# A fixed fuzzy point for fuzzy mapping in complete metric spaces*

D. TURKOGLU† AND B. E. RHOADES‡

**Abstract**. *In this paper, we prove a fixed fuzzy point theorem for fuzzy mappings over a complete metric space.*

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## 1. Introduction

After the introduction of the concept of a fuzzy set by Zadeh [11], several researches were conducted on the generalizations of the concept of a fuzzy set. The idea of an intuitionistic fuzzy set is due to Atanassov [1], [2], [3] and Çoker [5] has defined the concept of fuzzy topological spaces induced by Chang [4]. Heilpern [7], introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [8]. Estruch and Vidal [6] give a fixed point theorem for fuzzy contraction mappings over a complete metric spaces which is a generalization of the given Heilpern's fixed point theorem. In this paper we give a fixed point theorem for a fuzzy contraction mapping over a complete metric space which is a generalization of the fixed point theorem given by Estruch and Vidal. They give a fixed point theorem under the condition $D_\alpha(F(x), F(y)) \leq qd(x, y)$ for each $x, y \in X$. We give a fixed point theorem under the condition $D_\alpha(F(x), F(y)) \leq K(M(x, y))$ and some examples, where $D_\alpha, F, q, X, K$ and $M$ are following defined. One of these example has been an extended version of their main theorem.

## 2. Preliminaries

Let $X$ be a nonempty set and $I = [0, 1]$. A fuzzy set of $X$ is an element of $I^X$. For $A, B \in I^X$ we denote $A \subseteq B$ if and only if $A(x) \leq B(x)$ for each $x \in X$.

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†Department of Mathematics, Faculty of Science and Arts, University of Gazi, Besevler, 06530-Ankara, Turkey, e-mail: `dturkoglu@gazi.edu.tr`

‡Department of Mathematics, Indiana University, Bloomington, Indiana 47405 - 7106, USA, e-mail: `rhoades@indiana.edu`

**Definition 1 [3].** *An intuitionistic fuzzy set (i−fuzzy set) $A$ of $X$ is an object having the form $A = \langle A^1, A^2 \rangle$ where $A^1, A^2 \in I^X$ and $A^1(x) + A^2(x) \leq 1$ for each $x \in X$.*

We denote by $IFS(X)$ the family of all $i-$fuzzy sets of $X$.

**Remark 1.** *If $A \in I^X$, then $A$ is identified with the i−fuzzy set $\langle A, 1 - A\rangle$ denoted by $[A]$.*

For $x \in X$ we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of $X$. For $\alpha \in (0,1]$ the fuzzy point [9] $x_\alpha$ of $X$ is the fuzzy set of $X$ given by $x_\alpha(x) = \alpha$ and $\alpha \neq x$. Now we give the following definition.

**Definition 2 [6].** *Let $x_\alpha$ be a fuzzy point of $X$. We will say that $\langle x_\alpha, 1 - x_\alpha\rangle$ is an i−fuzzy point of $X$ and it will be denoted by $[x_\alpha]$. In particular $[x] = \langle \{x\}, 1 - \{x\}\rangle$ will be called an i−point of $X$.*

**Definition 3 [3].** *Let $A, B \in IFS(X)$. Then $A \subset B$ if and only if $A^1 \subset B^1$ and $B^2 \subset A^2$.*

**Remark 2.** *Notice $[x_\alpha] \subset A$ if and only if $x_\alpha \subset A^1$.*

Let $(X, d)$ be a metric linear space (i. e., a complex or real vector space). The $\alpha-$ level set of $A$, denoted by $A_\alpha$, is defined by

$$\begin{aligned} A_\alpha &= \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0,1], \\ A_0 &= \overline{\{x : A(x) > 0\}} \end{aligned}$$

where $\overline{B}$ denotes the closure of the (non fuzzy)set $B$. Heilpern [7] called a fuzzy mapping a mapping from the set of $X$ into a family $W(X) \subset I^X$ defined as follows: $A \in W(X)$ if and only if $A_\alpha$ is compact and convex in $X$ for each $\alpha \in [0,1]$ and $\sup\{Ax : x \in X\} = 1$. In this context we give the following definitions.

**Definition 4 [7].** *Let $A, B \in W(X)$, $\alpha \in [0,1]$. Define*

$$\begin{aligned} p_\alpha(A,B) &= \inf\{d(x,y) : x \in A_\alpha,\ y \in B_\alpha\}, \\ D_\alpha(A,B) &= H(A_\alpha, B_\alpha), \\ D(A,B) &= \sup_\alpha D_\alpha(A,B), \end{aligned}$$

*where $H$ is the Hausdorff distance.*

For $x \in X$ we write $p_\alpha(x, B)$ instead of $p_\alpha(\{x\}, B)$.

**Definition 5 [6].** *Let $X$ be a metric space and $\alpha \in [0,1]$. Consider the following family $W_\alpha(X)$ :*

$$W_\alpha(X) = \{A \in I^X : A_\alpha \textit{ is nonempty, compact and convex}\}$$

*Now we define the family $\Phi_\alpha(X)$ of i−fuzzy sets of $X$ as follows:*

$$\Phi_\alpha(X) = \{A \in IFS(X) : A^1 \in W_\alpha(X)\}.$$

Clearly, for $\alpha \in I$, $W(X) \subset \Phi_\alpha(X)$ in the sense of *Remark 1.*

We need the following lemmas. Let $(X, d)$ be metric space.

**Lemma 1.** [7] *Let $x \in X$ and $A \in W(X)$. Then $x_\alpha \subset A$ if $p_\alpha(x, A) = 0$.*

**Lemma 2.** [7] *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for $x, y \in X$, $A \in W(X)$.*

**Lemma 3.** [7] *If $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$, for each $A, B \in W(X)$.*

## 3. Fixed fuzzy point theorem

In mathematical programming, problems are expressed as optimizing some goal function given certain constraints and there are real-life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution that is quite useful is the one using Fuzzy Sets. The idea is to relax the pretenses of optimization by means of a subjective gradation which can be modelled into fuzzy membership functions $\mu_i$. If $\cap\mu_i$ the objective will be to search $x$ such that $\max F = F(x)$. If $\max F = 1$, then there exists $x$ such that $F(x) = 1$, but if $\max F = \alpha$, $\alpha \in (0,1)$, the solution of the multiobjective optimization is a fuzzy point $x_\alpha$ and $F(x) = \dot{\alpha}$.

In a more general sense than the one given by Heilpern, a mapping $F : X \to I^X$ is a fuzzy mapping over $X$ ([10]) and $(F(x))(x)$ is the fixed degree of $x$ for $F$. In this context we give the following definition.

**Definition 6 [6].** *Let $x_\alpha$ be a fuzzy point of $X$. We will say that $x_\alpha$ is a fixed fuzzy point of the fuzzy mapping $F$ over $X$ if $x_\alpha \subset F(x)$ (i. e., the fixed degree of $x$ is at least $\alpha$). In particular, and according to* [7], *if $\{x\} \subset F(x)$, we say that $x$ is a fixed point of $F$.*

**Theorem 1.** *Let $\alpha \in (0,1]$ and $(X,d)$ be a complete metric space. Let $F$ be a continuous fuzzy mapping from $X$ into $W_\alpha(X)$ satisfying the following condition:*

*There exists $K : [0,\infty) \to [0,\infty)$, $K(0) = 0, K(t) < t$ for all $t \in (0,\infty)$ and $K$ is non-decreasing such that*

$$D_\alpha(F(x), F(y)) \le K(M(x,y)) \tag{1}$$

*for all $x, y \in X$, where*

$$M(x,y) = \max\{d(x,y), p_\alpha(x,F(x)), p_\alpha(y,F(y)), p_\alpha(x,F(y)), p_\alpha(y,F(x))\}.$$

*Then there exists $x \in X$ such that $x_\alpha$ is a fixed fuzzy point of $F$ if and only if there exists $x_0, x_1 \in X$ such that $x_1 \in (Fx_0)_\alpha$ with $\sum\limits_{n=1}^{\infty} K^n(d(x_0,x_1)) < \infty$. In particular, if $\alpha = 1$, then $x$ is a fixed point of $F$.*

**Proof.** If there exists $x \in X$ such that $x_\alpha$ is a fixed fuzzy point of $F$, then $x_\alpha \subset F(x)$ and $d(x,x) = 0$, $0 = K(0) = K^2(0) = ... = K^n(0) = ...$ and $\sum\limits_{n=1}^{\infty} K^n(d(x,x)) = 0$. Let $x_0 \in X$ and suppose that there exists $x_1 \in (F(x_0))_\alpha$ such that $\sum\limits_{n=1}^{\infty} K^n(d(x_0,x_1)) < \infty$. Since $(F(x_1))_\alpha$ is a nonempty compact subset of $X$, then there exists $x_2 \in (F(x_1))_\alpha$ such that

$$d(x_1,x_2) = p_\alpha(x_1, F(x_1)) \le D_\alpha(F(x_0), F(x_1))$$

by *Lemma 3*. By induction we construct a sequence $\{x_n\}$ in $X$ such that $x_n \in (F(x_{n-1}))_\alpha$ and $d(x_n, x_{n+1}) \le D_\alpha(F(x_n), F(x_{n-1}))$.

Noting that $K$ is nondecreasing and using inequality (1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq D_\alpha(F(x_n), F(x_{n-1})) \\ &\leq K(\max\{d(x_{n-1}, x_n), p_\alpha(x_{n-1}, F(x_{n-1})), p_\alpha(x_n, F(x_n)), \\ &\quad p_\alpha(x_{n-1}, F(x_n)), p_\alpha(x_n, F(x_{n-1}))\}) \end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) \leq K(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \tag{2}$$

Suppose $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ for some $n$. Then from (2) and $K(t) < t$ for all $t \in (0, \infty)$, we have

$$d(x_n, x_{n+1}) \leq K(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$$

which is a contradiction. Therefore, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq D_\alpha(F(x_{n-1}), F(x_n)) \\ &\leq K(d(x_{n-1}, x_n)) \\ &= K(p_\alpha(x_{n-1}, F(x_{n-1}))) \\ &\leq K(D_\alpha(F(x_{n-1}), F(x_{n-2}))) \\ &\leq K(K(d(x_{n-1}, x_{n-2}))) \\ &\quad \vdots \\ &\leq K^n(d(x_0, x_1)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{n+m-1}, x_{n+m}) \\ &\leq K^n(d(x_0, x_1)) + \ldots + K^{n+m-1}(d(x_0, x_1)) \\ &= \sum_{k=n}^{n+m-1} K^k(d(x_0, x_1)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} K^n(d(x_0, x_1)) < \infty$, it follows that there exists $r$ such that $d(x_n, x_{n+m}) < r$. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in $X$. Suppose $\{x_n\}$ converges to $x \in X$. Now by *Lemmas 1* and *2* we have

$$\begin{aligned} p_\alpha(x, F(x)) &\leq d(x, x_n) + p_\alpha(x_n, F(x)) \\ &\leq d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \\ &\leq d(x, x_n) + K(d(x_{n-1}, x)). \end{aligned}$$

consequently, $p_\alpha(x, F(x)) = 0$ and by *Lemma 1*, $x_\alpha \subset F(x)$. □

**Remark 3.** *K needs only be defined on the range d. If one replaces the metric with an equivalent metric with* $d(x, y) < 1$*, then clearly* Theorem 4 *holds for* $K$ :

$[0,1) \to [0,\infty)$. *To apply* Theorem 4*, one needs a nondecreasing function* $K$ *and* $x$ *in* $X$ *with* $\sum_{n=1}^{\infty} K^n(p_\alpha(x, F(x))) < \infty$.

The following examples satisfy these conditions and therefore illustrate the generality of *Theorem 4*. Let $(X,d)$ be a complete metric space.

**Example 1.** *Suppose* $0<q<1$. *Let* $K(t) = qt$ *for* $t \geq 0$. *Then* $D_\alpha(F(x), F(y)) \leq K(M(x,y)) = qM(x,y)$ *and* $K^n(p_\alpha(x, F(x))) = q^n p_\alpha(x, F(x))$ *for any* $x$ *in* $X$. *It is known that there exist* $x \in X$ *such that* $x_\alpha \subset F(x)$.

**Remark 4.** Example 1 *is a generalization of Theorem 3.2. of* [6].

**Remark 5.** *Theorem 3.1. of* [7]. *A fixed point of the fuzzy mapping* $F : X \to W(X)$ *exists whenever* $D(F(x), F(y)) \leq qd(x,y)$ *for each* $x, y \in X$*, being* $q \in (0,1)$. *Now, in* Example 1*, it is clear that the condition* $D(F(x), F(y)) \leq qd(x,y)$ *can be weakened to* $D_1(F(x), F(y)) \leq qd(x,y)$.

**Example 2.** *Suppose that* $F$ *satisfies* $D_\alpha(F(x), F(y)) \leq \phi(M(x,y))d(x,y)$ *for all* $x, y$ *in* $X$*, where* $\phi : [0,\infty) \to [0,1)$ *and* $\phi$ *is nondecreasing . Then* $K(t) = t\phi(t)$ *is nondecreasing and* $K : [0,\infty) \to [0,\infty)$. *It follows by induction that* $K^n(t) \leq t[\phi(t)]^n$*, since* $\phi(t) < 1$ *and* $\sum_{n=1}^{\infty} K^n(t) < \infty$.

**Example 3.** *Consider* $K(M(x,y)) = M(x,y)\phi(M(x,y))$ *for all* $x, y$ *in* $X$*, where* $\phi : [0,\infty) \to [0,\infty)$, $\phi(t) \leq t$ *for* $t \leq 1$. *If* $t < 1$*, it follows that* $K^n(t) \leq t[\phi(t)]^n$. *If* $K$ *is nondecreasing, then* Theorem 4 *can be applied.*

**Example 4.** $K(M(x,y)) = M(x,y)\phi(M(x,y))$ *for all* $x, y$ *in* $X$*, where* $\phi : [0,\infty) \to [0,\infty)$, $\phi(qM(x,y)) \leq q\phi(M(x,y))$ *for* $q \in (0,1)$. *If* $\phi(t) < 1$*, then* $K^n(t) \leq K(t)[\phi(t)]^n$ *for all* $n \geq 2$.

**Example 5.** *Assume that* $K$ *is nondecreasing,* $K$ *is convex on* $[0,1)$ *and* $K(M(x,y)) < M(x,y)$ *for all* $x, y$ *in* $X$. *If* $t < 1$*,* $K(t) < t$ *for all* $0 < t < 1$*, then* $K(t) = qt$ *for some* $0 < q < 1$. *It can be shown that* $K^n(t) \leq q^n t$ *for all* $n$ *and* $\sum_{n=1}^{\infty} K^n(t) < \infty$.

**Example 6.** *Let* $X = [0,1]$ *and let* $d : X \times X \to IR^+$ *be the Euclidean metric. Let* $\alpha \in \left(0, \frac{1}{2}\right)$ *and suppose* $F : X \to I^X$ *defined by*

$$F(0)(x) = \begin{cases} 1 & , x = 0 \\ \alpha, & x \in (0, \frac{1}{2}] \\ \frac{\alpha}{2}, & x \in (\frac{1}{2}, 1] \end{cases}$$

$$F(1)(x) = \begin{cases} 1 & , x = 0 \\ 2\alpha, & x \in (0, \frac{1}{2}] \\ \frac{\alpha}{2}, & x \in (\frac{1}{2}, 1] \end{cases}$$

*and for* $z \in (0,1)$

$$F(z)(x) = \begin{cases} 1 & , x = 0 \\ \alpha, & x \in (0, \frac{1}{2}] \\ 0, & x \in (\frac{1}{2}, 1] \end{cases}.$$

*Then* $F(0)_1 = F(z)_1 = F(1)_1 = \{0\}$, $F(0)_\alpha = F(z)_\alpha = F(1)_\alpha = \left[0, \frac{1}{2}\right]$*, and*

$F(0)_{\frac{\alpha}{2}} = F(1)_{\frac{\alpha}{2}} = [0,1]$, $F(z)_{\frac{\alpha}{2}} = \left[0, \frac{1}{2}\right]$. *Consequently*

$$D_1(F(x),F(y)) = H(F(x)_1, F(y)_1) = 0, \quad \forall x,y \in X,$$
$$D_\alpha(F(x),F(y)) = H(F(x)_\alpha, F(y)_\alpha) = 0, \quad \forall x,y \in X,$$
$$D_{\frac{\alpha}{2}}(F(x),F(y)) = H(F(x)_{\frac{\alpha}{2}}, F(y)_{\frac{\alpha}{2}}) = 0, \quad \forall x,y \in \{0,1\} \text{ and } \forall x,y \in (0,1),$$
$$D_{\frac{\alpha}{2}}(F(x),F(y)) = H(F(x)_{\frac{\alpha}{2}}, F(y)_{\frac{\alpha}{2}}) = \frac{1}{2}, \quad \forall x \in \{0,1\} \text{ and } \forall y \in (0,1).$$

*Let* $\phi : [0,\infty) \to [0,\infty)$, $\phi(t) = \frac{t^2}{1+t}$. *Thus* $\phi(t) = t\frac{t}{1+t} < t$. *Let* $K(t) = \frac{t^3}{1+t} = t\phi(t)$. *If* $t \le 1$, *it follows that* $K^n(t) \le t[\phi(t)]^n = \frac{t^{2n+1}}{(1+t)^n}$ *and* $\sum_{n=1}^{\infty} K^n(t) < \infty$. *Therefore* $D_\alpha(F(x),F(y)) = 0 \le K(d(x,y))$ *for each* $x,y \in X$. *By* Theorem 4, *there exists a fixed fuzzy point (a fixed point ) of the fuzzy mapping* $F$. *We can see by the definition of* $F$ *that* 0 *is a fixed point. Nevertheless, Heilpern's theorem is not useful in this example because* $D_{\frac{\alpha}{2}}(F(0),F(\frac{1}{3})) = H(F(0)_{\frac{\alpha}{2}}, F(\frac{1}{3})_{\frac{\alpha}{2}}) = \frac{1}{2} > \frac{1}{3} = d(0,\frac{1}{3})$ *and then* $D(F(0),F(\frac{1}{3})) = \sup_k H(F(0)_k, F(\frac{1}{3})_k) > d(0,\frac{1}{3})$.

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