Cangelaris, A. C. Note on the definition and calculation of the per-unit-length internal impedance of a uniform conducting wire. // IEEE Transactions on Electromagnetic Compatibility, 49, 3(2007), 677-681.
- [10] Giacoletto, L. J. Frequency- and time-domain analysis of skin effects. // IEEE Transactions on Magnetics, 32, 1(1996), 220-229.
- [11] Vujević, S.; Boras, V.; Sarajčev, P. A novel algorithm for internal impedance computation of solid and tubular cylindrical conductors. // International Review of Electrical Engineering (IREE), 4, 6(2009), 1418-1425.
- [12] Lovrić, D.; Vujević, S.; Boras, V. Accuracy of approximate formulas for internal impedance of tubular cylindrical conductors for large parameters. // Progress in Electromagnetics Research M (PIERM), 16, (2011), 171-184.
- [13] Vujević, S.; Lovrić, D. Exponential approximation of the Heidler function for the reproduction of lightning current waveshapes. // Electric Power System Research, 80, 10(2010), 1293-1298.

**Appendix A:
Optimal truncation of Giacoletto infinite sum for hybrid algorithm**

The truncation of the Giacoletto infinite sum to $N = 150$ terms was chosen as the optimal value of terms due to the fact that, even for different height-width ratios of various rectangular conductors, a similar maximum absolute percent error of 0,05 % is obtained when using the proposed hybrid algorithm. As discussed in Section 3, the rectangular conductor can be substituted by a cylindrical conductor for the purpose of computing the pul internal impedance. Moreover, this substitution yields results of higher accuracy for $k \cdot b \geq S$, where S is the switching value.

It can be observed in Fig. A1 that, for $N = 150$, absolute percent error of the skin effect internal impedance ratio magnitude peaks at approximately 0,05 % when the cylindrical conductor becomes more accurate. This is valid for various height-width ratios of the rectangular conductor of which only three are depicted in Fig. A1 for brevity purposes – $d \in \{1; 4; 8\}$. In other words, choosing $N = 150$ as the optimal number of Giacoletto sum terms yields a maximum absolute percent error of 0,05 % for the skin effect internal impedance ratio magnitude of the hybrid algorithm.

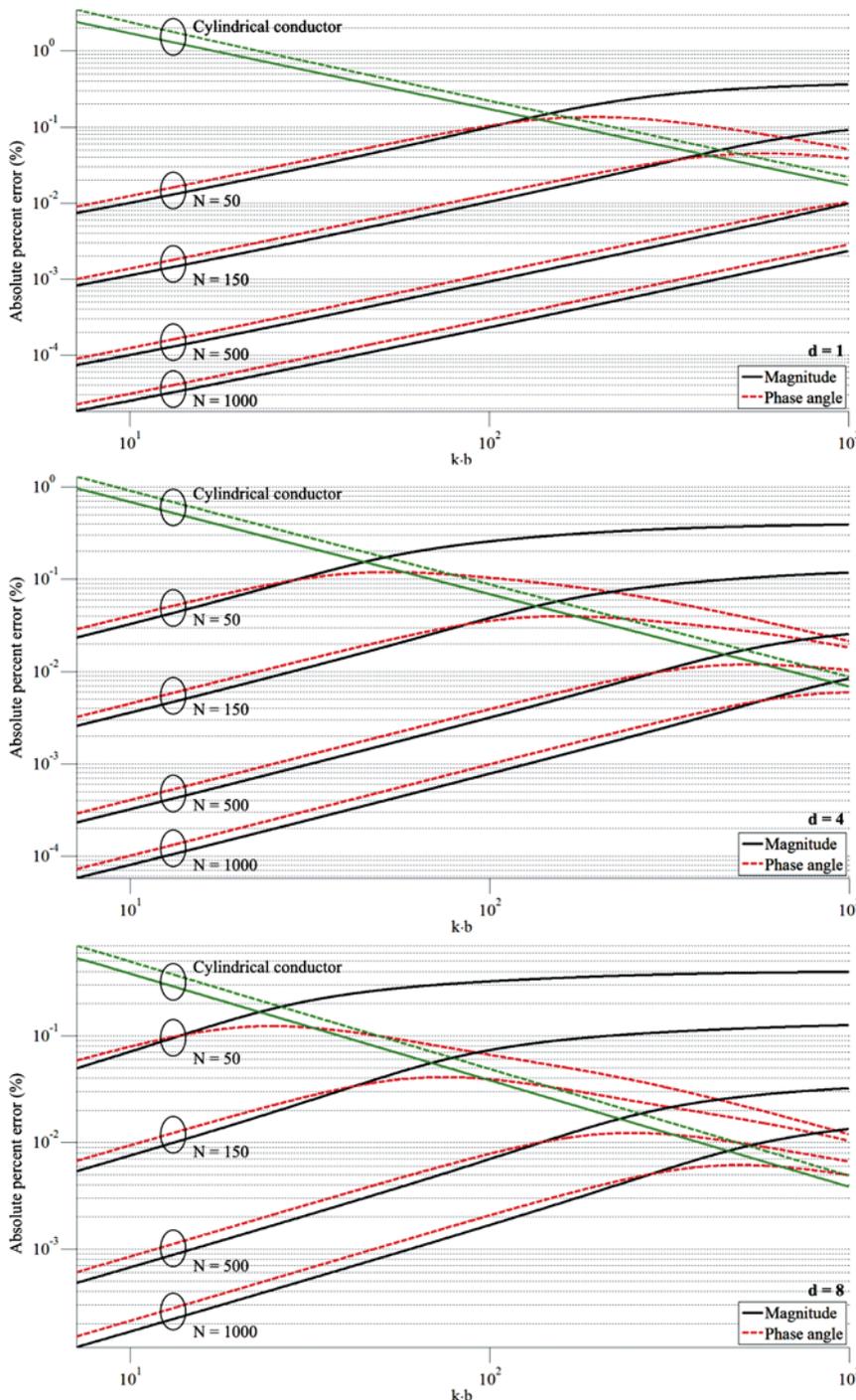


Figure A1 Absolute percent errors of the Giacoletto formula (8) and cylindrical conductor formula (13) relative to the exact value ($N = 5 \times 10^4$) for $d \in \{1; 4; 8\}$

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