On the periodic solutions of certain fourth and fifth order vector differential equations

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Abstract. *The aim of the present paper is to establish some sufficient conditions which ensure that equations (1.1) and (1.2) have no periodic solution other than the trivial solution* $X = 0$.

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1. Introduction

The problems related to the periodic behaviour of solutions of a higher order nonlinear scalar differential equation have been treated by many investigators. The papers achieved in Ezeilo [4], Tiryaki [9], Bereketoğlu [2, 3] and Tejumola [8] can be given as good examples on this subject. However, with respect to our observations, only a few studies were carried out on the same topic for the solutions of ordinary nonlinear vector differential equations of higher orders. In this aspect studies fulfilled by Ezeilo [5] and Tunç [13] could be given as examples.

In this paper, taking into account the results obtained for the ordinary nonlinear scalar differential equations

$$x^{(4)} + f_1(\ddot{x})\,\dddot{x} + f_2(\dot{x})\,\ddot{x} + f_3(\dot{x}) + f_4(x) = 0$$

and

$$x^{(5)} + b_1 x^{(4)} + g_1(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\,\dddot{x} + g_2(\dot{x})\,\ddot{x}$$
$$+ g_3(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) + g_4(x) = 0,$$

by Tiryaki [9], we establish two new results on the same topic for the nonlinear vector differential equations as follows:

$$X^{(4)} + \Phi(\ddot{X})\,\dddot{X} + \Psi(\dot{X})\,\ddot{X} + F(\dot{X}) + G(X) = 0 \qquad (1.1)$$

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and

$$X^{(5)} + AX^{(4)} + \Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\, \dddot{X} + \Psi(\dot{X})\, \ddot{X}$$
$$+ \Omega(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\, \dot{X} + \Theta(X) = 0. \tag{1.2}$$

in which $X \in R^n$; A is a constant $n \times n$-symmetric matrix; Φ, Ψ and Ω are continuous $n \times n$-symmetric matrices depending, in each case, on the arguments shown; $F, G, \Theta : R^n \to R^n$ are continuous n-vector functions. It will be assumed

$$F(0) = 0, \quad G(0) = 0 \tag{1.3}$$

and

$$\Omega(X, 0, Z, U, V) = 0, \quad \Theta(0) = 0 \tag{1.4}$$

for an arbitrary value of X, Z, U and V. Let $J_G(X)$ denote the Jacobian matrix corresponding to the function $G(X)$, that is, $J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right)$, $(i, j = 1, 2, ..., n)$ where $(x_1, x_2, ..., x_n)$ and $(g_1, g_2, ..., g_n)$ are the components of X and G, respectively. Other than these, it will also be assumed that the Jacobian matrices $J_G(X)$ exist and are symmetric and continuous. The symbol $\langle X, Y\rangle$ is used to denote the usual scalar product in R^n for given any X, Y in R^n, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\|X\|^2 = \langle X, X\rangle$. The matrix A is said to be negative-definite, when $\langle AX, X\rangle < 0$ for all non-zero X in R^n, and $\lambda_i(A)$ $(i = 1, 2, ..., n)$ are the eigenvalues of the $n \times n$-matrix A.

In what follows we use the following differential systems which are equivalent to the equations (1.1) and (1.2):

$$\begin{gathered} \dot{X} = Y, \dot{Y} = Z, \dot{Z} = U \\ \dot{U} = -\Phi(Z)U - \Psi(Y)Z - F(Y) - G(X) \end{gathered} \tag{1.5}$$

and

$$\begin{gathered} \dot{X} = Y, \dot{Y} = Z, \dot{Z} = U, \dot{U} = V, \\ \dot{V} = -AV - \Phi(X, Y, Z, U, V)U - \Psi(Y)Z - \Omega(X, Y, Z, U, V)Y - \Theta(X), \end{gathered} \tag{1.6}$$

respectively.

2. Main result

We shall establish here the following theorems.

Theorem 1. *In addition to the basic assumptions on the Φ, Ψ, F and G, suppose that there are constants a_2 and a_4 with $a_4 > \frac{1}{4}a_2^2$ such that*

(i) $0 \le \lambda_i(\Psi(Y)) \le a_2$ *for all* $Y \in R^n, (i = 1, 2, ..., n)$

(ii) $\lambda_i(J_G(X)) \geq a_4$ *for all* $X \in R^n, (i = 1, 2, ..., n)$.

Then equation (1.1) has no periodic solution whatsoever other than $X = 0$ *for all arbitrary* Φ.

Theorem 2. *In addition to the basic assumptions on the* A, Φ, Ψ, Ω *and* Θ, *suppose that*

(i) $\Theta(X) \neq 0$ *for* $X \neq 0$

(ii) $\lambda_i(\Omega(X, Y, Z, U, V)) \geq \frac{1}{4}\left[\lambda_i(\Phi(X, Y, Z, U, V))\right]^2$ *for arbitrary* X, Y, Z, U, V *then the equation (1.2) has no periodic solution whatsoever other than* $X = 0$ *for all arbitrary* A, Ψ.

Now, we dispose of some well known algebraic results which will be required in the proof of theorems. The first of these is a quite standard one:

Lemma 1. *Let* A *be a real symmetric* $n \times n$ *matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1, 2, ..., n), \text{ where } a', a \text{ are constants.}$$

Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle .$$

Proof. See [7]. □

Lemma 2. *Let* Q, D *be any two real* $n \times n$ *commuting symmetric matrices. Then*

(i) The eigenvalues $\lambda_i(QD)$ $(1, 2, ..., n)$ *of the product matrix* QD *are real and satisfy*

$$\max_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D)$$

(ii) The eigenvalues $\lambda_i(Q + D)$ $(1, 2, ..., n)$ *of the sum of matrices* Q *and* D *are real and satisfy*

$$\left\{\max_{1\leq j\leq n} \lambda_j(Q) + \max_{1\leq k\leq n} \lambda_k(D)\right\} \geq \lambda_i(Q + D) \geq \left\{\min_{1\leq j\leq n} \lambda_j(Q) + \min_{1\leq k\leq n} \lambda_k(D)\right\}.$$

Proof. See [1]. □

Proof of the Theorem 1. Let $(X, Y, Z, U) = (X(t), Y(t), Z(t), U(t))$ be an arbitrary α-periodic solution of (1.5), that is

$$(X(t), Y(t), Z(t), U(t)) = (X(t + \alpha), Y(t + \alpha), Z(t + \alpha), U(t + \alpha)) \tag{2.1}$$

for some $\alpha > 0$. It will be shown that, subject to the conditions in *Theorem 1*,

$$X = Y = Z = U = 0.$$

Our main tool in the proof of *Theorem 1* is the function $\Gamma = \Gamma(X, Y, Z, U)$ given by:

$$\begin{aligned}\Gamma = &\int_0^1 \langle \sigma\Phi(\sigma Z)Z, Z\rangle \, d\sigma + \int_0^1 \langle \Psi(\sigma Y)Y, Z\rangle \, d\sigma + \langle U, Z\rangle \\ &+ \langle Y, G(X)\rangle + \int_0^1 \langle F(\sigma Y), Y\rangle \, d\sigma.\end{aligned} \tag{2.2}$$

Consider the function

$$\psi(t) \equiv \Gamma(X(t), Y(t), Z(t), U(t)).$$

Since Γ is continuous and X, Y, Z, U are periodic in t, $\psi(t)$ is clearly bounded. An elementary differentiation will show that

$$\begin{aligned}\dot{\Gamma} = &\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Z)Z, Z\rangle \, d\sigma + \frac{d}{dt}\int_0^1 \langle \Psi(\sigma Y)Y, Z\rangle \, d\sigma + \langle U, U\rangle - \langle Z, \Phi(Z)U\rangle \\ &- \langle Z, \Psi(Y)Z\rangle - \langle Z, F(Y)\rangle + \langle Y, J_G(X)Y\rangle + \frac{d}{dt}\int_0^1 \langle F(\sigma Y), Y\rangle \, d\sigma.\end{aligned} \tag{2.3}$$

But

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle F(\sigma Y), Y\rangle \, d\sigma &= \int_0^1 \sigma \langle J_F(\sigma Y)Z, Y\rangle \, d\sigma + \int_0^1 \langle F(\sigma Y), Z\rangle \, d\sigma \\ &= \int_0^1 \sigma \frac{\partial}{\partial\sigma} \langle F(\sigma Y), Z\rangle \, d\sigma + \int_0^1 \langle F(\sigma Y), Z\rangle \, d\sigma \\ &= \sigma \langle F(\sigma Y), Z\rangle \Big|_0^1 = \langle F(Y), Z\rangle ,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Z)Z, Z\rangle \, d\sigma &= \int_0^1 \langle \sigma\Phi(\sigma Z)U, Z\rangle \, d\sigma + \int_0^1 \sigma^2 \langle J_\Phi(\sigma Z)ZU, Z\rangle \, d\sigma \\ &\quad + \int_0^1 \langle \sigma\Phi(\sigma Z)Z, U\rangle \, d\sigma \\ &= \int_0^1 \sigma \frac{\partial}{\partial\sigma} \langle \sigma\Phi(\sigma Z)U, Z\rangle \, d\sigma + \int_0^1 \langle \sigma\Phi(\sigma Z)U, Z\rangle \, d\sigma \\ &= \sigma^2 \langle \Phi(\sigma Z)U, Z\rangle \Big|_0^1 = \langle \Phi(Z)U, Z\rangle\end{aligned} \tag{2.5}$$

and similarly we have

$$\frac{d}{dt}\int_0^1 \langle \Psi(\sigma Y)Y, Z\rangle \, d\sigma = \langle \Psi(Y)Z, Z\rangle + \int_0^1 \langle \Psi(\sigma Y)Y, U\rangle \, d\sigma. \tag{2.6}$$

Upon gathering the estimates (2.4), (2.5) and (2.6) into (2.3) we obtain

$$\begin{aligned}\dot{\Gamma} &= \langle U, U\rangle + \int_0^1 \langle \Psi(\sigma Y)Y, U\rangle\, d\sigma + \langle Y, J_G(X)Y\rangle \\ &\geq \|U\|^2 - a_2 \|Y\| \|U\| + a_4 \|Y\|^2 \\ &= \left(\|U\| - \tfrac{1}{2}a_2 \|Y\|\right)^2 + a_4 \|Y\|^2 - \tfrac{1}{4}a_2^2 \|Y\|^2 \\ &= \left(\|U\| - \tfrac{1}{2}a_2 \|Y\|\right)^2 + \left(a_4 - \tfrac{1}{4}a_2^2\right) \|Y\|^2 \geq 0.\end{aligned} \tag{2.7}$$

Hence $\dot{\psi}\ (t) \geq 0$, so that $\psi(t)$ is monotone in t, and therefore, being bounded, tends to a limit, ψ_0 say, as $t \to \infty$. It is readily checked that

$$\psi(t) \equiv \psi_0 \quad \text{for all } t. \tag{2.8}$$

From by (2.1),

$$\psi(t) = \psi(t + m\alpha) \tag{2.9}$$

for any arbitrary fixed t an for arbitrary integer m, and then letting $m \to \infty$ in the right-hand side of (2.9) leads to (2.8).

The result (2.8) itself implies that

$$\dot{\psi}\ (t) = 0 \quad \text{for all } t$$

from which, by (2.7), it follows from assumptions onΨ and G , that

$$Y = 0 \quad \text{for all } t, \tag{2.10}$$

which in turn implies that

$$X = \xi \text{ (constant)},\ Y = 0 = Z = U \quad \text{for all } t. \tag{2.11}$$

Since (X, Y, Z, U) is a solution of (1.5), it is evident from (2.10) and (2.11) that $G(\xi) = 0$, so that $\xi = 0$, by (1.3). Hence

$$(X, Y, Z, U) = (0, 0, 0, 0)$$

This completes the proof of *Theorem 1.* □

Proof of Theorem 2. Let $(X, Y, Z, U, V) = (X(t), Y(t), Z(t), U(t), V(t))$ be an arbitrary ω-periodic solution of (1.6), that is

$$(X(t), Y(t), Z(t), U(t), V(t)) = (X(t+\omega), Y(t+\omega), Z(t+\omega), U(t+\omega), V(t+\omega))$$

for some $\omega > 0$.

Consider the function $W = W(X, Y, Z, U, V)$ defined by

$$\begin{aligned}W = &\ \tfrac{1}{2}\langle AZ, Z\rangle + \langle Z, U\rangle - \langle Y, V\rangle - \langle Y, AU\rangle \\ &- \int_0^1 \langle \sigma\Psi(\sigma Y)Y, Y\rangle\, d\sigma - \int_0^1 \langle \Theta(\sigma X), X\rangle\, d\sigma.\end{aligned} \tag{2.12}$$

It is clear that W is bounded. An elementary differentiation from (1.6) and (2.12) yields

$$\begin{aligned}\dot{W} = &\langle U,U\rangle + \langle Y,\Phi(X,Y,Z,U,V)U\rangle + \langle Y,\Omega(X,Y,Z,U,V)Y\rangle + \langle Y,\Theta(X)\rangle \\ &+ \langle Y,\Psi(Y)Z\rangle - \tfrac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Y)Y,Y\rangle\, d\sigma - \tfrac{d}{dt}\int_0^1 \langle \Theta(\sigma X),X\rangle\, d\sigma.\end{aligned} \tag{2.13}$$

But

$$\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Y)Y,Y\rangle\, d\sigma = \langle \Psi(Y)Z,Y\rangle \tag{2.14}$$

and

$$\frac{d}{dt}\int_0^1 \langle \Theta(\sigma X),X\rangle\, d\sigma = \langle \Theta(X),Y\rangle\,. \tag{2.15}$$

Using the estimates (2.14) and (2.15) in (2.13) we obtain

$$\begin{aligned}\dot{W} &= \langle U,U\rangle + \langle Y,\Phi(X,Y,Z,U,V)U\rangle + \langle Y,\Omega(X,Y,Z,U,V)Y\rangle \\ &= \left\|U + \tfrac{1}{2}\Phi(X,Y,Z,U,V)Y\right\|^2 + \langle Y,\Omega(X,Y,Z,U,V)Y\rangle \\ &\quad -\tfrac{1}{4}\langle \Phi(X,Y,Z,U,V)Y,\Phi(X,Y,Z,U,V)Y\rangle \\ &\geq \langle Y,\Omega(X,Y,Z,U,V)Y\rangle - \tfrac{1}{4}\langle \Phi(X,Y,Z,U,V)Y,\Phi(X,Y,Z,U,V)Y\rangle \geq 0\end{aligned}$$

Therefore, the rest of the proof, can be shown in the same way as the proof of *Theorem 1*, which gives

$$X = Y = Z = U = V = 0.$$

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