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(1,n) Congruences

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ABSTRACT

The first order algebraic congruences are classified into two basic classes which depend on their directing curves. By the method of synthetic geometry, we investigated the basic properties for each of these classes: the construction of rays, singularities, decomposition into developable surfaces, focal properties and the types of rays. The paper ends with a short analytical approach, which enables the visualizations of these congruences in the program *Mathematica*. Some exmples are shown.

Key words: congruence, decomposition on developable surfaces, focal lines, singularities, visualization

MSC 2000: 51M15,51M30,51N10,51N35,68U05

(1,n) kongruencije

SAŽETAK

Algebarske kongruencija prvoga reda razvrstane su u dvije osnovne klase, ovisno o njihovim ravnalicama. Za svaku od tih klasa, metodologijom sintetičke geometrije, istražuju se osnovna svjstva: konstrukcija zraka, singulariteti, dekompozicija na razvojne plohe, žarišne osobine te vrste zraka. Na kraju se daje i kratki analitički pristup koji je omogućio izradu programa za vizualizaciju ovih kongruencija u programu *Mathematica*. Pokazano je nekoliko primjera.

Ključne riječi: kongruencija, dekompozicija na razvojne plohe, žarišna linija, singulariteti, vizualizacija

1 Introduction

A congruence $\mathcal{C}$ is a double infinite line system, i.e. it is a set of lines in a three-dimensional space (projective, affine or Euclidean) depending on two parameters. A line $l \in \mathcal{C}$ is said to be a *ray* of the congruence.

The *order* of a congruence is the number of its rays which pass through an arbitraty point; the *class* of a congruence is the number of its rays which lie in an arbitrary plane. mth *order*, nth *class* congruence is signed $\mathcal{C}_n^m$.

A point is called the *singular point* of a congruence if ∞^1 rays pass through it. A plane is called the *singular plane* of a congruence if it contains ∞^1 rays.

Rays in a congruence can be decomposed in two ways into a one-parameter family of developable surfaces (torses) so that through every ray $l \in \mathcal{C}$ pass two torses that are real and different (the case of *hyperbolic* ray), or imaginary (an *elliptic* ray), or real and coincident (a *parabilic* ray).

The points of contact of a ray $l \in \mathcal{C}$ with the edges of regression (cuspidal edges) of these torses are called the *foci* of l. The foci of l are the intersection points of l with consecutive rays of a congruence. The surfaces formed by the foci of the rays of a congruence are called its *focal surfaces*. Each ray of a congruence touches its focal surface at the foci. Two planes defined as the planes containing a ray and consecutive rays are the *focal planes* of a congruence. The focal surface is the envelope of the focal planes. A congruence clearly reciprocates into a congruence. The focal planes and points are interchanged and the focal surface reciprocates into the new focal surface.
[11], [5], [9], [1]

Since the lines of the congruence are bitangents of the focal surface, every congruence of lines may be regarded as the system of bitangents of a surface. The surface may, however, break up into two separate surfaces, and the original surface, or each or either of the component surfaces may degenerate into a curve; we have thus as congruences the following systems of lines:

1. the bitangents of a surface,
2. common tangents of two surfaces,
3. tangents to a surface from the points of a curve,
4. common transversales of two curves,
5. lines “through two points” of a curve,

where the last four cases being degenerate cases of the first, which is the general one. [9, p.37]

2 1st order congruences

It was proved that the rays of the first order congruences are always tranversales of two curves, or they intersect the same space curve twice. Beside that it was proved that the only congruence of the first order, consisting of a system of lines meeting a proper curve twice, is when the curve is a twisted cubic. ([9, p. 64], [14, pp. 1184-1185], [13, p. 32])

If a congruence $\mathcal{C}$ is a system of lines meeting two directing curves of the orders m and m' which have α common points, the order of a congruence is $mm' - \alpha$. The only congruence of the first order of this kind is when the directing curves are a curve of the nth order and a straight line meeting it $n-1$ times. [9, p. 64]

Therefore, we have only two types of the congruences of the first order:

Type I 1st odrer nth class congruences $\mathcal{C}_n^1$ are the systems of lines which intersect a space curve c^n of the order n and a straight line d, where c^n and d have $n-1$ common points.

Type II 1st order 3rd class congruence $\mathcal{B}_3^1$ is the system of lines which meet a twisted cubic twice.

2.1 Congruences of the type I

2.1.1 Directing lines of $\mathcal{C}_n^1$

The directing lines of a congruence $\mathcal{C}_n^1$ are a space curve c^n of the order n and a straight line d which intersects c^n in $n-1$ points. If all intersection points are the regular points of c^n we will sign them $D_1^1, \ldots\ldots, D_{n-1}^1$. Some of these points can coincide. There are cases when the line d is the tangent line of c^n, the tangent at inflection, etc. If c^n and d have s-ple contact at one regular common point it is signed $D_i^{1,s}$, where $i \leq n-s$. (See Fig. 1)

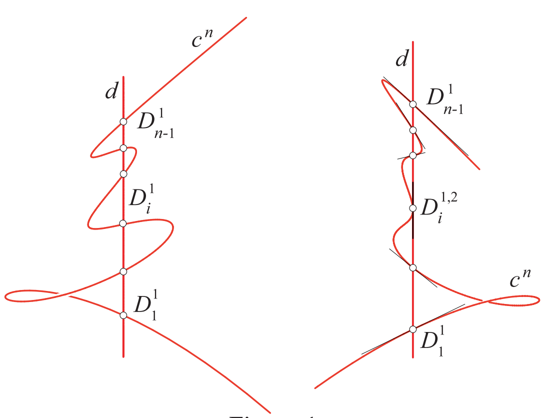

Figure 1

The nth order space curve can possess singular points with the highest multiplicity $n-2$. If the directing curve c^n has a multiple point it must lie on the line d because if c^n had a mutiple point out of the line d, the plane through that point and the line d would cut the curve c^n in more than n points, which is impossible. The k-ple point of the curve c^n which lies on the line d is signed D_i^k, where $i \leq n-k$. Three examples of 7th and 8th order curves with dobule and triple points are shown in Fig. 2.

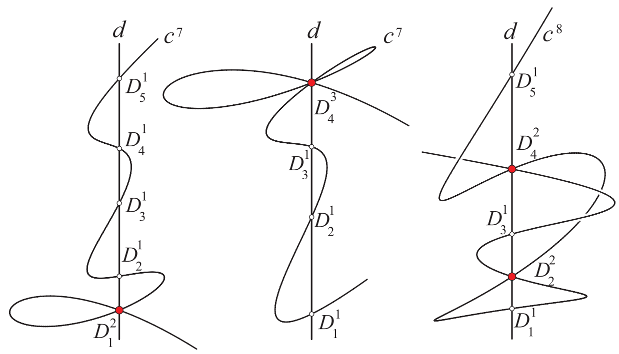

Figure 2

2.1.2 Singular points of $\mathcal{C}_n^1$

a) All singular points of $\mathcal{C}_n^1$ (the points which contain ∞^1 rays of $\mathcal{C}_n^1$) lie on its directing lines c^n and d.

If a point C lies on the curve c^n and $C \neq D_i^j$, then the rays of C_n^1 which pass trough C form the pencil of lines (C) in the plane $\delta \in [d]$ which contains C and d. (See Fig. 3)

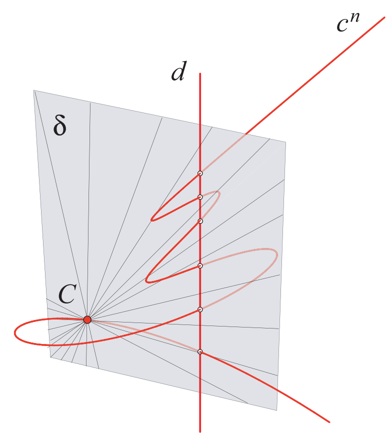

Figure 3

b) If a point D lies on the line d and $D \neq D_i^j$, then all the lines which join D with the points of the curve c^n are the rays of $\mathcal{C}_n^1$. They form nth degree cone ζ_D^n with the vertex D. Since c^n and d have $n-1$ common points, this cone intersects (or tuoches) itself $n-1$ times through the line d, thus the line d is $(n-1)$-ple generatrix of ζ_D^n. (See Fig. 4)

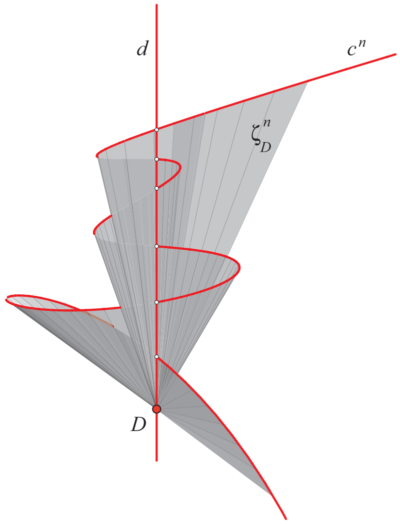

Figure 4

c) If a point D_i^k is the intersection point of c^n and d and if it is a k-ple point of the curve c^n, then the rays through D_i^k which cut c^n form $(n-k)$th degree cone $\zeta_{D_i^k}^{n-k}$ with the vertex D_i^k. The line d is $(n-k-1)$-ple genetartix of $\zeta_{D_i^k}^{n-k}$. Besides that the rays through the point D_i^k form k pencils of lines (D_i^k) in the planes determined by the line d and k tangent lines of c^n at D_i^k. If the intersection point is $D_i^{1,s}$, then the pencil of lines $(D_i^{1,s})$ in the rectifying plane of c^n at $D_i^{1,s}$ are also the rays of a congruence.

The other lines of the sheefs $\{D_i^k\}$ and $\{D_i^{1,s}\}$ are not regarded as the rays of a congruence.

The example of the rays through the regular intersection point D_1^1 is shown in Fig. 5.

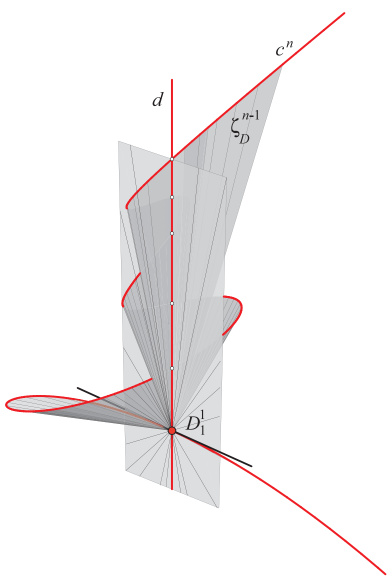

Figure 5

2.1.3 Rays of $\mathcal{C}_n^1$ through an arbitrary point

Every point A which is not the singular point of $\mathcal{C}_n^1$, i.e. $A \notin c^n, A \notin d$, determines the plane $\delta_A \in [d]$ which cuts c^n in only one point C which in general does not lie on the line d. The line AC, which cut d in one point D, is the unique ray of $\mathcal{C}_n^1$ through the point A. If the plane δ_A contains one of the the tangent lines of c^n at the intersection point D_i^k (or if it is the rectifying plane at $D_i^{1,s}$), then the points C and D cioncide with D_i^k $(D_i^{1,s})$ and the line AD_i^k $(AD_i^{1,s})$ is the unique ray of $\mathcal{C}_n^1$ through A. (See Fig. 6)

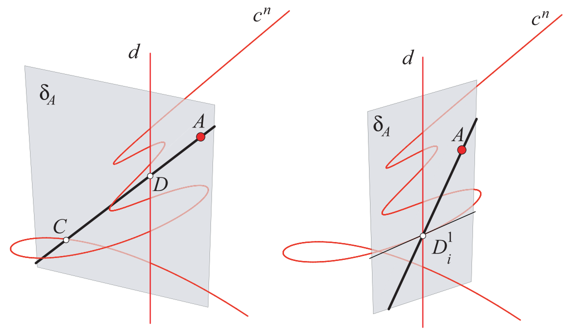

Figure 6

2.1.4 Singular planes of $\mathcal{C}_n^1$

All singular planes of $\mathcal{C}_n^1$ (the planes which contain ∞^1 rays of $\mathcal{C}_n^1$) are the planes of the pencil $[d]$. From 2.1.2. it is clear that in every plane $\delta \in [d]$ lie thes pencil of rays (C) or (D_i^k) or $(D_i^{1,s})$. (See Fig. 7)

It is possible that some of the tangent lines at intersection points D_j^k lie in the same plane of the pencil $[d]$. In such case there is more than one pencil of lines in the plane determined by these coplanar tangent lines.

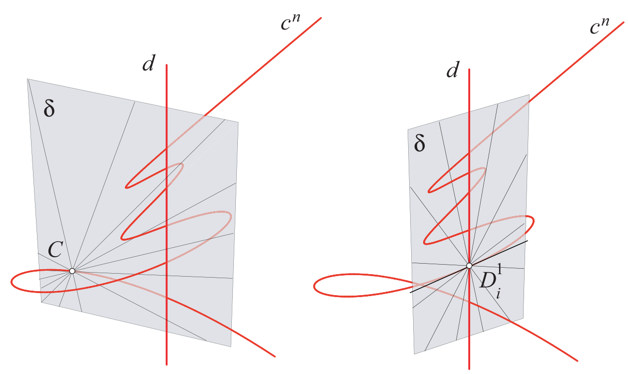

Figure 7

2.1.5 Rays of $\mathcal{C}_n^1$ through an arbitrary plane

Every plane α which is not the singular plane of $\mathcal{C}_n^1$, i.e. $\alpha \notin [d]$, contains n rays of the congruence. The plane α cuts line d in one point D and nth order space curve c^n in n points $C_j, j = 1,\dots,n$. The lines DC_j are n rays of the congruence $\mathcal{C}_n^1$ in the plane α. They are the intersection of the plane α and nth degree cone ζ_D^n and can be real and different, coinciding or imaginary. If α cuts the line d in D_i^k, then $n-k$ rays are the intersection of α and the cone $\zeta_{D_i^k}^{n-k}$ and other k rays are the intersection of α and the planes through d determined by the tangent lines of c^n at D_i^k. (See Fig 8)

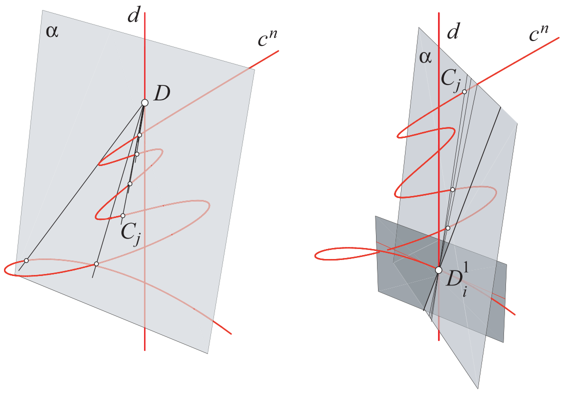

Figure 8

2.1.6 Decomposition of $\mathcal{C}_n^1$ into developable surfaces

As mentioned in the introduction every congruence can be decomposed in two ways into a one-parameter family of developables. These two families arise if one of the two parameters of which a congruence depends, is fixed.

In the case of the 1st order congruences of the type I the devlopables are the sets of rays through singular points, i.e. one family is formed by the nth degree cones ζ_D^n, and the other by the planes of the pencil $[d]$. Every ray of $\mathcal{C}_n^1$ is the intersection of two developables, one from each family. Since d is $(n-1)$-ple generatrix of ζ_D^n, every plane of $[d]$ cuts it into only one more generatrix which is the ray of a congruence. For the rays through the intercestion points D_i^k the nth degree cones split into $(n-k)$th degree cone and k planes through the line d. (See Figures 4, 5 and 7)

2.1.7 Focal properties – hyperbolic, elliptic and parabolic rays of $\mathcal{C}_n^1$

In general case the ray of a congruence touches the focal surface at foci which lie on the cuspidal edges of the developables. If the developables through the ray are real and different, the ray is hyperbolic and the foci are real and distinct. If the developables are imaginary, the ray is elliptic and the foci are imaginary. If the developables coincide, the ray is parabolic and the foci coincide. A congruence or the partition thereof is said to be hyperbolic, elliptic or parabolic if its rays are hyperbolic, elliptic or parabolic.

In the case of the 1st order surfaces of the type I the focal surface degenerates into the directing curves c^n and d. The developables have not the cuspidal edges, only cuspidal points: the verteces D of the cones ζ_D^n and the points C which are the intersections of the planes $\delta \in [d]$ and the curve c^n. Thus, each ray of $\mathcal{C}_n^1$ has the foci on the directing lines c^n and d. If they are real, the congruence is hyperbolic with parabolic rays in $n-1$ planes through the points D_i^k where the developables and foci coincide. If the directing lines are imaginary the congruence is elliptic.

2.2 Congruences of the type II

If a congruence $\mathcal{B}$ is the set of lines which cut a proper curve twice, this curve must be a twisted cubic. Since through an arbitrary point only one ray of the 1st order congruence passes, the projection of the directing curve from this point onto an arbitrary plane has only one double point. Thus this projection is the 3rd order plane curve. As the original curve and its projection have the same order, then the directing line of a congruence $\mathcal{B}$ is a twisted cubic. The projection of a twisted cubic onto a plane from a point on a secant line yields a nodal cubic and from a point on a tangent line a cuspidal cubic [4, p. 54]. (See Fig. 9)

The tangent and a secant lines of a twisted cubic b^3 fill up the projective space and are pairwise disjoint, except at points at curve itself [4, p. 90]. Thus through an arbitrary point unique ray of the congruence $\mathcal{B}$ passes.

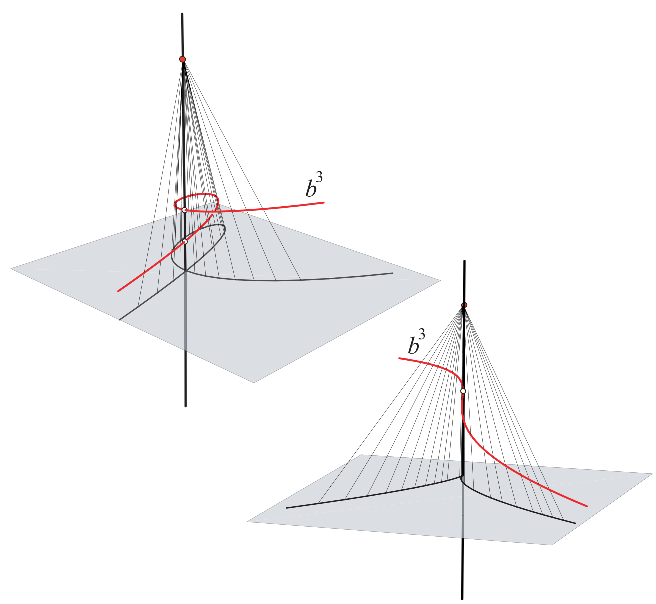

Figure 9

It is clear that such congruence is of the 3rd class ($\mathcal{B}_3^1$), because every plane cuts the curve b^3 in 3 points and the lines joining them are three rays of a congruence. These three rays can be: three real and different (a), one real and two imaginary (b), three real where two of them coincide (c) and three real and coinciding (d). (See Fig. 10)

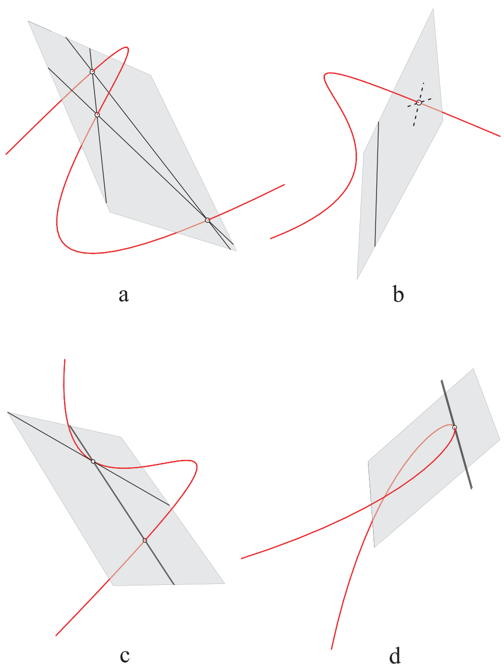

Figure 10

All singular points of $\mathcal{B}_3^1$ lie on the twisted cubic b^3. The lines which join the point $B \in b^3$ with the other points of b^3 form 2nd degree cone ζ_B^2. (See Fig. 11)

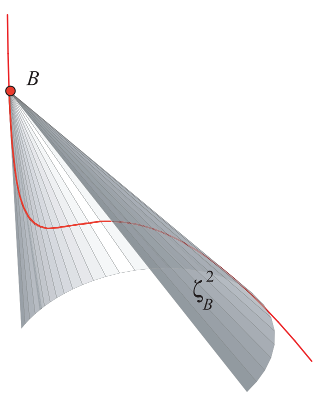

Figure 11

Since every plane contains exactly three rays of $\mathcal{B}_3^1$ there are no singular planes of $\mathcal{B}_3^1$.

$\mathcal{B}_3^1$ can be decomposed into one family of developables. This family consists of the cones ζ_B^2, $B \in b^3$. Every ray of $\mathcal{B}_3^1$ which cuts b^3 at the points B_1, B_2 is the part of the intersecion of the cones $\zeta_{B_1}^2$, $\zeta_{B_2}^2$. Namely, the intersection of $\zeta_{B_1}^2$ and $\zeta_{B_2}^2$ is $b^3 \cup B_1B_2$.

The curve b^3 is the focal curve of $\mathcal{B}_3^1$, it contains the cuspidal points of ζ_B^3, $B \in b^3$. The ray of $\mathcal{B}_3^1$ is hyperbolic if the intersection points $B_1, B_2 \in b^3$ are real and different, it is parabolic if they coincide (the ray is a tangent of b^3) and it is elliptic if they are imaginary.

The parabolic rays of $\mathcal{B}_3^1$ form the tangent developable of b^3. (See Fig. 12)

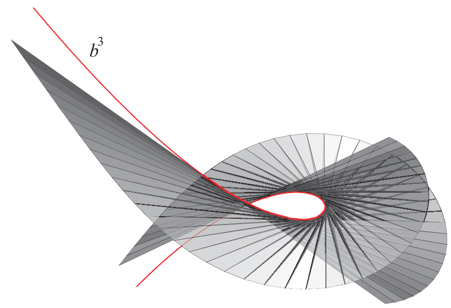

Figure 12

The rays of such congruences are also the intersections of the corresponding elements of two collinear bundles of planes $\{B_1\}$, $\{B_2\}$, [7, p. 135], [14, p. 1185]. In this case the basic points of the bundles (B_1, B_2) lie on a twisted cubic b^3, and the unique ray through an arbitrary point can be constructed as the part of the intersection of two ruled quadrics. These quadrics pass through b^3 and their rulings are determined by the collineation between the bundles $\{B_1\}$ and $\{B_2\}$, [7, p. 136]. In the special case when one plane in the collineation between $\{B_1\}$ and $\{B_2\}$ corresponds to itself, the basic cubic b^3 splits into one straight line and a conic which have one common point, and the congruence $\mathcal{B}_3^1$ splits into the 2nd class congruence $\mathcal{C}_2^1$ and the field of lines in the plane of the conic. These congruences are elaborated in detail in [2].

3 Analytical approach and *Mathematica* visualizations

If two algebraic space curves c_1 and c_2 are given by the following parametric equations

$$
\begin{aligned}
c_1 \ldots\, & x = x_1(u), \quad y = y_1(u), \quad z = z_1(u), \\
& x_1, y_1, z_1 : I_1 \to \mathbb{R},\ I_1 \subseteq \mathbb{R},\ x_1, y_1, z_1 \in C^1(I_1) \\
c_2 \ldots\, & x = x_2(v), \quad y = y_2(v), \quad z = z_2(v), \\
& x_2, y_2, z_2 : I_2 \to \mathbb{R},\ I_2 \subseteq \mathbb{R},\ x_2, y_2, z_2 \in C^1(I_2), \qquad (1)
\end{aligned}
$$

then the set of lines which join the points of c_1 and c_2 are given by the following equations

$$\frac{x-x_1(u)}{x_1(u)-x_2(v)}=\frac{y-y_1(u)}{y_1(u)-y_2(v)}=\frac{z-z_1(u)}{z_1(u)-z_2(v)}, \qquad (u,v)\in I_1\times I_2\subseteq\mathbb{R}^2. \tag{2}$$

In the previous section, for drawing the directing lines of $\mathcal{C}_n^1$, we used the following parametric functions:

$$\begin{aligned} d \ldots x_1(u)&=0, \quad y_1(u)=0, \quad z_1(u)=u, \quad u\in\mathbb{R},\\ c^n \ldots x_2(v)&=a_x(v-v_1)\cdots(v-v_{n-1}),\\ y_2(v)&=a_y v x_2(v),\\ z_2(v)&=v, \qquad v,a_x,a_y,v_1,\ldots,v_{n-1}\in\mathbb{R}. \end{aligned} \tag{3}$$

It is clear that the line d is the axis z and c^n is the nth order space curve which cuts the axis z at the points $D_i(0,0,v_i)$, $i\in\{1,...,n-1\}$.

If the polynomial $x_2(v)$, from (3), contains the factor $(v-v_i)^s$, then $i\le n-s$ and d and c^n have $s-$ple contact at the point $D_i(0,0,v_i)$.

If the polynomial $z_2(v)$, from (3), takes the form

$$\begin{aligned} z_2(v)&=v(v-v_{i_1})\cdots(v-v_{i_k}),\ v_{i_j}\neq 0,\\ &i_1,\ldots,i_k\in\{1,\ldots,n-1\}, \quad k\le n-2, \end{aligned} \tag{4}$$

then $(0,0,0)$ is the $k-$ple singular point of c^n and the coordinates of intersection points of c^n and d are $(0,0,z_2(v_i))$.

(3) and (2) give the following equations of the rays of $\mathcal{C}_n^1$

$$\frac{x}{x_2(v)}=\frac{y}{y_2(v)}=\frac{u-z}{u-z_2(v)}, \quad (u,v)\in\mathbb{R}^2. \tag{5}$$

The above equations enable computer visualization of the rays of $\mathcal{C}_n^1$ and $\mathcal{B}_3^1$. Based on this we made the program in *webMathematica* which enables interactive visualizations of $\mathcal{C}_n^1$ on the internet. It is available at the following address:
www.grad.hr/itproject_math/Links/webmath/indexeng.html

3.1 Examples

In the following examples the graphics are produced with the program *Mathematica*.

EXAMPLE 1

Two dsiplays of the same 2nd class congruence are shown in Fig. 13. The directing lines of this $\mathcal{C}_2^1$ are the axis z and the circle given by the following parametric equations:

$$x(v)=\cos v+1, \ y(v)=\sin v, \ z(v)=0, \ v\in[0,2\pi].$$

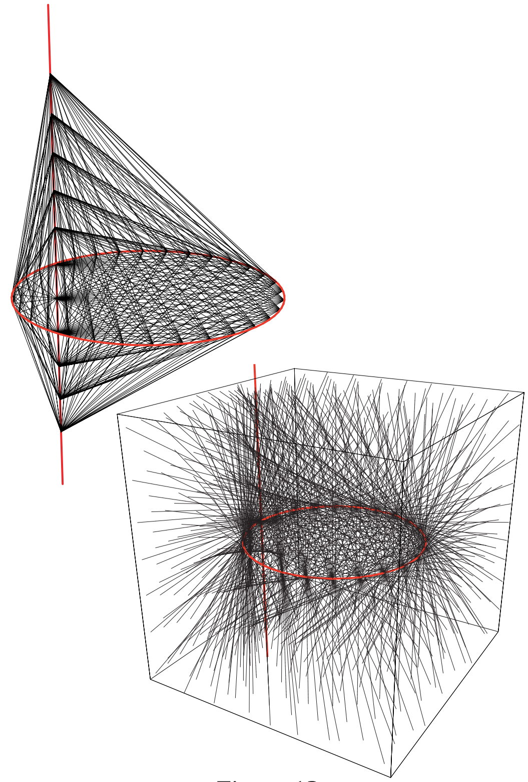

Figure 13

EXAMPLE 2

The rays of the 4th class congruence are shown in Fig. 14. The directing lines of this congruence are the Viviani's curve c^4 which cuts the axis z in two points, but one of the intersection points is the double point of c^4. The parametric equations of c^4 are:

$$x(v)=\frac{\sqrt{2}}{2}(\cos v-2\sin\frac{v}{2}-1), \quad y(v)=\sin v,$$
$$z(v)=\frac{\sqrt{2}}{2}(\cos v+2\sin\frac{v}{2}-1), \quad v\in[-2\pi,2\pi].$$

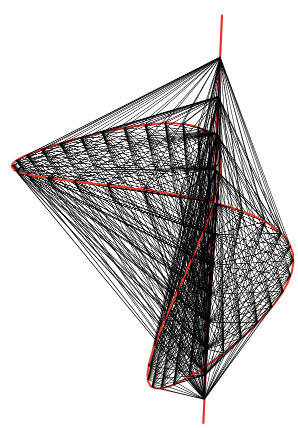

Figure 14

EXAMPLE 3

Two displays of the same 7th class congrence are shown in Fig. 15. The directing lines of this $\mathcal{C}_7^1$ are the axis z and the curve c^7 which is given by the following parametric equations:

$$x(v) = \frac{1}{10} v(v-1)(v-2)(v-3)(v-3.5)(v-4),$$
$$y(v) = 2vx_2(v),$$
$$z(v) = v, \quad v \in \mathbb{R}.$$

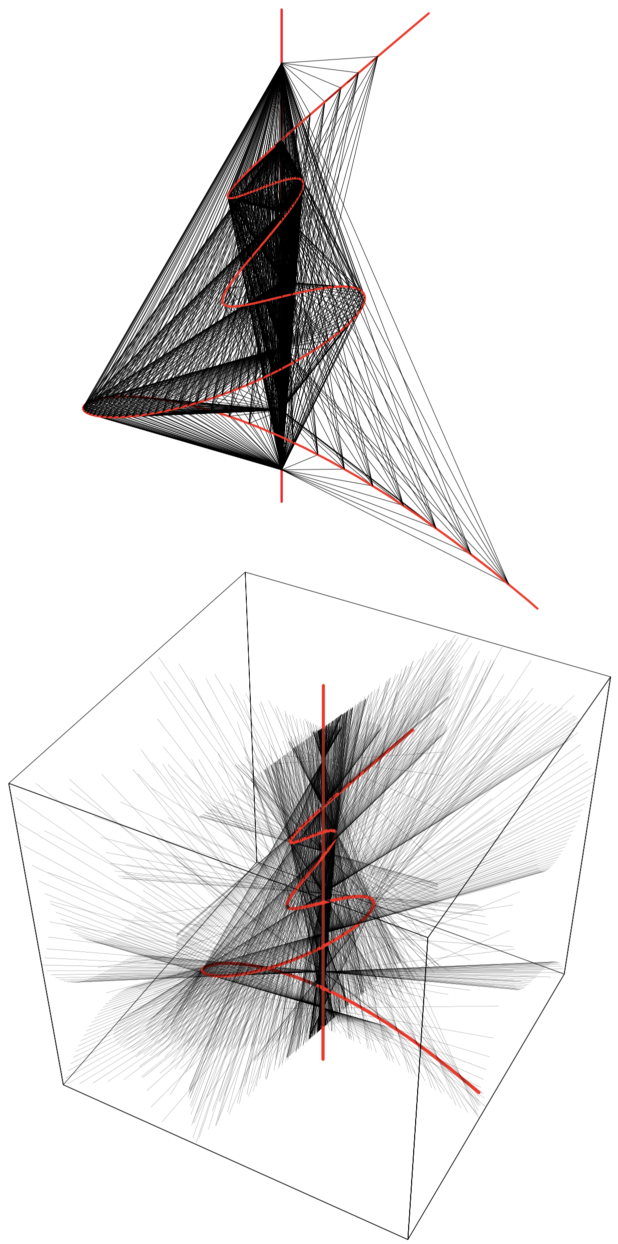

Figure 15

EXAMPLE 4

The visualization of $\mathcal{B}_3^1$ whose directing curve is given by the following parametric equations

$$x(v) = v,$$
$$y(v) = (v-1)(v+1),$$
$$z(v) = (v-1)^2(v+1), \quad v \in \mathbb{R}$$

is shown in Fig. 16. The same congruence, with red parabolic rays, is shown in Fig. 17 for two different view points.

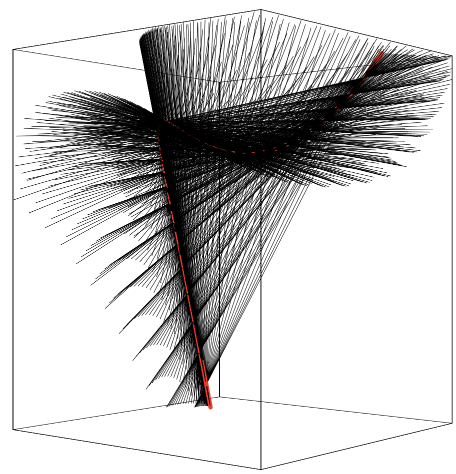

Figure 16

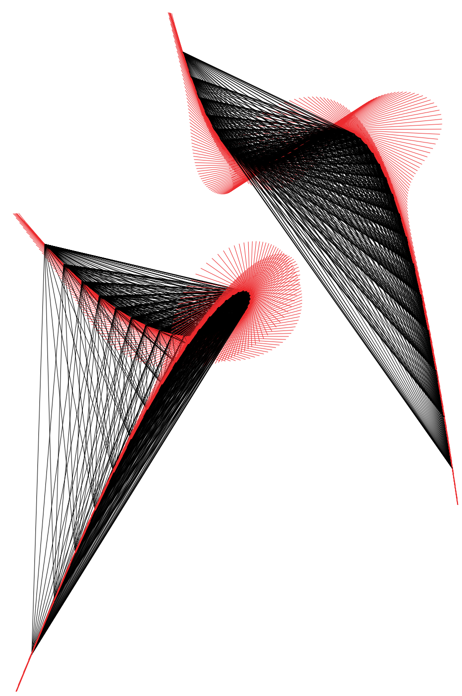

Figure 17

References

[1] S. P. FINIKOV: *Theorie der Kongruenzen*. Akademie-Verlag-Berlin, Berlin, 1959. (translation from Russian)

[2] S. GORJANC: *The Classification of the Pedal Surfaces of (1,2) Congruences*. Dissertation, Department of Mathematics, University of Zagreb, 2000. (in Croatian)

[3] A. GRAY: *Modern Differential Geometry of Curves and Surfaces with Mathematica*. CRC Press, Boca Raton, 1998.

[4] J. HARRIS: *Algebraic Geometry: A First Course* Springer, New York, 1992.

[5] C. M. JESSOP: *A Treatise on the Line Geometry*. Chelsea Publishing Company, New York 1969 (reprint).

[6] E. MÜLLER, J. L. KRAMES: *Konstruktive Behandlung der Regelflächen*. Franc Deuticke, Leipzig und Wien, 1931.

[7] V. NIČE: *Introduction to Synthetic Geometry*. Školska knjiga, Zagreb, 1956. (in Croatian)

[8] G. SALMON: *A Treatise on the Analytic Geometry of Three Dimensions, Vol.I* . Chelsea Publishing Company, New York, 1958. (reprint)

[9] G. SALMON: *A Treatise on the Analytic Geometry of Three Dimensions, Vol.II* . Chelsea Publishing Company, New York, 1965. (reprint)

[10] G. SALMON: *Higher Plane Curves*. Chelsea Publishing Company, New York, 1960. (reprint)

[11] *Encyclopaedia of Mathematics - SpringerLink*. Edited by M. Hazewinkel, available online: `http://eom.springer.de/C/c024910.htm`

[12] R. STURM: *Die Lehre von den geometrischen Verwandtschaften, Band IV*. B. G. Taubner, Leipzig-Berlin, 1909.

[13] R. STURM: *Liniengeometrie, II. Teil*. B. G. Taubner, Leipzig, 1893.

[14] K. ZINDLER: *Algebraische Liniengeometrie*. Encyklopädie der Mathematischen Wissenschaften, Band III, 2. Teil, 2. Hälfte. A., pp. 1184-1185, B. G. Teubner, Liepzig, 1921-1928.

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