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LINEAR AND NONLINEAR BUCKLING AND POST BUCKLING ANALYSIS OF A BAR WITH THE INFLUENCE OF IMPERFECTIONS

Stipica Novoselac, Todor Ergić, Pavo Baličević

Preliminary notes

This paper presents a linear and nonlinear buckling and post buckling numerical analysis of a bar with the influence of imperfections. In the first step, we used an approach of analytic and numerical linear buckling analysis of a bar with linear-elastic material. After linear buckling analysis of the bar, we performed nonlinear buckling analysis with the Riks method. Since the nonlinear analysis was performed in the case of ideal loading, to get correct and more realistic information of post buckling response, imperfections and plastification of material must be considered. In this case, the imperfections are eccentrical loads. This paper finally shows that the post buckling behaviour becomes unstable even for a very small value of eccentrical load in nonlinear analysis with elasto-plastic behaviour of material. Numerical analysis was performed in software Abaqus 6.10.

Keywords: *influences of imperfections, linear buckling, nonlinear buckling, numerical analysis, post buckling*

Linearna i nelinearna analiza izvijanja i poslijekritičnog izvijanja štapa s utjecajem imperfekcija

Prethodno priopćenje

U ovom radu prikazana je numerička analiza linearnog i nelinearnog izvijanja i poslijekritičnog izvijanja štapa s utjecajem imperfekcija. U prvom koraku pristupilo se analitičkoj i numeričkoj linearnoj analizi izvijanja štapa s linearno-elastičnim materijalom. Nakon linearne analize izvijanja štapa, provedena je nelinearna analiza izvijanja Riksovom metodom. Budući da je nelinearna analiza provedena u slučaju idealnog opterećenja, za dobivanje točnijih i realnijih informacija u poslijekritičnom području izvijanja nužno je uzeti u obzir imperfekcije i plastifikaciju materijala. Imperfekcije su u ovom slučaju ekscentričnosti opterećenja. Rezultati rada naposljetku prikazuju da poslijekritično ponašanje izvijanja postaje nestabilno čak i kod vrlo malih vrijednosti ekscentričnosti opterećenja u nelinearnoj analizi s elasto-plastičnim ponašanjem materijala. Numeričke analize provedene su u programskom paketu Abaqus 6.10.

Ključne riječi: *linearno izvijanje, nelinearno izvijanje, numerička analiza, poslijekritično izvijanje, utjecaj imperfekcija*

1 Introduction

Stability of a structure can be analysed by computing its critical load, i.e., the load corresponding to the situation in which a perturbation of the deformation state does not disturb the equilibrium between the external and internal forces [1]. Critical load is usually calculated in Eigenvalue linear buckling analysis. In an eigenvalue buckling problem we are searching for the loads for which the model stiffness matrix becomes singular. Eigenvalue buckling analysis is generally used to estimate the critical buckling loads of ideal structures. Their response usually involves very little deformation prior to buckling. From the detailed structural point of view, it is more accurate to use nonlinear analysis which includes post buckling behaviour with elasto-plastic behaviour of material. However, the unavoidable imperfections of the structures may influence their stability behaviour considerably, with respect to the value of the critical load, and even in terms of the characteristics of the deformation [1]. In order to find out about these influences, in this paper we deal with eigenvalue linear buckling analysis, nonlinear buckling analysis using the modified Riks method with elasto-plastic behaviour of material and with three cases of imperfection, i.e., eccentrical loading.

Nonlinear buckling analysis with effect of plastification will be used to investigate the post buckling behaviour and find if the yield transforms the stable post buckling behaviour into unstable, since, after the yield, an increase in the deformation causes a decrease of the corresponding load.

Since the post buckling behaviour may become unstable when the elasto-plastic deformations take place, it is very important to investigate the influence of imperfections on the bar loading capacity. Performed numerical study will investigate buckling and post buckling problems with the influence of imperfections in engineering structures in order to find out the most accurate, reliable and realistic numerical method for complex structures.

Besides bar structures, plates and shells are also sensitive to buckling. In many branches of engineering, stiffened plates are used as one of the main structural components in order to improve the strength/weight ratios and reduce costs of structures [4]. The global capacity of ships and offshore structure depends also on buckling strength of the stiffened panels. The buckling behaviour of single-walled carbon nanotubes under combined axial compression and torsion can be investigated by using molecular dynamics simulations [10].

If the concern about material nonlinearity, geometric nonlinearity, or unstable post buckling behaviour exists for a structure, the post buckling analysis should be performed [11].

2 Basics of buckling and post buckling analysis

Eigenvalue linear buckling analysis is generally used to estimate the critical buckling load of ideal structures. The buckling loads are calculated relative to the base of the structure. The base state can include preloads (e.g., bolt preload) but preloads are often zero in classical eigenvalue buckling problems. In an eigenvalue buckling problem we search for the loads for which the model stiffness matrix becomes singular, so that the problem:

$$\boldsymbol{K}^{MN}\boldsymbol{v}^{M}=0. \tag{1}$$

has nontrivial solutions [2] where $\boldsymbol{K}^{MN}$ is the tangent stiffness matrix when the loads are applied, and the $\boldsymbol{v}^{M}$ are nontrivial displacement solutions.

For hinged bar, Euler formula for estimation of buckling load is:

$$F_{\mathrm{cr}} = \frac{\pi^2 \cdot E \cdot I_{\min}}{l^2}, \tag{2}$$

where F_{cr} is critical (Euler) buckling load, E is elastic modulus of material, $I_{\min}$ is minimal moment of inertia of the cross-section and l is the (buckling) length of the structure.

For numerical post buckling analysis the Riks method is used. The Riks method is generally used to predict unstable, geometrically nonlinear collapse of a structure. Geometrically nonlinear static problems sometimes involve buckling or collapse behaviour, where the load-displacement response shows negative stiffness and the structure must release strain energy to remain in equilibrium. The Riks method uses the load magnitude as an additional unknown; it solves simultaneously for loads and displacements. For unstable problems, the load-displacement response can exhibit the type of behaviour shown in Fig. 1. That is, during periods of response, the load and/or the displacement may decrease as the solution evolves.

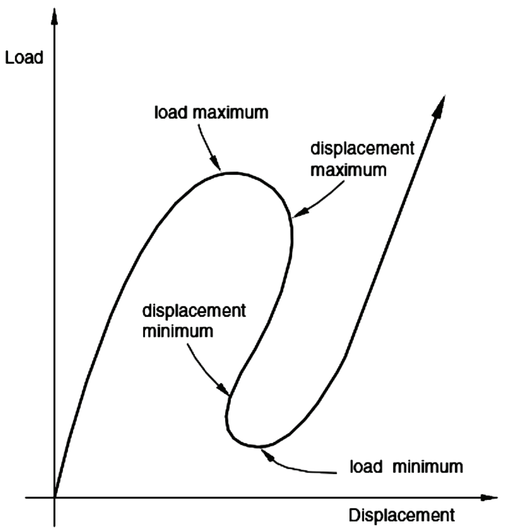

Figure 1 Typical unstable static response [2]

P^N (N = 1, 2, ... = the degrees of freedom of the model) is the loading pattern, as defined with one or more loads. Let λ be the load magnitude parameter, so at any time the actual load state is λP^N, and let u^N be the displacements at that time. The solution space is scaled to make the dimensions of approximately the same magnitude on each axis. In Abaqus this is done by measuring the maximum absolute value of all displacement variables, $\bar{u}$, in the initial (linear) iteration. We also define $\bar{P} = (P^N P^N)^{1/2}$. The scaled space is then spanned by:

- load = $\lambda \tilde{P}^N, \tilde{P}^N = P^N / \bar{P}$,
- displacements = $\tilde{u}^N = u^N / \bar{u}$,

and the solution path is then the continuous set of equilibrium points described by the vector $(\tilde{u}^N;\ \lambda)$ in this scaled space. The algorithm is shown in Fig. 2.

If we suppose that the solution has been developed to the point $\mathrm{A}^0 = (\tilde{u}_0^N;\ \lambda_0)$ then tangent stiffness, $\boldsymbol{K}_0^{NM}$, is formed, and we solve $\boldsymbol{K}_0^{NM} \boldsymbol{v}_0^M = P^N$. The increment size ($\mathrm{A}^0$ to A^1) is chosen from a specified path length, Δl, in the solution space, so that

$$\Delta\lambda_0^2 (\tilde{v}_0^N; 1) : (\tilde{v}_0^N; 1) = \Delta l^2 \tag{3}$$

and, hence

$$\Delta\lambda_0 = \frac{\pm \Delta l}{(\tilde{v}_0^N \tilde{v}_0^N + 1)^{1/2}}. \tag{4}$$

The value Δl is initially suggested by the user and is adjusted by the Abaqus automatic load incrementation algorithm for static problems, based on the convergence rate. The value of $\Delta\lambda_0$ is the direction of response along the tangent line, and is chosen for the dot product of $\Delta\lambda_0(\tilde{v}_0^N; 1)$ in the solution to the previous increment, $(\Delta\tilde{u}_{-1}^N;\ \Delta\lambda_{-1})$, and it is found positive:

$$\Delta\lambda_0 (\tilde{v}_0^N; 1) : (\Delta\tilde{u}_{-1}^N;\ \Delta\lambda_{-1}) > 0, \tag{5}$$

that is

$$\Delta\lambda_0 (\tilde{v}_0^N \Delta\tilde{u}_{-1}^N + \Delta\lambda_{-1}) > 0. \tag{6}$$

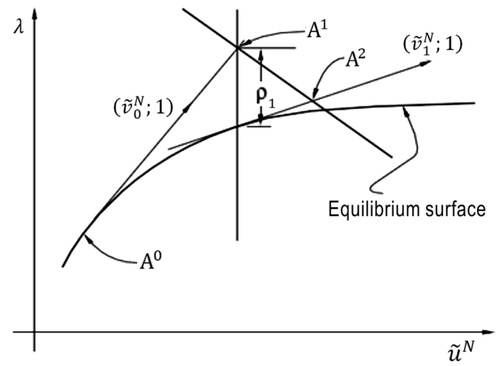

Figure 2 Modified Riks algorithm [3]

3 Effect of imperfections and effect of plastification of deformable elements

Actual members always have imperfections, both in the way the load is applied (eccentricity with respect to the centroid of the cross-section or inclination with respect to the bar axis) and with respect to the geometry of the bar (residual curvature, non constant cross-section, etc.). As a consequence of these unavoidable perturbations, the axial force causes bending even when it takes a value which is smaller than the critical load [1]. The post buckling curve of an initially perfect system does not by itself give sufficient information. To obtain correct information about post buckling behaviour, it is needed to consider imperfections of shape or/and eccentricities of loading which are present in all real structures [4]. When ideal load is applied, until the buckling load is reached, no internal forces are necessary in the elastic element to balance the applied load. The bending deformation introduces additional stresses, which become larger when the load gets close to the critical value. As a consequence, the critical load predicted by the Euler's formula is usually not reached, since plastic deformations or material failure take place before this point. Residual stresses introduced into the bar by the manufacturing process are also an imperfection, since fibres with a residual compressive stress may reach the limit of proportionality before the computed critical stress is attained, which would influence the value of the critical load [1].

The differential equation expressing the interaction between the deformation and the bending moment may be obtained from the relations between bending moment and curvature and between bending moment and deflection.

In eccentrically loaded structure maximum applicable load is always smaller than the critical buckling load, even in the case of long members, since, as a consequence of the additional stresses introduced by bending, the maximum allowable stress is reached before the load attains the critical value. Bending moment in eccentrically loaded structure is:

$$M_{\max} = F(\delta + e). \tag{7}$$

Where $M_{\max}$ is maximum bending moment, F is force, δ is displacement and e is eccentricity.

Unavoidable imperfections of the structures may influence their stability behaviour considerably, with respect to the value of the critical load, and even in terms of the characteristics of the deformation [1]. However, in actual structures the large deformations caused by the imperfection, when the load gets close to the critical value, do limit the loading capacity, even in the case of stable post-critical behaviour.

When the deformable elements of a compressed structure enter the elasto-plastic regime, the corresponding loss of stiffness usually causes a considerable reduction in the maximum load of the structure. Yielding transforms the stable post buckling behaviour into unstable, since, after yielding, an increase in the deformation causes a decrease of the corresponding permissible load [1]. Since the post buckling behaviour may become unstable when elasto-plastic deformations take place, it is very important to investigate the influence of imperfections on the loading capacity of the structure. Influence of plastification on the buckling failure of a bar has the characteristics of a brittle failure, since its axial strength after buckling is practically reduced to zero, even in the case of ductile material behaviour.

4 Arc-length method

In arc-length method, the load proportionality factor (i. e. *LPF*) at each iteration is modified so that the solution follows some specified path until convergence is achieved. Since the method treats the LPF as a variable, it becomes an additional unknown in equilibrium equations resulting from finite element procedure, and gives $N + 1$ unknowns, where N is the number of elements in the displacement vector. The solution of $N + 1$ unknowns requires an additional constraint equation expressed in terms of current displacement, load-factor and arc-length. Two approaches, fixed arc-length and varying arc-length are generally used. In the former the arc-length is kept fixed for current increment, whereas in the latter case, new arc-length is evaluated at the beginning of each load step to ensure the achievement of the solution procedure. Simplification of the constraint equation leads to a quadratic equation, whose roots are used for determining the load-factor. Generally, for the first increment, the trial value of the load-factor is assumed as 1/5 or 1/10 of total load. For further increments the load-factor is computed according to the rate of convergence of the solution process. In case of divergence from the solution path, the arc-length is reduced and computations are done again. The equilibrium equation of nonlinear system can be written as:

$$\boldsymbol{g}_i(\lambda_i) = \boldsymbol{f}_i - \lambda_i \boldsymbol{q}, \tag{8}$$

where $\boldsymbol{f}_i$ is vector of internal equivalent nodal forces, $\boldsymbol{q}$ is the external applied load vector, λ_i is the load-level parameter, and $\boldsymbol{g}_i$ is out-of-balance force vector. The arc-length method is aimed to find the intersection of Eq. (8) with constant s termed as the arc-length, and can be written in differential form as:

$$s = \int \sqrt{\mathrm{d}\boldsymbol{p}^{\mathrm{T}}\mathrm{d}\boldsymbol{p} + \mathrm{d}\lambda^2 \psi^2 \boldsymbol{q}^{\mathrm{T}} \boldsymbol{q}}\,. \tag{9}$$

5 FE modelling

FE modelling was done in software package Abaqus 6.10. FE mesh was built up on existing CAD geometry. For analysis, continuum tetraedar elements with 10 nodes (C3D10) were used.

CAD geometry and FEM model are shown in Fig. 3. The total height of the bar is 400 mm. Cut view of CAD is shown in Fig. 4, with dimensions of width and thickness.

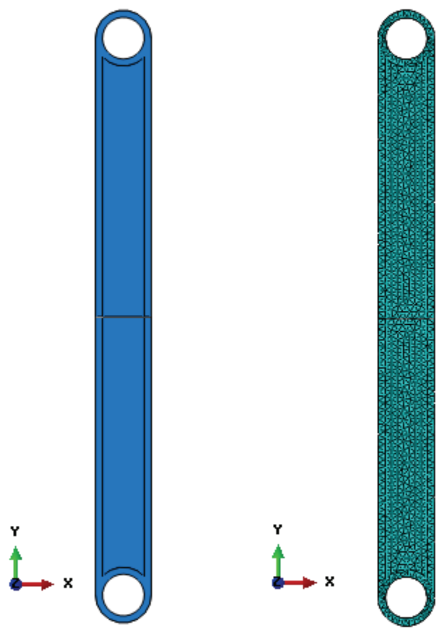

Figure 3 CAD geometry and FEM model

Boundary conditions are modelled like *single point constraint* (i.e., SPC) via *kinematic coupling constraint*.

Master nodes for *kinematic coupling constraints* are created at the centre of the hole in both bar ends. The bar is hinged in both ends. Load is applied via *kinematic coupling constraint*. Concentrated force was applied in master node (written with command *CFORCE in Abaqus input file). FEM model with applied force in kinematic coupling is shown in Fig. 5.

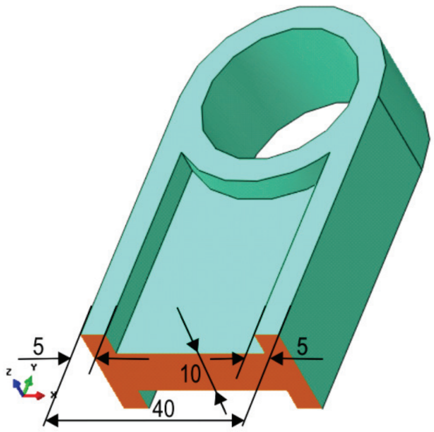

Figure 4 CAD cut view (in mm)

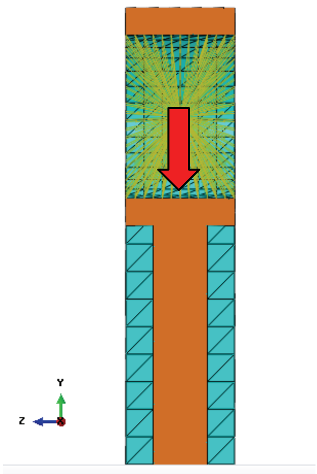
Figure 5 FEM model with Kinematic Coupling constraint

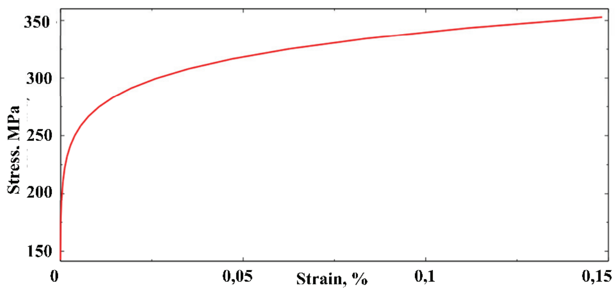

Figure 6 Stress – Strain diagram for S235 steel

The bar is made from S235 steel. The S235 steel is structural steel with elastic modulus E = 210 GPa and Poisson ratio ν = 0,3. Fig. 6 shows the stress – strain diagram for S235 steel.

6 Results

The first analysis done was eigenvalue linear analysis for determination of critical buckling load and buckling modes. After eigenvalue linear analysis, the nonlinear Riks static analysis was done for elastic-linear and nonlinear elasto-plastic material definition with ideal axial compressive load.

To investigate the imperfection sensitivity for the bar, analyses of three eccentrical load cases were performed. Eccentrical loads were in order of applying: 0,1; 0,5 and 1 mm, respectively.

6.1 Results of linear eigenvalue analysis

From Eq. (2), the obtained critical load is F_{cr} = 118,74 kN. From Abaqus eigenvalue linear buckling analysis critical buckling load is F_{cr} = 118,56 kN. The results of numerical analysis are in excellent correlation with theory. First three buckling modes are shown in Fig. 7. State block is shown for the first buckling mode where one can see the result for critical buckling load.

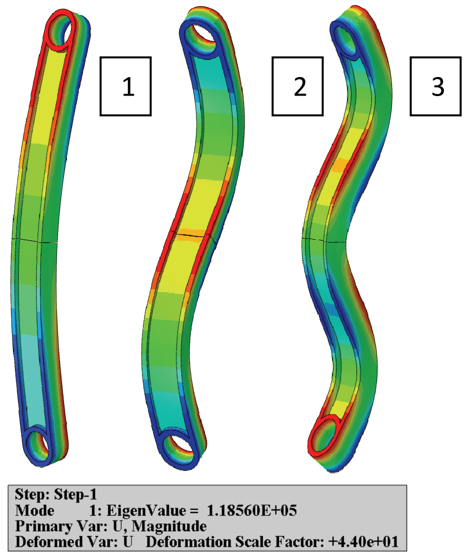

Figure 7 First three buckling modes

6.2 Results of linear-elastic and nonlinear elasto-plastic FEM model with the Riks method

Analyses with linear-elastic and nonlinear elasto-plastic material were done with the Abaqus command *STATIC, RIKS. In these analyses the load is acting in ideal state (i. e., in the centre of the bar). To end the Riks analysis we must specify a way for completing this analysis because the loading magnitude is a part of the solution. We can specify a maximum value of the *LPF* or a maximum displacement value at a specified degree of freedom. In these analyses we specify the displacement of 8 mm in y as one degree of freedom.

Load-displacement curves of the linear-elastic and nonlinear elasto-plastic FEM model are shown in Fig. 8 via history plot of the reference node (i.e. the node in which the load is applied).

With linear-elastic FEM model results are the same as for eigenvalue linear analysis in way of critical buckling load. Post buckling behaviour for linear model shows that the bar cannot increase its strength beyond bifurcation point because it has zero stiffness after buckling has occurred.

With nonlinear elasto-plastic FEM model results show that critical buckling load is significantly smaller (i.e., $F_{cr} \approx 90$ kN). The estimation of the critical buckling load is based upon the results in Fig. 8 for nonlinear FEM model which show that bifurcation point is approximately at 90 kN. Post buckling behaviour of the bar with ideal load shows stiffness increase after bifurcation point. The results show that when the bar enters in the elasto-plastic area, the corresponding loss of stiffness causes a significant reduction of critical buckling load compared to linear model (i.e., decreasing from the F_{cr} = 118,56 kN to the $F_{cr} \approx 90$ kN).

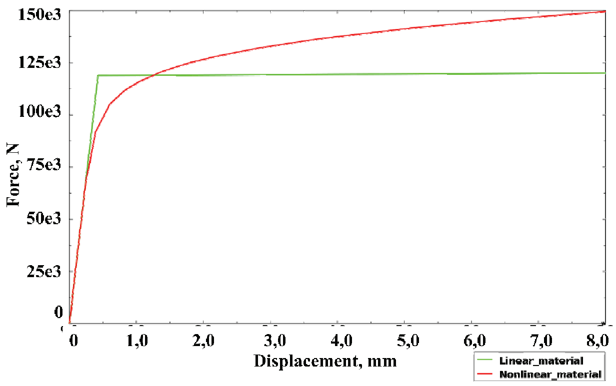

Figure 8 Load-displacement curve of linear and nonlinear FEM model

For maximum displacement of reference node at 8 mm, the equivalent von Mises stresses for linear-elastic model are shown in Fig. 9. From results we can conclude that the stresses are too high for the stated displacement, because the material has a linear-elastic definition.

For accurate stress fields at the stated displacement, the nonlinear elasto-plastic model is shown in Fig. 10. The maximum equivalent von Mises stress is 326,5 MPa. Also, the equivalent plastic strains are shown in Fig. 11. Maximum equivalent plastic strain is 0,031 %. Both figures are shown in real deformation scale.

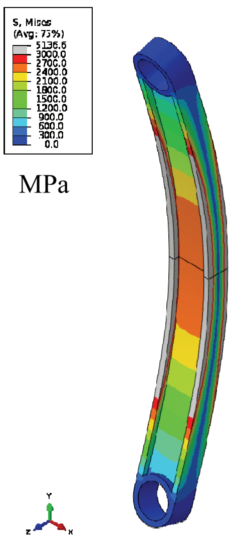

Figure 9 Equivalent von Mises stresses for linear material definition

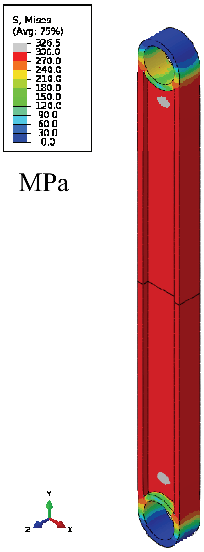

Figure 10 Equivalent von Mises stresses for elasto-plastic material definition

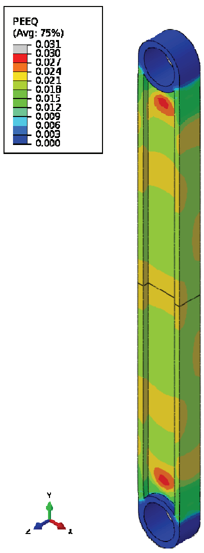

Figure 11 Equivalent plastic strains for elasto-plastic material definition

6.3 Results of linear-elastic and nonlinear elasto-plastic model with eccentric loads with the Riks method

The effect of eccentric loads is investigated in three cases: 0,1; 0,5 and 1 mm, respectively, for linear-elastic and nonlinear elasto-plastic FEM model. Eccentric loads are eccentric in z axis. Eccentric loads are also applied with *STATIC, RIKS step.

In Fig. 12 are shown all load-displacement curves. From results we can see the influence of eccentric applied loads. For linear-elastic model, eccentric load of 0,1 mm has a very small influence. Eccentric loads of 0,5 mm and 1 mm have influence only in way of decreasing the critical load. Post buckling behaviour is very similar for all three cases with linear-elastic material.

Nonlinear elasto-plastic analysis results of this model show that the influence of eccentric loads is much larger compared to the linear model. Even a small eccentric load of 0,1 mm has large influence both on critical buckling load and on post buckling behaviour. Critical buckling load for eccentric load of 0,1 mm is $F_{cr} \approx 76$ kN (compared to $F_{cr} \approx 90$ kN for ideal state load). Further increasing of eccentric loads has influence just in way of decreasing the critical buckling load. For eccentric load of 0,5 mm, critical buckling load is $F_{cr} \approx 69$ kN, and for 1 mm is $F_{cr} \approx 63$ kN. Post buckling behaviour from the results shows that in the case of eccentric loads, the behaviour of a bar becomes unstable and that the stiffness of the bar decreases after bifurcation point. Post buckling behaviour is the same for all three eccentric loads (due to identical post buckling results, the colours of the nonlinear curves shown in Fig. 12 overlap and it is impossible to present them in another way in the same figure).

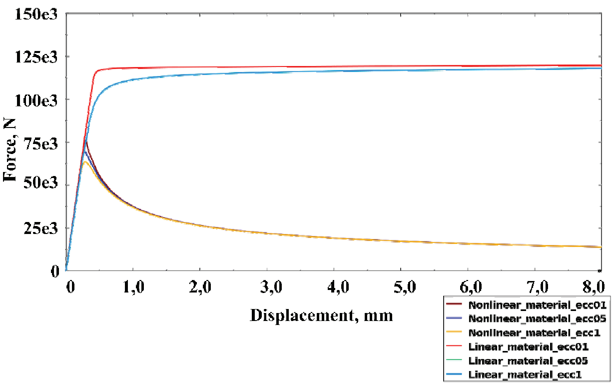

Figure 12 Load-displacement curve of linear and nonlinear eccentrically loaded FEM model

The equivalent von Mises stresses for the stated displacement of 8 mm for linear-elastic FEM model are shown in Fig. 13. Stresses are unrealistic due to linear-elastic behaviour of material.

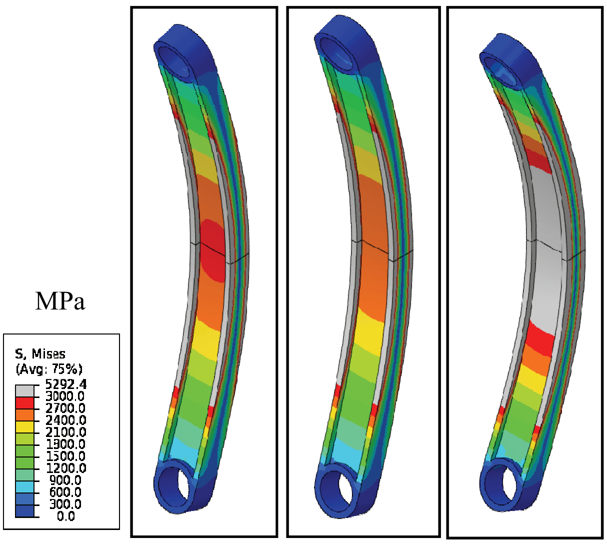

Figure 13 Equivalent von Mises stresses for linear behaviour of material with eccentric load of 0,1; 0,5 and 1mm, respectively

The equivalent von Mises stresses for stated displacement of 8 mm for nonlinear elasto-plastic FEM model are shown in Fig. 14. The highest stress field is in the middle of the bar with values above 400 MPa for stated displacement. From the results we can conclude that plastic hinge is in the middle of the bar.

The equivalent plastic strains for the stated displacement of 8 mm for nonlinear elasto-plastic FEM model are shown in Fig. 15. The highest strains area is in the middle of the bar with values 0,055 % for the stated displacement.

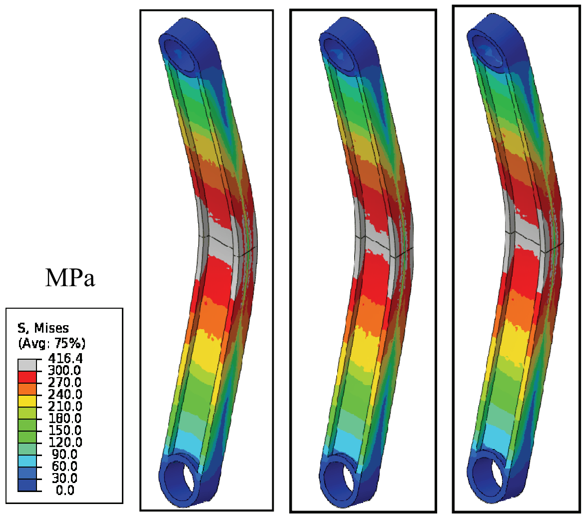

Figure 14 Equivalent von Mises stresses for elasto-plastic behaviour of material with eccentric load of 0,1; 0,5 and 1mm, respectively

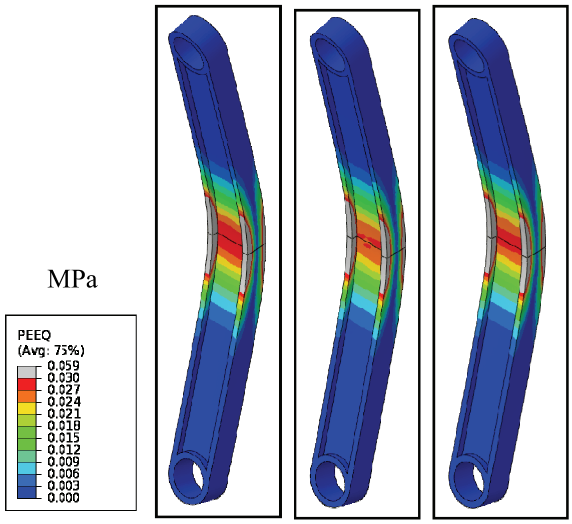

Figure 15 Equivalent plastic strain for elasto-plastic behaviour of material with 0,1 mm eccentric load

7 Conclusion

Linear eigenvalue buckling analysis is usually used for fast determination of critical buckling load. High safety factors, approximately 2,5 to 5, are used in this type of analysis, because imperfections and elasto-plastic behaviour of material are not considered. However, structures always have imperfections due to eccentric loads and/or geometry imperfections. Therefore, post buckling analysis should always be performed with elasto-plastic material and imperfections. From the results of the numerical analysis we can conlude:

- Linear eigenvalue analysis can be used just for evaluation of structure critical buckling load,
- For the evaluation of the post buckling response, the Riks method in Abaqus can be used,
- Post buckling response with elasto-plastic material becomes stable only in case of ideal applied load,
- Post buckling response with linear-elastic behaviour of material and eccentric loads shows that only

critical buckling load is influenced (i. e., bigger imperfection results with smaller critical buckling load),

- Post buckling response with elasto-plastic behaviour of material and eccentric loads shows that the imperfection of very small values causes a significant decrease of the critical buckling load and that post buckling response becomes unstable,
- When the deformable structure under the compressive axial load enters the elasto-plastic area, the corresponding loss of stiffness causes a significant reduction of the critical buckling load,
- Yielding of material transforms the stable post buckling behaviour into unstable. An increase in the displacement causes the decrease of the corresponding load capacity after yielding.

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Authors' addresses

Stipica Novoselac, MEng. Mechanical E.
AVL-AST d.o.o.
Avenija Dubrovnik 10
10000 Zagreb, Croatia
e-mail: stipica.novoselac@avl.com

prof. dr. sc. Todor Ergić
Josip Juraj Strossmayer University of Osijek
Faculty of Mechanical Engineering in Slavonski Brod
Trg Ivane Brlić-Mažuranić 2
35000 Slavonski Brod, Croatia
e-mail: Todor.Ergic@sfsb.hr

doc. dr. sc. Pavo Baličević
Josip Juraj Strossmayer University of Osijek
Faculty of Agriculture
Trg Kralja Petra Svačića 1d
31000 Osijek, Croatia
e-mail: pbalicevic@pfos.hr