GS–deltoids in GS–quasigroups

ZDENKA KOLAR–BEGOVIĆ* AND VLADIMIR VOLENEC†

Abstract. *A "geometric" concept of the GS–deltoid is introduced and investigated in the general GS–quasigroup and geometrical interpretation in the GS–quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$ is given. The connection of GS–deltoids with parallelograms, GS–trapezoids, DGS–trapezoids and affine regular pentagons in the general GS–quasigroup is obtained.*

Key words: *GS–quasigroup, GS–deltoid*

AMS subject classifications: 20N05

Received April 20, 2004 Accepted September 30, 2005

In [1] the concept of a GS–quasigroup is defined. A quasigroup $(Q,\cdot)$ is said to be a GS–quasigroup if it is idempotent and if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b, \qquad a\cdot(a\cdot bc)c=b. \tag{1, 1'}$$

The considered GS–quasigroup $(Q,\cdot)$ satisfies the identitites of mediality, elasticity, left and right distributivity, i.e. we have the identities

$$ab\cdot cd=ac\cdot bd \tag{2}$$

$$a\cdot ba=ab\cdot a \tag{3}$$

$$a\cdot bc=ab\cdot ac, \qquad ab\cdot c=ac\cdot bc. \tag{4, 4'}$$

Further, the identities

$$a(ab\cdot b)=b, \qquad (b\cdot ba)a=b \tag{5, 5'}$$

$$a(ab\cdot c)=b\cdot bc, \qquad (c\cdot ba)a=cb\cdot b \tag{6, 6'}$$

$$a(a\cdot bc)=b(b\cdot ac), \qquad (cb\cdot a)a=(ca\cdot b)b \tag{7, 7'}$$

and equivalencies

$$ab=c\Leftrightarrow a=c\cdot cb, \qquad ab=c\Leftrightarrow b=ac\cdot c. \tag{8, 8'}$$

*Department of Mathematics, University of Osijek, Gajev trg 6, HR-31 000 Osijek, Croatia, e-mail: `zkolar@mathos.hr`

†Department of Mathematics, University of Zagreb, Bijenička 30, HR-10 000 Zagreb, Croatia, e-mail: `volenec@math.hr`

also hold.

Let C be the set of points of the Euclidean plane. For any two different points a, b we define $ab = c$ if the point b divides the pair a, c in the golden section ratio. In [1] it is proved that $(C, \cdot)$ is a GS–quasigroup. We shall denote that quasigroup by $C(\frac{1}{2}(1+\sqrt{5}))$because we have $c = \frac{1}{2}(1+\sqrt{5})$ if $a = 0$ and $b = 1$. Figures in this quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$ can be used for illustration of "geometrical" relations in any GS–quasigroup.

From now on let $(Q, \cdot)$ be any GS–quasigroup. The elements of the set Q are said to be points. Points a, b, c, d are said to be the vertices of a parallelogram and we write $Par(a, b, c, d)$ if the identity $a \cdot b(ca \cdot a) = d$ holds. In [1] numerous properties of the quaternary relation Par on the set Q are proved. Let us mention just the following characterization which we shall use afterwards.

Lemma 1. *If* (e, f, g, h) *is any cyclic permutation of* (a, b, c, d) *or of* (d, c, b, a), *then* $Par(a, b, c, d)$ *implies* $Par(e, f, g, h)$.

We shall say that b is the midpoint of the pair of points a, c and write $M(a, b, c)$ iff $Par(a, b, c, b)$. In [1] it is proved that the statement $M(a, b, c)$ holds iff $c = ba \cdot b$.

In [2] the concept of the GS–trapezoid is defined. Points a, b, c, d are said to be the vertices of the golden section trapezoid and it is denoted by $\mathrm{GST}(a, b, c, d)$ if the identity $a \cdot ab = d \cdot dc$ holds. Because of (8), this identity is equivalent with the identity $d = (a \cdot ab)c$.

In [2] it is proved that any two of the five statements

$$(9) GST(a, b, c, d),\ GST(b, c, d, e),\ GST(c, d, e, a),\ GST(d, e, a, b),\ GST(e, a, b, c)$$

imply the remaining statement.

In [4] the concept of an affine regular pentagon is defined. Points a, b, c, d, e are said to be the vertices of the affine regular pentagon and it is denoted by $ARP(a, b, c, d, e)$ if any two (and then all five) of the five statements (9) are valid.

The concept of the DGS–trapezoid is introduced in [3]. Points a, b, c, d are said to be the vertices of the double golden section trapezoid or shorter a DGS–trapezoid and we write $DGST(a, b, c, d)$ if the equality $ab = dc$ holds.

GS–deltoids in GS–quasigroups

Points o, a, b, c are said to be the vertices of a golden section deltoid and we write $\mathrm{GSD}(o, a, b, c)$ if and only if the identity

$$c = oa \cdot b$$

is valid (*Figure 1*).

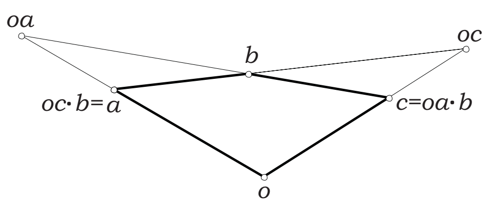

Figure 1.

Obviously the following theorem holds.

Theorem 1. *$GSD(o,a,b,c) \Rightarrow GSD(o,c,b,a)$ (Figure 1).*

Proof. From $c = oa \cdot b$ it follows $oc \cdot b = o(oa \cdot b) \cdot b \overset{(1)}{=} a$. □

Theorem 2. *Any two of the three statements $GSD(o,a,b,c)$, $GSD(o',a',b',c')$ and $GSD(oo',aa',bb',cc')$ imply the remaining statement (Figure 2).*

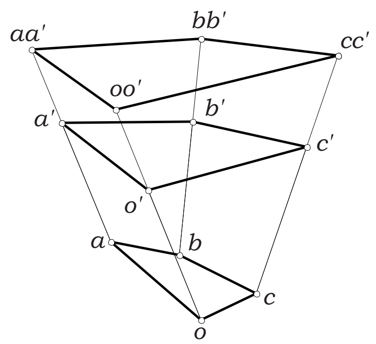

Figure 2.

Proof. Because of (2) we have successively

$$(oo' \cdot aa') \cdot bb' = (oa \cdot o'a') \cdot bb' = (oa \cdot b)(o'a' \cdot b')$$

and then it is obvious that any two of the three equalities $oa \cdot b = c$, $o'a' \cdot b' = c'$ and $(oo' \cdot aa') \cdot bb' = cc'$ imply the remaining equality. □

For any point p we have obviously GSD(p,p,p,p) and from *Theorem 2* it follows further:

Corollary 1. *For any point p the statements $GSD(o,a,b,c)$, $GSD(po,pa,pb,pc)$ and $GSD(op,ap,bp,cp)$ are equivalent.*

Theorem 3. *If the statements $GSD(o,a,b,c)$, $GSD(o,b,c,d)$ hold, then $ab = dc = e$, i.e. $DGST(a,b,c,d)$ and $Par(o,a,e,d)$ hold (Figure 3).*

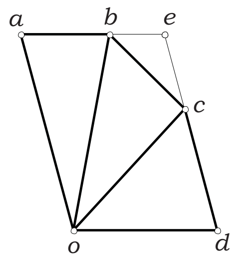

Figure 3.

Proof. From $c = oa \cdot b$ and $d = ob \cdot c$ there follows $d = ob \cdot (oa \cdot b) \overset{(4)'}{=} (o \cdot oa)b$ which gives

$$dc = (ob \cdot c)c \overset{(7)'}{=} (oc \cdot b)b = [o(oa \cdot b) \cdot b]b \overset{(1)}{=} ab$$

and the first statement is proved.

Because of

$$\begin{aligned} o\cdot d(eo\cdot o) &= o[(o\cdot oa)b\cdot(ab\cdot o)o] \overset{(2)}{=} o[(o\cdot oa)(ab\cdot o)\cdot bo] \\ &\overset{(2)}{=} o[(o\cdot ab)(oa\cdot o)\cdot bo] \overset{(3)}{=} o[(o\cdot ab)(o\cdot ao)\cdot bo] \\ &\overset{(4)}{=} o[o(ab\cdot ao)\cdot bo] \overset{(4)}{=} o[o(a\cdot bo)\cdot bo] \overset{(1)'}{=} a \end{aligned}$$

we get the statement $\mathrm{Par}(o,d,e,a)$ out of which, according to *Lemma 1*, the second statement of the theorem follows. □

Theorem 4.

(i) Any two of the three statements $GSD(o,a,b,c)$, $GSD(o,b,c,d)$, $GST(o,a,b,d)$ imply the remaining statement (Figure 4).

(ii) Any two of the three statements $GSD(o,a,b,c)$, $GSD(o,b,c,d)$, $GST(o,d,c,a)$ imply the remaining statement (Figure 4).

Proof. (i) It is necessary to prove that any two of the three statements $oa\cdot b=c$, $ob\cdot c=d$, $(o\cdot oa)b=d$ imply the remaining statement. However, it becomes obvious because of $(4)'$ we have the equality $ob\cdot(oa\cdot b)=(o\cdot oa)b$.
(ii) The statement follows from (i) and *Theorem 1.* □

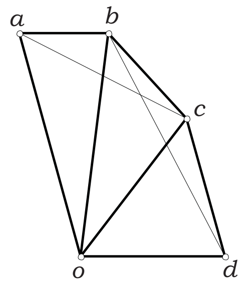

Figure 4.

Corollary 2. *From $GSD(o,a,b,c)$, $GSD(o,b,c,d)$, $GSD(o,c,d,e)$ it follows $ARP(o,a,b,d,e)$ (Figure 5).*

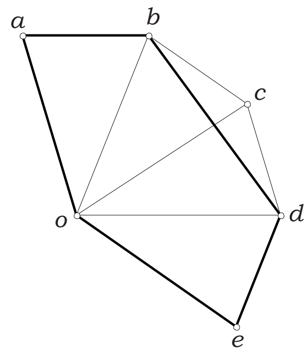

Figure 5.

Proof. Because of *Theorem 4 (i)* and the definition of an affine regular pentagon the following implications are valid

$$\begin{aligned} GSD(o,a,b,c)\,GSD(o,b,c,d) &\Rightarrow GST(o,a,b,d)\\ GSD(o,e,d,c)\,GSD(o,d,c,b) &\Rightarrow GST(o,e,d,b)\\ GST(o,a,b,d)\,GST(o,e,d,b) &\Rightarrow ARP(o,a,b,d,e). \end{aligned}$$ □

Theorem 5. *Any two of the three statements $GST(a,b,c,d)$, $GSD(b,d,c,e)$, $M(a,b,e)$ imply the third statement (Figure 6).*

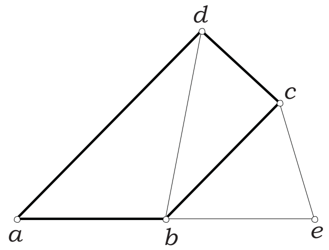

Figure 6.

Proof. We must prove that any two of the three equalities

$$(a\cdot ab)c = d, \quad bd\cdot c = e, \quad ba\cdot b = e$$

imply the remaining equality. That holds because we get successively

$$\begin{aligned} [b\cdot(a\cdot ab)c]c &\overset{(6)'}{=} b(a\cdot ab)\cdot(a\cdot ab) \overset{(2)}{=} ba\cdot(a\cdot ab)(ab)\\ &\overset{(4)}{=} ba\cdot a(ab\cdot b) \overset{(5)}{=} ba\cdot b. \end{aligned}$$ □

Theorem 6. *Any two of the three statements $GSD(o,a,b,c)$, $GSD(o,c,d,e)$, $GST(a,b,d,e)$ imply the third statement (Figure 7).*

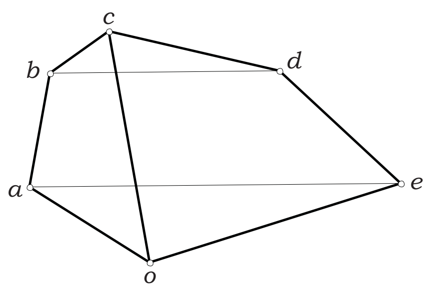

Figure 7.

Proof. Because of symmetry $a \leftrightarrow e$, $b \leftrightarrow d$, it is sufficient to prove that under assumption GSD(o,a,b,c), i.e. $c = oa\cdot b$, the statements GST(a,b,d,e) and GSD(o,e,d,c), i.e. $c = oe\cdot d$ i.e. $oa\cdot b = oe\cdot d$ are equivalent. However, this holds due to Theorem 6(i) from [2]. □

References

[1] V. Volenec, *GS–quasigroups*, Čas. pěst. mat. **115**(1990), 307–318.

[2] V. Volenec, Z. Kolar, *GS–trapezoids in GS–quasigroups*, Mathematical Communications **7** (2002), 143–158.

[3] Z. Kolar–Begović, V. Volenec, *DGS–trapezoids in GS–quasigroups*, Mathematical Communications **8** (2003), 215–218.

[4] V. Volenec, Z. Kolar–Begović, *Affine regular pentagons in GS–quasigroups*, Quasigroups and related systems **12** (2004), 103–112.