Incarcerating the Euler-Mascheroni constant γ

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Abstract. *In this work, we develop new tight bounds for the Euler-Mascheroni constant γ in terms of other well-known constants. For finding various upper and lower bounds, we use Gauss-Legendre and Lobatto quadrature rules.*

Key words: *Euler-Mascheroni Constant; error bound; Euler constant; mathematical constant*

1. Introduction

The Euler-Mascheroni constant also referred to as the Euler constant is defined by the following limit:

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) \right] \tag{1}$$

[see 1, and references therein]. The Euler-Mascheroni constant appears, among other places, in analysis and number theory. There are major open questions regarding the Euler-Mascheroni constant. For example, it is not known whether the number γ is algebraic or transcendental and rational or irrational.

In this work, we present new upper and lower bounds for this constant. We will notice that for bounding the constant γ, we need to bound $\ln(1+\frac{1}{n})$. For example, see the inequality (30). Thus, we will develop bounds for both γ and $\ln(1+\frac{1}{n})$ through quadrature rules. Let us first review the quadrature rule, and develop two quadrature inequalities.

2. Lobatto and Gauss-Legendre quadrature rules

Figure 1 presents a graph of the function $\frac{1}{x}$. The area under the graph, and between the vertical lines $x = n$ and $x = n + 1$ is given by the integral:

$$\int_{n}^{n+1} \frac{1}{x}\,\mathrm{d}x$$

The exact value of this integral is $\ln(1+\frac{1}{n})$.

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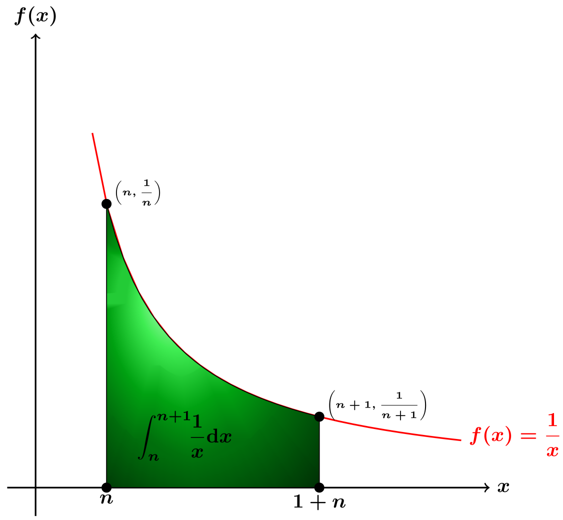

Slika 1: Graph of $f(x) = \frac{1}{x}$. The shaded area is equal to $\ln\left(1 + \frac{1}{n}\right)$.

Let us briefly discuss about the Lobatto quadrature [2–4]. Integral of a function $f(x)$ between the limits a and b through n points Lobatto quadrature is given as:

$$\int\limits_a^b f(x)\,\mathrm{d}x = k \sum_{i=1}^{n} \omega_i\, f(c + k\, x_i) - \mathcal{E} \tag{2}$$

Here, ω_i, x_i and $\mathcal{E}$ are weights, abscissa and error of the quadrature, respectively. The error is given as:

$$\mathcal{E} = \frac{n(n-1)^3 2^{2n-1} \left[(n-2)!\right]^4}{(2n-1)\left[(2n-2)!\right]^3} f^{(2n-2)}(\xi) \tag{3}$$

Here, $\xi \in [a, b]$. For the function $f(x) = \frac{1}{x}$, the derivative $f^{(2n-2)}(\xi)$ is given as follows:

$$f^{(2n-2)}(\xi) = \frac{(2n-2)!}{\xi^{2n-1}}$$

and which is strictly positive for all $\xi > 0$. Thus error is positive for a positive interval of integration. Consequently for positive interval of integration the equation (2) results in the following inequality:

$$\int\limits_a^b \frac{1}{x}\,\mathrm{d}x < k \sum_{i=1}^{n} \omega_i\, f(c + k\, x_i). \tag{4}$$

We may call it the Lobatto inequality. We will use it forming upper bounds for $\ln\left(1 + \frac{1}{n}\right)$, and lower bounds for the constant γ. The constants k and c are defined from a and b:

$$c = \frac{a+b}{2} \qquad \text{and} \qquad k = \frac{b-a}{2} \tag{5}$$

For our purpose $a = n$ and $b = n+1$ (see the Figure 1), thus:

$$c = \frac{2n+1}{2} \qquad \text{and} \qquad k = \frac{1}{2} \tag{6}$$

For the Lobatto quadrature, boundary abscissas are fixed. Thus,

$$x_1 = -1 \quad \text{and} \quad x_n = 1$$

The free abscissas x_i for $i = 2, 3, \ldots, n-1$ are the roots of $P'_{n-1}(x)$. Here, $P_n(x)$ is a Legendre polynomial of degree n [4]. We are using the **Maple** software package for finding the free abscissa [4] through the following commands:

1. First we specify the Legendre polynomial $P_{n-1}(x)$ of degree $n-1$:

```
Pn  := simplify(LegendreP(n-1,x));
```

2. Then we find the derivative $P'_{n-1}(x)$ of the above polynomial:

```
dPn := diff(Pn,x);
```

3. Finally free abscissas x_i, $i = 1, \ldots, n-2$ are obtained by solving $P'_{n-1}(x) = 0$:

```
solve(dPn=0,x);
```

The weights of the free abscissas are given as:

$$\omega_i = \frac{2}{n\,(n-1)P^2_{n-1}(x_{i-1})}, \quad i = 2, \ldots, n-1 \tag{7}$$

while the weights for the fixed abscissas are the following:

$$\omega_i = \frac{2}{n\,(n-1)}, \quad i = 1, n. \tag{8}$$

Let us now discuss about the Gauss-Legendre quadrature [5]. Integral of a function $f(x)$ between the limits a and b through n points Gauss-Legendre quadrature is given as follows:

$$\int_a^b f(x)\,\mathrm{d}x = k \sum_{i=1}^{n} \omega_i\, f(x_i) + \mathcal{E} \tag{9}$$

Here, ω_i, x_i and $\mathcal{E}$ are weights, abscissa and error of the quadrature, respectively. The error is given as:

$$\mathcal{E} = \frac{2^{2n+1}\,(n!)^4}{(2n+1)\,\left[(2n)!\right]^3}\, f^{(2n)}(\xi) \tag{10}$$

Here, $\xi \in [a, b]$. For the function $f(x) = \frac{1}{x}$, the derivative $f^{(2n)}(\xi)$ is given as follows:

$$f^{(2n)}(\xi) = \frac{(2n)!}{\xi^{2n+1}}$$

and which is strictly positive for all $\xi > 0$. Thus error is positive for a positive interval of integration. Consequently for a positive interval of integration, equation (9) results in the following inequality:

$$\int_a^b \frac{1}{x}\,\mathrm{d}x > k \sum_{i=1}^{n} \omega_i\, f(x_i) \tag{11}$$

Here, $k = \frac{(n+1-n)}{2} = \frac{1}{2}$. We may call it the Gauss-Legendre inequality. We will use it forming lower bounds for $\ln(1+\frac{1}{n})$, and upper bounds for the constant γ. For the Gauss-Legendre quadrature the weights are taken from the literature [5].

2.1. Approximation of $\ln(1+\frac{1}{n})$ through the three Point Lobatto quadrature

The weights and abscissa are found through the Maple package. Upon substituting these weights and abscissa in the Lobatto inequality (4), we get the following upper bound:

$$\ln\left(1+\frac{1}{n}\right) < \frac{12n^2+12n+1}{12n^3+18n^2+6n} \tag{12}$$

2.2. Approximation of $\ln(1+\frac{1}{n})$ through the four Point Lobatto quadrature

The weights and abscissa are found through the Maple package. Upon substituting these weights and abscissa in the Lobatto inequality (4), we get the following upper bound:

$$\ln\left(1+\frac{1}{n}\right) < \frac{60\,n^3+90\,n^2+32\,n+1}{60\,n^4+120\,n^3+72\,n^2+12\,n} \tag{13}$$

Now let us use the Gauss-Legendre inequality (11) for developing lower bounds for the $\ln(1+\frac{1}{n})$.

2.3. Approximation of $\ln(1+\frac{1}{n})$ through the 2-point Gauss-Legendre quadrature

$$\ln\left(1+\frac{1}{n}\right) > \frac{6\,n+3}{6\,n^2+6\,n+1} \tag{14}$$

2.4. Approximation of $\ln\left(1+\frac{1}{n}\right)$ through the 3-point Gauss-Legendre quadrature

$$\ln\left(1+\frac{1}{n}\right) > \frac{1}{3}\left[\frac{(60\,n^2+60\,n+11)}{20\,n^3+30\,n^2+12\,n+1}\right] \tag{15}$$

In the next section, we develop bounds on the constant γ. First twenty digits of the constant γ are

$$\gamma = 0.57721566490153286061\ldots$$

3. Trapezoidal rule and lower bound on γ

Now let us develop lower bound for γ through Trapezoidal rule. By the Trapezoidal rule, the area under the graph and between the vertical lines $x = n$ and $x = n+1$ is given as (see the Figure 1):

$$\int_{n}^{n+1} \frac{1}{x}\,\mathrm{d}x = \frac{h}{2}\left[f(x_1)+f(x_2)\right] - \frac{1}{12}f''(c)\,h^2 \tag{16}$$

Here, $h = 1$, $x_1 = n$, $x_2 = n+1$, and the number c lies somewhere between x_1 and x_2. The second derivative of the function: $f''(x) = \frac{2}{x^3}$, which is always positive for positive values of x. Thus,

$$[\ln x]_n^{n+1} < \frac{1}{2}\left[\frac{1}{n}+\frac{1}{n+1}\right] \tag{17}$$

$$\ln\left(\frac{n+1}{n}\right) < \frac{1}{n}\left[\frac{1}{1+\frac{0.5}{n+0.5}}\right] \tag{18}$$

$$\frac{1}{n} > \left[1+\frac{0.5}{n+0.5}\right]\ln\left(\frac{n+1}{n}\right) \tag{19}$$

$$\Rightarrow \frac{1}{n} > \ln(n+1) - \ln(n) + \left[\frac{0.5}{n+0.5}\right]\ln\left(1+\frac{1}{n}\right) \tag{20}$$

Substituting $n = 1, 2, 3, \ldots, n$ in the above inequality gives

$$\frac{1}{1} > \ln(2) - \ln(1) + \left[\frac{0.5}{1+0.5}\right] \ln\left(1+\frac{1}{1}\right) \tag{21}$$

$$\frac{1}{2} > \ln(3) - \ln(2) + \left[\frac{0.5}{2+0.5}\right] \ln\left(1+\frac{1}{2}\right) \tag{22}$$

$$\frac{1}{3} > \ln(4) - \ln(3) + \left[\frac{0.5}{3+0.5}\right] \ln\left(1+\frac{1}{3}\right) \tag{23}$$

$$\vdots \tag{24}$$

$$\frac{1}{n-1} > \ln(n) - \ln(n-1) + \left[\frac{0.5}{n-1+0.5}\right] \ln\left(1+\frac{1}{n-1}\right) \tag{25}$$

$$\frac{1}{n} > \ln(n+1) - \ln(n) + \left[\frac{0.5}{n+0.5}\right] \ln\left(1+\frac{1}{n}\right) \tag{26}$$

Adding all these inequations, we get the following:

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1) + \sum_{k=1}^{n} \left[\frac{0.5}{k+0.5}\right] \ln\left(1+\frac{1}{k}\right) \tag{27}$$

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) > \sum_{k=1}^{n} \left[\frac{0.5}{k+0.5}\right] \ln\left(1+\frac{1}{k}\right) \tag{28}$$

The Euler constant γ is defined as:

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1)\right] \tag{29}$$

From the equation (28) and definition of γ, we get the following:

$$\gamma > \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{0.5}{k+0.5}\right) \ln\left(1+\frac{1}{k}\right)\right] \tag{30}$$

Now from the above inequation and the lower bound (14) give the following:

$$\gamma > \lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{0.5}{k+0.5}\right) \frac{6\,k+3}{6\,k^2+6\,k+1}\right]$$

We used the following Maple command for finding the sum on the right hand side of the above inequation: `sum((6*n+3)/((2*n+1)*(6*n*n+6*n+1)), n = 1..Infinity)`.

$$\gamma > -3 + \frac{\sqrt{3}\,\pi}{2}\,\tan\frac{\pi\sqrt{3}}{6}.$$

The right side of the above inequality evaluates to 0.4775199588. From equations (30) and (15), we get the following lower bound:

$$\gamma > \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{0.5}{k+0.5}\right)\frac{1}{3}\left[\frac{2\,(60k^2+60k+11)}{40k^3+60k^2+24k+2}\right]\right]$$

Using the following Maple command for finding the sum on the right hand side of the above inequation:

```
sum((2/3)*(60*n*n+60*n+11)/((2*n+1)*((40*n*n)*n+60*n*n+24*n+2)), n = 1 ..  Infinity):
```

$$\gamma > -\frac{11}{3}+\frac{5\sqrt{15}\,\pi}{54}\tan\frac{\pi\sqrt{15}}{10}+\frac{\pi^2}{9}$$

The right side of the above inequality evaluates to 0.4778051568.

Three point Lobatto quadrature and lower bound for γ

From equation (12), we can write the following:

$$\frac{1}{n} > \ln(1+n)-\ln n+\left(\frac{6\,n+5}{12\,n^2+12\,n+1}\right)\ln\left(1+\frac{1}{n}\right) \quad (31)$$

Substituting $n = 1, 2, 3, \ldots, n$ in the above inequation, and adding those inequations:

$$\sum_{k=1}^{n}\left[\frac{1}{k}-\ln(1+n)\right] > \sum_{k=1}^{n}\left(\frac{6\,k+5}{12\,k^2+12\,k+1}\right)\ln\left(1+\frac{1}{k}\right) \quad (32)$$

Using the definition of the Euler constant (1) and the above inequation:

$$\gamma > \sum_{k=1}^{\infty}\left(\frac{6\,k+5}{12\,k^2+12\,k+1}\right)\ln\left(1+\frac{1}{k}\right) \quad (33)$$

Now using the lower bound (14) and the above inequation:

$$\gamma > \sum_{k=1}^{\infty}\left(\frac{6\,k+5}{12\,k^2+12\,k+1}\right)\left(\frac{6\,k+3}{6\,k^2+6\,k+1}\right) \quad (34)$$

Using the following Maple command:
`sum((6*k+5)*(6*k+3)/((12*k*k+12*k+1)*(6*k*k+6*k+1)), k = 1 ..  infinity)`. The sum of infinite series on the right of the above expression is given as follows:

$$\gamma > -15+\pi\tan\frac{\pi\sqrt{6}}{6}+\frac{\sqrt{6}\,\pi}{2}\tan\frac{\pi\sqrt{6}}{6}+2\,\Psi\left(\frac{1}{2}+\frac{\sqrt{3}}{6}\right)-2\,\Psi\left(\frac{1}{2}+\frac{\sqrt{6}}{6}\right)$$
$$-\frac{\sqrt{3}\,\pi}{2}\tan\frac{\sqrt{3}\,\pi}{6}-\pi\tan\frac{\pi\sqrt{3}}{6}. \quad (35)$$

The expression on the right side of the above inequality can be evaluated by using the Maple command `evalf` command. And, its value is 0.574823010.

Four point Lobatto quadrature and lower bound for γ

From equation (13), we can write the following:

$$\frac{1}{n} > \ln(1+n) - \ln(n) + \left(\frac{150\,n^2+200\,n+55}{300\,n^3+450\,n^2+160\,n+5}\right)\ln\left(1+\frac{1}{n}\right) \qquad (36)$$

Substituting $n = 1, 2, 3, \ldots, n$ in the above inequation, and adding those inequations:

$$\sum_{k=1}^{n}\frac{1}{k} - \ln(1+n) > \sum_{k=1}^{n}\left(\frac{150\,k^2+200\,k+55}{300\,k^3+450\,k^2+160\,k+5}\right)\ln\left(1+\frac{1}{k}\right) \qquad (37)$$

Using the definition of the Euler constant (1) and the above inequation:

$$\gamma > \sum_{k=1}^{\infty}\left(\frac{150\,k^2+200\,k+55}{300\,k^3+450\,k^2+160\,k+5}\right)\ln\left(1+\frac{1}{k}\right) \qquad (38)$$

Now using the lower bound (14) and the above inequation:

$$\gamma > \sum_{k=1}^{\infty}\left(\frac{150\,k^2+200\,k+55}{300\,k^3+450\,k^2+160\,k+5}\right)\left(\frac{6\,k+3}{6\,k^2+6\,k+1}\right) \qquad (39)$$

Upper bound on γ through the 2-point Gauss-Legendre quadrature

From equation (14):

$$\ln\left(1+\frac{1}{n}\right) > \frac{6\,n+3}{6\,n^2+6\,n+1} \qquad (40)$$

$$\Rightarrow \frac{1}{n} < \ln(1+n) - \ln n + \left(\frac{3n+1}{6\,n^2+3\,n}\right)\ln\left(1+\frac{1}{n}\right) \qquad (41)$$

Substituting $n = 1, 2, 3, \ldots, n$ in the above inequation, and adding those inequations, we get

$$\sum_{k=1}^{n}\left[\frac{1}{k} - \ln(1+n)\right] < \sum_{k=1}^{n}\left(\frac{3k+1}{6\,k^2+3\,k}\right)\ln\left(1+\frac{1}{k}\right) \qquad (42)$$

Now from the above inequation and the definition of the Euler constant (1):

$$\gamma < \sum_{k=1}^{\infty}\left(\frac{3k+1}{6\,k^2+3\,k}\right)\ln\left(1+\frac{1}{k}\right) \qquad (43)$$

Using the upper bound (12) and the above inequation gives our first upper bound:

$$\gamma < \sum_{k=1}^{n} \left(\frac{3k+1}{6\,k^2+3\,k}\right)\left(\frac{12k^2+12k+1}{12k^3+18k^2+6k}\right) \tag{44}$$

Using the following Maple command for finding the sum on the right hand side of the above inequation: `sum((3*n+1)*(12*n*n+12*n+1)/((6*n*n+3*n)*((12*n*n)*n+18*n*n+6*n)), n = 1 .. infinity)`:

$$\gamma < \frac{5}{9} + \frac{7}{108}\,\pi^2 - \frac{8}{9}\ln 2 \tag{45}$$

The right side of the above expression evaluates to 0.5791213099.

Upper bound on γ through the 3-point Gauss-Legendre quadrature

From equation (15), we can write:

$$\ln\left(1+\tfrac{1}{n}\right) < \tfrac{1}{n}\left[\frac{1}{1+\left(\frac{30\,n+25+\frac{3}{n}}{60\,n^2+60\,n+11}\right)}\right] \tag{46}$$

$$\Rightarrow \tfrac{1}{n} < \ln(1+n) - \ln(n) + \left(\tfrac{30\,n+25+\frac{3}{n}}{60\,n^2+60\,n+11}\right)\ln\left(1+\tfrac{1}{n}\right) \tag{47}$$

Substituting $n = 1, 2, 3, \ldots, n$ in the above inequation, and adding those inequations, we get

$$\sum_{k=1}^{n}\left[\tfrac{1}{k} - \ln(1+n)\right] < \sum_{k=1}^{n}\left(\tfrac{30\,k+25+\frac{3}{k}}{60\,k^2+60\,k+11}\right)\ln\left(1+\tfrac{1}{k}\right) \tag{48}$$

Now from the above inequation and the definition of the Euler constant (1):

$$\gamma < \sum_{k=1}^{\infty}\left(\tfrac{30\,k+25+\frac{3}{k}}{60\,k^2+60\,k+11}\right)\ln\left(1+\tfrac{1}{k}\right) \tag{49}$$

Using the upper bound (12) and the above inequation gives the following upper bound:

$$\gamma < \sum_{k=1}^{\infty}\left(\tfrac{30\,k+25+\frac{3}{k}}{60\,k^2+60\,k+11}\right)\left(\tfrac{12k^2+12k+1}{12k^3+18k^2+6k}\right) \tag{50}$$

Using the following command in the computer algebra system Maple: `sum((30*n+25+3/n)*(12*n*n+12*n+1)/((60*n*n+60*n+11)*((12*n*n)*n+18*n*n+6*n)), n = 1 .. infinity)`, we can find the sum of the series on the right hand side of the above inequation. Accordingly, we obtain

$$\gamma < \frac{136}{1331} + \frac{\pi^2}{132} - \frac{30}{121}\gamma - \frac{30}{121}\Psi\left(\frac{1}{2} + \frac{\sqrt{15}}{15}\right) + \frac{15\,\pi}{121}\tan\frac{\pi\sqrt{15}}{15} + \frac{\sqrt{15}\,\pi}{242}\tan\frac{\pi\sqrt{15}}{15} - \frac{4}{3}\ln 2. \quad (51)$$

Here, Ψ is the digamma function. Further simplifying the above expression, we obtain

$$\gamma < \frac{136}{1661} + \frac{11\,\pi^2}{1812} - \frac{484}{453}\ln(2) - \frac{30}{151}\Psi\left(\frac{1}{2} + \frac{\sqrt{15}}{15}\right) + \frac{15}{151}\pi\tan\left(\frac{\sqrt{15}}{15}\pi\right) + \frac{15}{302}\sqrt{15}\pi\tan\left(\frac{\sqrt{15}}{15}\pi\right). \quad (52)$$

The right side of the above inequality is 0.57772570 which is γ accurate til five decimal places.

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