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Variational inequality and complementarity problem in locally convex Hausdorff topological vector space

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Abstract. *The purpose of this paper is to study variational inequality and complementarity problem in a locally convex Hausdorff topological vector space.*

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Let X be a real locally convex Hausdorff topological vector space (lc Htvs) with a continuous seminorm p and let X^* be its dual. Let K be a closed convex subset of X and $T : X \to X^*$ a mapping. Let $\eta : K \times K \to X$.

Definition 1. *T is said to be*

(i) η-monotone *if* $(Tx - Ty, \eta(x, y)) \geq 0$, $\forall x, y \in K$,

(ii) strictly η-monotone *if* $(Tx - Ty, \eta(x, y)) > 0$, $\forall x, y \in K, x \neq y$,

(iii) strongly η-monotone *if there exists a constant* $C > 0$ *such that*

$$(Tx - Ty, \eta(x, y)) \geq C[p(\eta(x, y))]^2,$$

(iv) η-coercive *if* $(Tx, Ty)/p(\eta(x, x)) \to \infty$ *as* $p(\eta(x, x)) \to \infty$.

We consider the nonlinear variational inequality (NVI) which is defined as follows:

$$x \in K : (Tx, \eta(y, x)) \geq 0 \forall y \in K. \tag{1}$$

Another NVI can be stated as follows:

$$x \in K : (Ty, \eta(y, x)) \geq 0 \forall y \in K. \tag{2}$$

Let S_1 and S_2 denote the solutions of (1) and (2) respectively. These can be generalized as follows:

$$x \in K : (Tx - Sx, \eta(y, x)) \geq 0 \forall y \in K \tag{3}$$

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$$x \in K : (Ty - Sy, \eta(y,x)) \geq 0 \forall y \in K. \tag{4}$$

We have

Theorem 1. *If η is antisymetric and T is strictly η-monotone, then S_1 is empty or singleton.*

Proof. Assume that $x_1, x_2 \in S_1$. Then

$$(Tx, \eta(x_2, x_1)) \geq 0, \tag{5}$$

and $(Tx_2, \eta(x_1, x_2)) \geq 0$. From (1), η is antisymmetric. We have $(Tx_1, \eta(x_1, x_2)) \leq 0$. Hence $(Tx_1 - Tx_2, \eta(x_1, x_2)) \leq 0$. Because T is strictly η-monotone, this is impossible unless $x_1 = x_2$ and this completes the proof. □

Theorem 2. *Let T be η-monotone and semicontinuous, $\eta(x,x) = 0, \eta$ positive homogeneous. Then $S_1 = S_2$.*

Proof. Let $x \in S_1$. Since T is η-monotone, $(Ty, \eta(y,x)) \geq (Tx, \eta(y,x)) \geq 0 \Rightarrow x \in S_2$. Let $x \in S_2$.Let $y \in K$. Since K is convex, for $0 < t < 1$, y_t = (1-t)x + ty = x - t(y - x)$\in K$. Hence $(Ty_t, \eta(Y_t, x)) \geq 0$. But $\eta(y_t, x) = -t\eta(y,x)$. Now letting $t \to 0$ we get $(Tx, -t\eta(y,x)) \geq 0$. Thus $x \in S_1$ and this completes the proof. □

We introduce the concept of the complementarity problem(CP) in real lcHtvs. Let K be a closed convex cone in X. Let K^* be the subset of X^* defined by $K^* = \{y \in X^* : (y, \eta(y,x)) \geq 0 \forall x \in K\}$. Then $x \in K, Tx \in K^*, (Tx, \eta(x,x)) = 0$ will be called the generalised complementarity problem (GCP). Let C denote the set of all solutions of GCP.

Theorem 3. *Let K be a closed convex cone, η is antisymmetric, then $S_1 = C$.*

Proof. Let $x \in S_1$. Take $y = x$. Then $(Tx, \eta(x,x)) \geq 0$. Since η is antisymmetric $(Tx, \eta(x,x)) \leq 0 \Rightarrow (Tx, \eta(x,x)) = 0$. Thus $x \in C$ and hence $S_1 \subset C$. Clearly $C \subset S_1$ and this completes the proof. □

We shall now prove the existence theorem for variational inequality in lcHtvs. For this purpose we need the following results which are due to Tarafdar[2].

Lemma 1. *Let K be a nonempty compact and convex subset of a Hausdorff tvs X and S: K→P(K) be a multivalued mapping such that*

(i) for each $x \in K$, Sx is a nonempty convex subset of K,

(ii) for each $y \in K$, $S_y^{-1} = \{x \in K : y \in Sx\}$ contains an open subset U_y of K where U_y may be empty.

(iii) $\bigcup \{U_y : y \in K\} = K$.

Then there exists an element $x_0 \in K$ such that x_0 belongs to Sx_0.

Theorem 4. *Let K be a nonempty compact convex subset of lcHtvs X and let $T : K \to X^*$ be strongly η-monotone. Let η be continuous. Suppose η satisfies $\eta(y,x) = \eta(y,z) + \eta(z,x)$. Then NVI(1) has a solution in K.*

Proof. Suppose NVI has no solution in K. Then for each $x \in$K, there exists a $y \in K$ such that $(Tx, \eta(y,x)) < 0$. Define a multivalued map $F : K \to P(K)$ by $F(x) = \{y \in K : (Tx, \eta(y,x)) < 0\}$. Clearly $F(x)$ is nonempty and convex for each $x \in K$. It follows that $F^{-1}(y) = x \in K : (Tx, \eta(y,x)) < 0$. Since T is strongly η-monotone, for each $y \in K$, the complement of $F^{-1}(y)$ is in K, i.e.

$$\begin{aligned}\left(F^{-1}(y)\right)^c = K - F^{-1}(y) &= \{x \in K : (Tx, \eta(y,x)) \geq 0\} \\ &\subseteq \{x \in K : (Ty, \eta(y,x)) \geq C[(p(\eta)(y,x))^2]\} = H(y).\end{aligned}$$

It is easy to show that $H(y)$ is convex. We now show that $H(y)$ is relatively closed in K. For this purpose, let $\{x_\alpha\}$ be a Moore-Smith sequence in $H(y)$. Then $(Ty, \eta(y, x_\alpha)) \geq C[p(\eta(y, x_\alpha))]^2$. Let $x_\alpha \to x \in K$. We claim that $x \in H(y)$. Since η is continuous, $\eta(X \times X)$ is dense in X, p is a continuous seminorm. We have

$$\begin{aligned}(Ty, \eta(y, x)) &= (Ty, \eta(y, x_\alpha)) + (Ty, \eta(x_\alpha, x)) \\ &\geq C[p(\eta(y, x_\alpha))]^2 + (Ty, \eta(x_\alpha, x)) \\ &\geq C[p(\eta(y, x_\alpha))]^2\end{aligned}$$

$\Longrightarrow x \in$ H(y).
Now

$$\begin{aligned}K - H(y) &= \{x \in K : (Ty, \eta(y, x)) < C(p(\eta(y, x)))^2\} \\ &\subseteq \{x \in K : (Tx, \eta(y, x)) < 0\} \\ &= F^{-1}(y).\end{aligned}$$

This implies for each $y \in K$ there is an element $x \in$ K such that $\bigcup$(K-H(y))=K. But by *Lemma 1*, there exists an element $x \in K$ such that $x \in F(x)$, which means $0 > (Tx, \eta(x, x)) = 0$. This contradiction completes the proof. □

Let D be a nonempty compact, convex subset of X and $F : D \to Y = X^*$. The following existence theorem on variational inequality was established by Karamardian [1].

Proposition 1. *Let the mapping $(u, v) \to (u, F(v))$ be continuous on $D \times D$. Then there exists a point $\bar{x} \varepsilon D$ such that for all $x \in D$, $(x - \bar{x}, F(\bar{x})) \geq 0$.*

We now obtain the following theorem on the complementarity problem, by using the results of Karamardian stated above.

Theorem 5. *Let K be a closed and convex cone in X and let $F : K \to Y = X^*$ be such that*

(i) the mapping $(u, v) \to (u, F(v))$ is continuous on $K \times K$,

(ii) there exists $\bar{x} \in K$ such that $F(x) \in intK^*$.

Then there exists $x \in X$ such that $\bar{x} \in K$, $F(\bar{x}) \in K^*$ and $(\bar{x}, F(\bar{x})) = 0$.

Proof. For any $u \in K$ define

$$\begin{aligned}D_u &= \{x \in D :< x, Fx > \leq < u, Fx >\} \\ {D_u}^0 &= \{x \in D :< x, Fx > < < u, Fx >\} \\ S_u &= \{x \in D :< x, Fx > = < u, Fx >\}.\end{aligned}$$

For each $u \in$ K, D_u is convex. From the continuity assumption it follows that D_u is a closed subset of the compact convex set D for each $u \in K$ and hence is compact. Thus for each $u \in K$, D_U is a nonempty, compact, convex set in X, therefore by *Proposition 1* it follows that for each $u \in K$, there is $x_u \in D_u$ such that

$$< y - x_u, Fx_u > \geq 0, \qquad \text{for all } y \in D_u. \tag{6}$$

Since $0 \in D_u$, $< x_u, Fx_u >\leq 0$.
Case1: Let $x_u \in {D_u}^0$. Then there is a $\lambda > 1$ such that $\lambda x_u \in S_u \subset D_u$. Then we have $< x_u, Fx_u >\leq< \lambda x_u, Fx_u >= \lambda < x_u, Fx_u >$. Since $< x_u, Fx_u >\leq 0$, it is impossible unless $< x_u, Fx_u >=0$. Thus (6) holds.
Case2. Let $x_u \in S_u$ for all $u \in K$. Let $u \in K$ be such that F$x_u \in intK^*$. Then $< u, Fx_u >> 0$. By the hypothesis there is $x \in K$ such that $Fx \in intK^*$. Thus for this x we have $< x, Fx >> 0$. Choose u such that $< u, Fx >>< x, Fx >> 0$. Thus $x \in {D_u}^0$. Now $x_u \in S_u$. Hence $< x_u, Fx_u >>=< u, Fx_u >> 0$. This contradicts $< x_u, Fx_u >\leq 0$ and thus case 2 cannot occur and this completes the proof. □

Remark 1. *Observe that in the above theorem Du is convex and (compact) if D is convex and (compact): Du need not be convex if D is any compact (non -convex)set. For example, take $F : R^+ \to R$, $F(x) = \sin x$, $D = [\frac{\pi}{4}, 2\pi]$. Then*

$$Du = \{x \in D : (x, Fx) \leq (u, Fx)\} = \{x : x \sin x \leq u \sin x\}.$$

For $u = \pi/2$, $Du = [\pi/4, \pi/2] \bigcup [\pi, 2\pi]$ which is not convex.

References

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