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Metrical relationships in a standard triangle in an isotropic plane

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Abstract. *Each allowable triangle of an isotropic plane can be set in a standard position, in which it is possible to prove geometric properties analytically in a simplified and easier way by means of the algebraic theory developed in this paper.*

Key words: *isotropic plane, triangle, standard triangle*

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1. Terms of elementary geometry in I_2

Let $\mathbf{P}_2(\mathbb{R})$ be a real projective plane, $\mathbf{f}$ a real line in $\mathbf{P}_2$, and $\mathbf{A}_2 = \mathbf{P}_2 \setminus \mathbf{f}$ the associated affine plane. The *isotropic plane* $\mathbf{I}_2(\mathbb{R})$ is a real affine plane $\mathbf{A}_2$ where the metric is introduced with a real line $\mathbf{f} \subset \mathbf{P}_2$ and a real point F incidental with it. The main facts about the isotropic plane can be found in [1], [2], [3].
We will first define some terms and point out some properties of triangles and circles in I_2 that are going to be used further on.

All straight lines through the point F are called *isotropic straight lines* (*isotropic lines*). All other straight lines are simply called *straight lines*. Two points A, B ($A \neq B$) are called *parallel* if they lie on the same isotropic line. For two non-parallel points $A = (a_1, a_2)$, $B = (b_1, b_2)$, the *isotropic distance* is defined by $d(A, B) = b_1 - a_1$. Note that the isotropic distance is directed. For two parallel points $A = (a_1, a_2)$, $B = (b_1, b_2)$, $a_1 = b_1$, the quantity known as an *isotropic span* is defined by $s(A, B) = b_2 - a_2$. A straight line $\mathcal{P}$ through two points A and B will be denoted by $\mathcal{P} = AB$. Furthermore, it will be recognized from the context whether, for example BC, refers to the straight line passing through points B and C or to the length of the line segment with endpoints B, C.

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Each non-isotropic straight line $\mathcal{G} \subset \mathbf{I}_2$ can be written in the normal form $y = ux + v$, that is, in line coordinates, $\mathcal{G} = (u, v)$. For two straight lines $\mathcal{G}_1 = (u_1, v_1)$, $\mathcal{G}_2 = (u_2, v_2)$ the *isotropic angle* is defined by $\varphi = \angle(\mathcal{G}_1, \mathcal{G}_2) = u_2 - u_1$. Note that the isotropic angle is directed as well. The Euclidean meaning of the isotropic angle can be understood from the affine model given in *Figure 1*.

For two parallel straight lines $\mathcal{G}_1 = (u_1, v_1)$, $\mathcal{G}_2 = (u_2, v_2)$ there exists an isotropic invariant defined by $\varphi^*(\mathcal{G}_1, \mathcal{G}_2) = v_2 - v_1$ (see *Figure 2*).

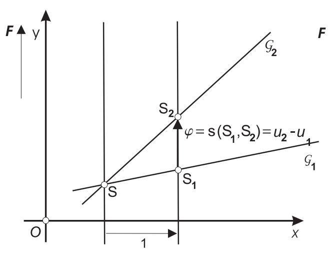

Figure 1.

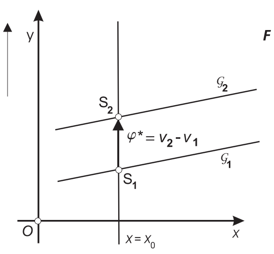

Figure 2.

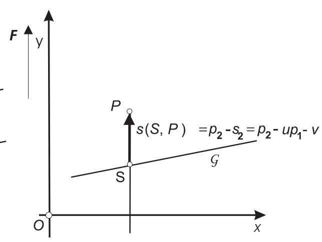

Figure 3.

An isotropic normal to the straight line $\mathcal{G} = (u, v)$ from the point $P = (p_1, p_2)$, $P \notin \mathcal{G}$ is an isotropic line through P. The inverse holds as well, i.e. each straight line $\mathcal{G} \subset \mathbf{I}_2$ is a normal for each isotropic straight line.

Denoting by $S = (s_1, s_2)$ the point of intersection of the isotropic normal from the point P with the straight line $\mathcal{G}$, the isotropic span of the point P from the line $\mathcal{G}$ is given by $s(S, P) = p_2 - s_2 = p_2 - up_1 - v$ (see *Figure 3*).

2. Triangle in a standard position

Under a *triangle* in $\mathbf{I}_2$ an ordered set of three non-collinear points (A, B, C) is understood. A, B, C are called *vertices*, and BC, CA, AB *sides* of a triangle. A triangle is called *allowable* if none of its sides is isotropic. In an allowable triangle the *lengths* of the sides are defined by $BC = d(B, C)$, $CA = d(C, A)$, $AB = d(A, B)$, with $BC \neq 0$, $CA \neq 0$, $AB \neq 0$, $BC + CA + AB = 0$. For the directed angles we have $\angle A = \angle(AB, AC)$, $\angle B = \angle(BC, BA)$, $\angle C = \angle(CA, CB)$, $\angle A + \angle B + \angle C = 0$ (see *Figure 4*).

Isotropic altitudes $\mathcal{H}_{BC}$, $\mathcal{H}_{CA}$, $\mathcal{H}_{AB}$ associated with sides BC, CA and AB are isotropic straight lines passing through vertices A, B, C, i.e. normals to sides BC, CA and AB. Their lengths are defined by $h_a = s(A_h, A)$ where $A_h = BC \cap \mathcal{H}_{BC}$, etc. The Euclidean meaning is given in *Figure 5*.

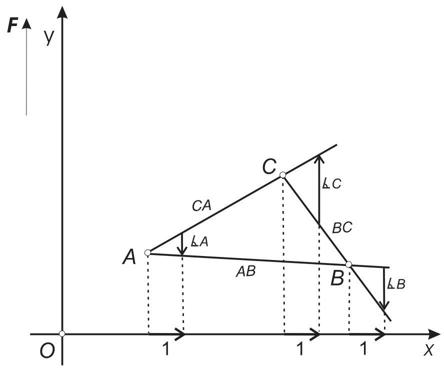

Figure 4.

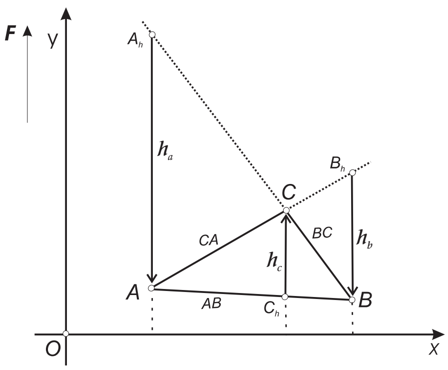

Figure 5.

According to [1], to any allowable triangle in $\mathbf{I}_2$ exactly one circle can be circumscribed. The equation of the circumcircle is of the form

$$y = ux^2 + vx + w, \quad u \neq 0.$$

The coordinate system can be moved to the position that reduces above equation to the form $y = ux^2$. Without loss of generality, by using the substitution $\frac{y}{u} \to y$, we will assume that the equation of this circle is given by

$$\mathcal{K}_c \quad \ldots \quad y = x^2. \tag{1}$$

Excepting as arranged, vertices of a given allowable triangle $\triangle ABC$ are

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{2}$$

a, b, c being mutually different numbers. In writing relations, equations, etc., the following abbreviations will be useful:

$$a + b + c = s, \quad abc = p, \quad ab + bc + ca = q. \tag{3}$$

Besides, by choosing without loss of generality, that

$$s = a + b + c = 0 \tag{4}$$

is fulfilled, it follows that the y–axis coincides with the diameter of the circle passing through the centroid

$$G = \left(\frac{a+b+c}{3}, \frac{a^2+b^2+c^2}{3}\right) = \left(0, -\frac{2}{3}q\right) \tag{5}$$

of $\triangle ABC$, while the x–axis is a tangent line of the circumscribed circle at the endpoint of the diameter through G.

For each allowable triangle ABC it can be achieved in the described way, that its circumscribed circle has equation (1) and its vertices are of the form (2) while equalities (3) and (4) are satisfied. We shall say that this triangle is in the *standard position* or shorter triangle ABC is a *standard triangle*. To prove geometric facts for each allowable triangle it is sufficient to give a proof for a standard triangle.

3. Algebraic relationships in the standard triangle

For the systematic studying of the triangle in an isotropic plane various consequences of equations (3) and (4) will be useful.

First of all, the following equalities

$$a^2 = bc - q, \quad b^2 = ca - q, \quad c^2 = ab - q \tag{6}$$

are valid, because of e.g.

$$a^2 = -a(b + c) = bc - q,$$

so it follows

$$a^2 + b^2 + c^2 = -2q, \tag{7}$$

and it implies that $q < 0$. Equality (7) is used in (5). Then we have

$$q = -(b^2 + bc + c^2) = -(c^2 + ca + a^2) = -(a^2 + ab + b^2) \tag{8}$$

because of e.g.

$$b^2 + bc + c^2 = (b + c)^2 - bc = a^2 - bc = -q$$

is valid according to (6).

The formula

$$BC^2 + CA^2 + AB^2 = (b - c)^2 + (c - a)^2 + (a - b)^2 = -6q \tag{9}$$

is also valid because of

$$2(a^2 + b^2 + c^2) - 2(bc + ca + ab) = -4q - 2q = -6q.$$

The following equalities are also interesting

$$(b - c)^2 = -(q + 3bc), \quad (c - a)^2 = -(q + 3ca), \quad (a - b)^2 = -(q + 3ab), \tag{10}$$

$$(c-a)(a-b) = 2q-3bc, \quad (a-b)(b-c) = 2q-3ca, \quad (b-c)(c-a) = 2q-3ab, \tag{11}$$

and they are obtained for example in this way

$$\begin{aligned}(b - c)^2 &= (b + c)^2 - 4bc = a^2 - 4bc = bc - q - 4bc = -(q + 3bc),\\ (c - a)(a - b) &= -a^2 - bc + ca + ab = -(bc - q) + q - 2bc = 2q - 3bc.\end{aligned}$$

By means of (10), the equality

$$(b - c)^2(c - a)^2(a - b)^2 = -(27p^2 + 4q^3) \tag{12}$$

follows, and therefore the inequality $27p^2 + 4q^3 < 0$ is valid. Indeed, the left-hand side of (12) is equal to

$$\begin{aligned}-(q + 3bc)(q + 3ca)(q + 3ab) &= -q^3 - 3q^2(bc + ca + ab) - 9qabc(a + b + c) - 27a^2b^2c^2\\ &= -27p^2 - 4q^3.\end{aligned}$$

4. Elements of the standard triangle

The standard triangle ABC has vertices given by equalities (2). The sides of that triangle have the equations

$$\begin{aligned} BC \quad &\ldots \quad y = -ax - bc, \\ CA \quad &\ldots \quad y = -bx - ca, \\ AB \quad &\ldots \quad y = -cx - ab, \end{aligned} \tag{13}$$

since, for example, for the point $B = (b, b^2)$ and the first line (13) owing to (4) the equality $b^2 = -ab - bc$ is valid.

With $x = a$ from the equation of the line BC owing to (6) it follows

$$y = -a^2 - bc = q - 2bc,$$

and we obtain the first of the three feet of altitudes

$$A_h = (a, q - 2bc), \quad B_h = (b, q - 2ca), \quad C_h = (c, q - 2ab) \tag{14}$$

of the triangle ABC, and the remaining two are obtained analogously. Now, because of (6) and (11), we obtain, for example, for the altitude h_a

$$h_a = s(A_h, A) = a^2 - (q - 2bc) = 3bc - 2q = -(c - a)(a - b),$$

then equalities

$$h_a = -(c - a)(a - b), \quad h_b = -(a - b)(b - c), \quad h_c = -(b - c)(c - a) \tag{15}$$

are valid. Triangle ABC with vertices (2) has the (oriented) area $\triangle$, for which we obtain

$$2\triangle = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = (b - c)(c - a)(a - b) = -BC \cdot CA \cdot AB, \tag{16}$$

because of $BC = c - b$, $CA = a - c$, $AB = b - a$. From (15) and (16) the following equalities, analogous to those in Euclidean geometry, follow immediately

$$2\triangle = BC \cdot h_a = CA \cdot h_b = AB \cdot h_c. \tag{17}$$

If we want the equality $4\triangle R = -BC \cdot CA \cdot AB$ to be as in Euclidean geometry, we have to arrange that the radius of the circumscribed circle of the standard triangle is equal to $R = \frac{1}{2}$, i.e. this is the radius of the circle with equation (1). (In [1] the arrangement is slightly different; it is arranged that circle (1) has radius 1.) In various metric formulae, which are not seemingly homogenous, i.e. all members are not of the same dimension, it is necessary to replace factor 1 with $2R$ at some places so that the formula becomes homogenous. So, for example the first equation (15) should be written in the form $2Rh_a = -CA \cdot AB$.

As the absolute point F has the role of the orthocenter and the center of the circumscribed circle too, then the isotropic line through the centroid G of that triangle will be called an *Euler line* of that triangle (*Figure 6*). Because of (5), the standard triangle ABC has Euler line $\mathcal{E}$ with the equation $x = 0$.

The line with the equation

$$y = 2ax + 2bc - q$$

passes through points B_h and C_h from (14) since e.g. for the point B_h the equality $2ab + 2bc - q = q - 2ca$ is valid. For that reason it is the equation of the line B_hC_h. Triangle $A_hB_hC_h$ is called an *orthic triangle* of triangle ABC. Its sides have the equations

$$\begin{aligned} B_hC_h \quad &\ldots \quad y = 2ax + 2bc - q, \\ C_hA_h \quad &\ldots \quad y = 2bx + 2ca - q, \\ A_hB_h \quad &\ldots \quad y = 2cx + 2ab - q. \end{aligned} \tag{18}$$

Theorem 1. *The corresponding sides of the triangle and its orthic triangle intersect at three points which lie on the same line (Figure 6).*

Proof. If we add to the first equation in (18) the first equation from (13) multiplied by 2, we obtain the equation of the line $3y = -q$ on which the point $BC \cap B_hC_h$ lies, and the same is valid for the points $CA \cap C_hA_h$ and $AB \cap A_hB_h$. □

The line from *Theorem 1* is called, by the analogy with the Euclidean case, *orthic axis* of the observed triangle.

Corollary 1. *The orthic axis* $\mathcal{H}$ *of the standard triangle* ABC *has the equation*

$$\mathcal{H} \quad \ldots \quad y = -\frac{q}{3}. \tag{19}$$

The orthic axis of the triangle has one more interesting property.

Theorem 2. *The centroid of three points at which an arbitrary isotropic straight line intersects sides of an allowable triangle lies on the orthic axis of this triangle.*

Proof. Really, the arithmetic mean of the right-hand sides of equalities (13) is equal to $-\frac{q}{3}$ for each x, because of (4). □

In some way, the next theorem is related to *Theorem 2*.

Theorem 3. *The sum of the spans from vertices of the triangle to the line* $\mathcal{G}$ *is equal to zero if and only if that line passes through the centroid of that triangle.*

Proof. The line $\mathcal{G}$ with equation $y = ux + v$ passes through the centroid G from (5) of the standard triangle ABC under the assumption $v = -\frac{2}{3}q$. Spans of points A, B, C from (2) to the line $\mathcal{G}$ are, respectively, equal to

$$a^2 - au - v, \quad b^2 - bu - v, \quad c^2 - cu - v$$

and because of (4) and (7) we have the sum $-2q - 3v$, which is equal to zero under the same assumption $v = -\frac{2}{3}q$. □

Out of all the lines through the centroid of the triangle the one which is parallel to its orthic axis is specially interesting (*Figure 6*). That line will be called *inertial axis* of triangle.

Corollary 2. *The inertial axis* $\mathcal{G}$ *of the standard triangle has the equation*

$$\mathcal{G} \quad \ldots \quad y = -\frac{2}{3}q. \tag{20}$$

5. Complementarity and anticomplementarity

If G is the centroid of the allowable triangle ABC, then the homothecy $(G, -\frac{1}{2})$ and its inverse homothecy $(G, -2)$, by analogy with the Euclidean case, will be called *complementarity* resp. *anticomplementarity* with regard to triangle ABC. The image of the point T (or the curve $\mathcal{K}$) by these homothecies will be called a *complementary* resp. *anticomplementary* point (curve) of the point T (of the curve $\mathcal{K}$, respectively).

Theorem 4. *The point $(-\frac{x}{2}, -\frac{y}{2} - q)$ is a complementary point and the point $(-2x, -2y - 2q)$ is an anticomplementary point to the point $T = (x, y)$, with regard to triangle ABC.*

Proof. If the point $T' = (x', y')$ is complementary to the point $T = (x, y)$ then the equality $T + 2T' = 3G$ is valid, i.e. because of (5) we have equalities $x + 2x' = 0$ and $y + 2y' = -2q$, wherefrom $x' = -\frac{x}{2}$, $y' = -\frac{y}{2} - q$ follows. If the point $T'' = (x'', y'')$ is anticomplementary to the point $T = (x, y)$, then the equality $2T + T'' = 3G$ is valid, i.e. we have equalities $2x + x'' = 0$ and $2y + y'' = -2q$, from which $x'' = -2x$ and $y'' = -2y - 2q$ follow. □

Theorem 5. *If the curve $\mathcal{K}$ has the equation $\mathcal{K}(x, y) = 0$, then its complementary and anticomplementary curve, with regard to the standard triangle ABC, have successively the equations*

$$\mathcal{K}(-2x, -2y - 2q) = 0 \quad \textit{and} \quad \mathcal{K}\left(-\frac{x}{2}, -\frac{y}{2} - q\right) = 0. \tag{21}$$

Proof. If the point $T' = (x', y')$ is complementary to the point $T = (x, y)$ on the curve $\mathcal{K}$, then $x = -2x'$ and $y = -2y' - 2q$ is valid according to the proof of *Theorem 4*, $\mathcal{K}(-2x', -2y' - 2q) = 0$ is the equation of the curve which is described by the point T'. If we write x and y instead of x' and y', then we get the first equation (21). If the point T'' is anticomplementary to the point $T = (x, y)$ then we obtain $x = -\frac{x''}{2}$, $y = -\frac{y''}{2} - q$ by the proof of *Theorem 4* hence $\mathcal{K}(-\frac{x''}{2}, -\frac{y''}{2} - q) = 0$ is the equation of the curve which is described by the point T''. With the substitutions $x'' \to x$, $y'' \to y$ we obtain the second equation (21). □

The complementary point to the point $A = (a, a^2)$, because of

$$-\frac{a^2}{2} - q = -\frac{1}{2}(bc - q) - q = -\frac{q}{2} - \frac{bc}{2}$$

and according to *Theorem 4*, is the first of the three analogous points

$$A_m = (-\frac{a}{2}, -\frac{q}{2} - \frac{bc}{2}), \quad B_m = (-\frac{b}{2}, -\frac{q}{2} - \frac{ca}{2}), \quad C_m = (-\frac{c}{2}, -\frac{q}{2} - \frac{ab}{2}). \tag{22}$$

Anticomplementary point to the same point is the first one of the three points

$$A_n = (-2a, -2bc), \quad B_n = (-2b, -2ca), \quad C_n = (-2c, -2ab) \tag{23}$$

because of $-2a^2 - 2q = -2bc$. Triangles $A_mB_mC_m$ and $A_nB_nC_n$ are *complementary* and *anticomplementary* triangles of the standard triangle ABC. The points

A_m, B_m, C_m are in fact the midpoints of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of triangle ABC. Really, points B and C from (2) have the midpoint A_m because of

$$\frac{1}{2}(b+c) = -\frac{a}{2}, \quad \frac{1}{2}(b^2+c^2) = \frac{1}{2}(-q-bc)$$

which is valid according to (4) and (8).

A complementary resp. anticomplementary line to the line BC with the first equation (13) has the equation $-2y-2q = -a(-2x)-bc$ and $-\frac{y}{2}-q = -a(-\frac{x}{2})-bc$ according to *Theorem 4*, and this is the first of the three analogous equations of the sides of the triangle $A_mB_mC_m$ and $A_nB_nC_n$ in the form

$$\begin{aligned} B_mC_m &\quad \ldots \quad y = -ax - q + \frac{bc}{2}, \\ C_mA_m &\quad \ldots \quad y = -bx - q + \frac{ca}{2}, \\ A_mB_m &\quad \ldots \quad y = -cx - q + \frac{ab}{2}, \end{aligned} \tag{24}$$

respectively

$$\begin{aligned} B_nC_n &\quad \ldots \quad y = -ax + 2a^2, \\ C_nA_n &\quad \ldots \quad y = -bx + 2b^2, \\ A_nB_n &\quad \ldots \quad y = -cx + 2c^2, \end{aligned} \tag{25}$$

because of $bc - q = a^2$. The lines (25) are the middle lines of the triangle ABC, and the lines (26) are parallels with the lines BC, CA, AB, which obviously pass successively through the points A, B, C from (2).

A complementary resp. anticomplementary circle to the circumscribed circle of triangle ABC with the equation (1) is the circle with the equation $-2y-2q = (-2x)^2$ resp. $-\frac{y}{2}-q = (-\frac{x}{2})^2$, i.e. $y = -2x^2 - q$ respectively $y = -\frac{1}{2}x^2 - 2q$. Thus we have proved these two theorems.

Theorem 6. *The complementary triangle $A_mB_mC_m$ to the standard triangle ABC has vertices (22), sides with equations (24) and the circumscribed circle with the equation*

$$y = -2x^2 - q. \tag{26}$$

Theorem 7. *The anticomplementary triangle $A_nB_nC_n$ to the standard triangle ABC has vertices (23), sides with equations (25) and the circumscribed circle with the equation*

$$y = -\frac{1}{2}x^2 - 2q. \tag{27}$$

As midpoints A_m, B_m, C_m of the sides of triangle ABC have the abscissas $-\frac{a}{2}$, $-\frac{b}{2}$, $-\frac{c}{2}$, then the perpendicular bisectors of sides BC, CA, AB have successively the equations

$$x = -\frac{a}{2}, \quad x = -\frac{b}{2}, \quad x = -\frac{c}{2}. \tag{28}$$

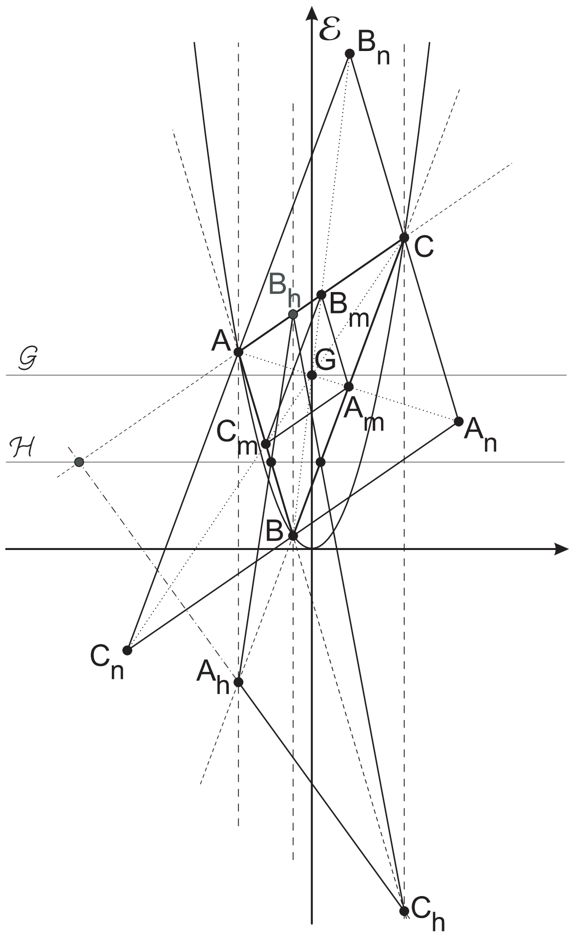

Figure 6.

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