A common unique fixed point result in metric spaces involving generalised altering distances

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Abstract. *In this paper we work out a unique common fixed point result for two self-mappings defined on a complete metric space. These mappings are assumed to satisfy a contractive inequality which involves two generalised altering distances.*

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1. Introduction

Fixed point theory in spaces has a vast literature. In particular, there has been a number of works on fixed points involving Altering Distance Functions. These are control functions which alter the distance between two points in a metric space. Such functions were introduced by M. S. Khan et al. in [4] and used in the same paper for defining and solving a new category of fixed points problems in metric spaces.

Definition 1 [see [4]]. *An* altering distance function *is a function* $\psi : [0, \infty) \to [0, \infty)$ *which is*

(i) monotone increasing and continuous and

(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

Afterwards a number of works has appeared in which altering distances has been used. In references [5], [6] and [7], for example, fixed points of single valued mappings and in [1] fixed points of multi-mappings have been obtained by using altering distance functions. Altering distances have been generalised to a two-variable function and in [3] a generalisation to a three-variable function has been introduced and applied for obtaining fixed point results in metric spaces.

In this paper we propose a generalisation of altering distances to a three-variable function and with the help of such function we derive a unique common fixed point result for two self-mappings in a complete metric space. We note that specific fixed

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point results follow by specific choices of the functional form of ψ. We propose the following definition.

Definition 2. *A function $\psi : [0,\infty)^3 \to [0,\infty]$ is said to be a* generalised altering distance function *if*

(i) $\psi(x,y,z)$ is continuous,

(ii) ψ is monotone increasing in all the three variables and

(iii) $\psi(x,y,z) = 0$ only if $x = y = z = 0$.

We define $\phi\ (x) = \psi(x,x,x)$ for $x \in [0,\infty)$. Clearly, $\phi(x) = 0$ if and only if $x = 0$. Examples of ψ are

$$\begin{aligned}
&\psi(a,b,c) = k \max\{a,b,c\}, \text{ for } k > 0,\\
&\psi(a,b,c) = a^p + b^q + c^r, \quad p,q,r \geq 1,\\
&\psi(a,b,c) = (a + \alpha\, b^q)^r + c^s, \text{ where } p,q,r,s \geq 1 \text{ and } \alpha > 0
\end{aligned}$$

Other examples may also be constructed.

2. Fixed point theorem

Theorem 1. *Let(X,d) be a complete metric space and S and T two self - mappings such that the following inequality is satisfied:*

$$\phi_1(d(Sx,Ty) \geq \psi_1(d(x,y),d(x,Sx),d(y,Ty)) - \psi_2(d(x,y),d(x,Sx),d(y,Ty)) \quad (1)$$

where ψ_1 and ψ_2 are generalised altering distance functions and $\phi_1(x) = \psi_1(x,x,x)$. Then S and T have a common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. For $n = 0,1,2,\ldots$

$$x_{2n+1} = Sx_{2n}$$

and

$$x_{2n+2} = Tx_{2n+1}$$

let

$$a_n = d(x_n, x_{n+1}). \quad (2)$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1), for all $n = 0,1,2,\ldots$ we get

$$\begin{aligned}
\phi_1(d(Sx_{2n},Tx_{2n+1})) &= \phi_1(d(x_{2n+1},x_{2n+2}))\\
&\leq \psi_1(d(x_{2n},x_{2n+1}),d(x_{2n},Sx_{2n}),d(x_{2n+1},Tx_{2n+1}))\\
&\quad -\psi_2(d(x_{2n},x_{2n+1}),d(x_{2n},Sx_{2n}),d(x_{2n+1},Tx_{2n+1}))\\
&= \psi_1(d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\\
&\quad -\psi_2(d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}))
\end{aligned}$$

or by(2), for all $n = 0,1,2,\ldots$

$$\phi_1(a_{2n+1}) \leq \psi_1(a_{2n},a_{2n},a_{2n+1}) - \psi_2(a_{2n},a_{2n},a_{2n+1}). \quad (3)$$

If $a_{2n+1} > a_{2n}$, then

$$\phi_1(a_{2n+1}) < \psi_1(a_{2n+1}, a_{2n+1}, a_{2n+1}). = \phi(a_{2n+1}) \tag{4}$$

This is due to the fact that ψ_1 is monotone increasing in all variables and $\psi_2(a_{2n}, a_{2n}, a_{2n+1}) \neq 0$ whenever $a_{2n+1} \neq 0$. Thus we arrive at a contradiction, so that

$$a_{2n+1} \leq a_{2n}, \qquad n = 0, 1, 2, \ldots \tag{5}$$

Putting $x = x_{2n}$ and $y = x_{2n-1}$ in (1) we obtain

$$\phi_1(a_{2n}) \leq \psi_1(a_{2n-1}, a_{2n-1}, a_{2n}) - \psi_2(a_{2n-1}, a_{2n-1}, a_{2n}) \tag{6}$$

By an identical argument we obtain

$$a_{2n+2} \leq a_{2n+1}, \quad n = 0, 1, 2 \ldots \tag{7}$$

From (5) and (7), we obtain for all $n = 0, 1, 2, \ldots$

$$a_{n+1} \leq a_n, \quad n = 0, 1, 2 \ldots \tag{8}$$

Then from (3) and (6), for all $n = 0, 1, 2, \ldots$ we obtain

$$\phi_1(a_{n+1}) \leq \phi_1(a_n) - \phi_2(a_{n+1}), \quad \text{where } \phi_2(x) = \psi_2(x, x, x)$$

or equivalently

$$\phi_2(a_{n+1}) \leq \phi_1(a_n) - \phi_1(a_{n+1}).$$

Summing up in (8) we obtain

$$\sum_{n=0}^{\infty} \phi_2(a_{n+1}) \leq \phi_1(a_0) < \infty$$

which implies

$$\phi_2(a_n) \to 0 \text{ as } n \to \infty. \tag{9}$$

Again from (8) $\{a_n\}$ is convergent and let $a_n \to a$ (say) as $n \to \infty$. Since ϕ is continuous, from (9) we obtain $\phi_2(a) = 0$ which implies that $a = 0$, that is

$$a_n = d(x_{n+1}, x_n) \to 0 \quad \text{as } n \to \infty. \tag{10}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. In view of (10) it is sufficient to prove that $\{x_{2r}\}_{r=1}^{\infty} \subset \{x_n\}$is a Cauchy sequence. If $\{x_{2r}\}_{r=1}^{\infty}$ is not a Cauchy sequence, then given $\in > 0$ we can find monotone increasing sequences of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that

$$n(k) > m(k), d(x_{2m(k)}, x_{2n(k)}) > \in$$

and

$$d(x_{2m(k)}, x_{2n(k)-1}) < \in \tag{11}$$

Then by (11)

$$\begin{aligned} \in &< d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \in + d(x_{2n(k)-1}, x_{2n(k)}) \end{aligned}$$

Making $k \to \infty$ in the above inequality by virtue of (10) we obtain

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \in \tag{12}$$

For all $k = 1, 2, \ldots$

$$d(x_{2n(k)+1}, x_{2m(k)}) \leq d(x_{2n(k)+1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)}). \tag{13}$$

Also for all $k = 1, 2, \ldots$

$$d(x_{2n(k)}, x_{2m(k)}) \leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)}). \tag{14}$$

Making $k \to \infty$ in (13) and (14) respectively, by using f (10) and (12) we have

$$\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) \leq \in$$

and

$$\in \leq \lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)})$$

that is

$$\lim_{k\to\infty} d(x_{2n(k)+1}, x_{2m(k)}) = \in . \tag{15}$$

For all $k = 1, 2, \ldots$

$$\begin{aligned} d(x_{2n(k)}, x_{2m(k)-1}) &\leq d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ d(x_{2n(k)}, x_{2m(k)}) &\leq d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)}). \end{aligned}$$

Making $k \to \infty$ in the above two inequalities and using (10) and (12) we obtain

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) = \in . \tag{16}$$

Putting $x = x_{2n(k)}\, and\, y = x_{2m(k)-1}$ in (1), for all $k = 1, 2, \ldots$ we obtain

$$\begin{aligned} &\phi_1 d(x_{2n(k)+1}, x_{2m(k)}) \\ &\qquad \leq \psi_1(d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), d(x_{2m(k)-1}, x_{2m(k)})) \\ &\qquad\quad - \psi_2(d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), d(x_{2m(k)-1}, x_{2m(k)})) \end{aligned}$$

Making $k \to \infty$ in the above inequality and taking into account the continuity of ψ_1 and ψ_2, by virtue of (10), (15) and (16) we have

$$\phi_1(\in) \;\leq\; \psi_1(\in, 0, 0) - \psi_2(\in, 0, 0) < \phi_1(\in)$$

This is due to the fact that ψ_1 is monotone increasing in its variables and by property of ψ_2 that $\psi(x, y, z) = 0$ if and only if $x = y = z = 0$.

The above inequality gives a contradiction so that $\in= 0$. This establishes the fact $\{x_{2n}\}_{n=1}^{\infty}$is a Cauchy sequence and hence in view of (10) $\{x_n\}$ is also a Cauchy sequence and hence convergent in (X,d).

Let

$$x_n \to z \quad \text{as } n \to \infty. \tag{17}$$

Putting $x = x_{2n}$ and $y = z$ in (1), for all $n = 1, 2, \dots$ we obtain

$$\begin{aligned}\phi_1(d(x_{2n+1}, Tz)) \le \psi_1(&d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz)) \\ &-\psi_2(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz)).\end{aligned}$$

Making $n \to \infty$ in the above inequality, by using (10) and (17) and continuity of ψ_1 and ψ_2 we obtain

$$\phi_1(d(z, Tz)) \le \psi_1(0, 0, d(z, Tz)) - \psi_2(0, 0, d(z, Tz)).$$

If $d(z, Tz) \ne 0$, then using the property that ψ_1 and ψ_2 are monotone increasing and $\psi_2(x, y, z) = 0$ if and only if $x = y = z = 0$, we obtain

$$\phi_1(d(z, Tz)) < \phi_1(d(z, Tz))$$

which is a contradiction. Hence, we obtain

$$d(z, Tz) = 0, \quad \text{ or } z = Tz. \tag{18}$$

In an exactly similar way we prove

$$z = Sz. \tag{19}$$

Equations (18) and (19) show that z is a common fixed point of S and T. □

Let z_1 and z_2 be two common fixed points of S and T and $z_1 \ne z_2$. Then $d(z_1, z_2) \ne 0$. From (1) we obtain

$$\phi_1(d(z_1, z_2)) \le \psi_1(d(z_1, z_2), 0, 0) - \psi_2(d(z_1, z_2), 0, 0) < \phi_1(d(z_1, z_2)).$$

This is again due to the fact that ψ_1 is monotonic increasing in all its variables and $\psi(x, y, z) \le 0$ if at least one of x, y, z is non-zero.

The above inequality is a contradiction which shows that $z_1 = z_2$. This establishes the uniqueness property of the fixed point.

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. In particular, fixed point results under various contractive conditions are obtainable from the above theorems. Contractive mappings are important in fixed point theory. A comprehensive survey of various types of contractive mappings and related fixed point theorems may be obtained in [8]. Here, for example, we derive the following corollary of our theorem.

Corollary 1. *Let S, $T : X \to X$where (X, d) is a complete metric space satisfying*

$$[d(Sx, Ty]^s \le k_1[d(x, y)]^s + k_2[d(x, Tx)]^s + k_3[d(y, Ty)]^s \tag{20}$$

where $0 < k_1 + k_2 + k_3 < 1$ *and* $s > 0$. *Then* S *and* T *have a common fixed point.*

Proof. We make particular choices of ψ_1 and ψ_2 as the follows:

$$\psi_1(a,b,c) = k_1 a^s + k_2 b^s + k_3 c^s$$
$$\psi_2(a,b,c) = (1-k)[k_1 a^s + k_2 b^s + k_3 c^s]$$

with $k = k_1 + k_2 + k_3$. Then (20) is implied by (1). The corollary then follows by an applying of *Theorem 1.* □

Other fixed point results may also be obtained under specific choices of ψ_1 and ψ_2. As a final remark we observe that there is no continuity assumption on the functions S and T.

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