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About Curvatures on Triangle Meshes

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ABSTRACT

A face-based curvature estimation on triangle meshes is presented in this paper. A flexible disk is laid on the mesh around a given triangle. Such a bent disk is used as a geodesic neighborhood of the face for approximating normal and principal curvatures. The radius of the disk is free input data in the algorithm. Its influence on the curvature values and the stability of estimated principal directions are investigated in the examples.

Key words: triangle mesh, curvature

MSC 2000: 68U05, 68U07, 65D20

O zakrivljenostima na trokutnim mrežama

SAŽETAK

U članku je prikazana procjena zakrivljenosti na trokutnim mrežama, bazirana na stranicama. Gipki disk položen je na mrežu oko danog trokuta. Takav prilagodljiv disk koristi se kao geodetska okolina stranice za aproksimaciju normalnih i glavnih zakrivljenosti. Polumjer diska je nezavisni ulazni podatak u algoritmu. U primjerima se istražuje njegov utjecaj na vrijednosti zakrivljenosti i na stabilnost procijenjenih glavnih smjerova.

Ključne riječi: trokutna mreža, zakrivljenost

1 Introduction

Triangle meshes are the most frequently used surface representations in many surface-oriented applications. Surface curvature properties have been successfully employed for solution of different practical problems, as smoothing or simplifying meshes in modeling and manufacturing, also for surface classification and 3D object recognition in computer vision research, etc. Discrete counterparts of continuous definitions of differential operators, curvature values, geodesic curves and Dirichlet energy, etc. have been given and derived for arbitrary triangle meshes in [3], [10], [13] and [14].

Almost all methods for surface derivative and curvature estimations on meshes have been vertex connectivity based. In these approaches a specified neighborhood of vertices formed by adjacent vertices, edges and faces is used to approximate the surface normal, surface derivatives and curvature values at a vertex. The algorithms use either analytic methods based on surface fitting, or they work with discrete differential operators. The crucial first step in these algorithms is the computation of a vector at each vertex that approximates the true normal vector at this point of the surface represented by the mesh. This problem is equivalent to the computation of the best tangent plane to the mesh at a given vertex. Most methods compute a weighted average of facet normals in a one-ring neighborhood of the vertex.

$$N = \frac{\sum \omega_j N_j}{m_i}, \quad j = 1 \dots m_i,$$

where m_i is the number of edges emanating from the vertex v_i, $\omega_j = k \cdot Area(triangle_j)$, $k > 0$ and $Area(triangle_j)$ is either surface area or Voronoi-surface area or a mixed surface area of the triangle $\{v_i, v_j, v_{j+1}\}$ (Fig. 1).

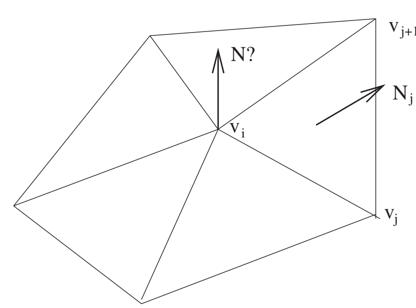

Figure 1: *One-ring neighborhood of a vertex*

A number of proposals have been published for choosing the weights or the neighborhood for determining the best normal vector ([15], [11], [9]).

An other problem is to estimate normal curvatures, which is equivalent to the definition of osculating circles producing a second-order approximation to those curvature values ([12], [19]).

A normal curvature estimation in a one-ring neighborhood can be given simply by defining the osculating circle in a normal plane through the vertices v_i, v_j and the normal vector N (Fig. 2).

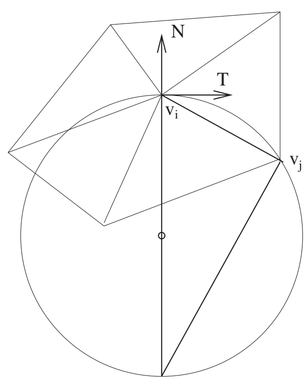

Figure 2: *Osculating circle in a normal plane*

The curvature of this circle is

$$\kappa_n(v_i) \approx \frac{2 < N, (v_j - v_i) >}{|v_j - v_i|^2},$$

where $<,>$ denotes the dot product.

Instead of circle in a normal plane, interpolating quadratic polynomial curve is also used e.g. in [7]. The one-ring neighborhood of a specified vertex is replaced by a Voronoi or mixed surface area [10], it is extended in [7] and geodesic neighborhoods are also used ([12], [16]) in the computations. The selection of neighborhood size can affect results significantly: small neighborhoods provide better estimates for clean data, while increasing the neighborhood size smoothes the estimates, leading to less sensitivity to noise. Obviously, small errors in these approximations lead rapidly to unreliable, noisy curvature values. Comparisons of five frequently used methods are given in [5].

Analytic methods are also applied for curvature estimations by fitting a surface to the mesh in the neighborhood of the point of interest and evaluating its curvatures ([5], [6], [19]). Principal curvatures and principal directions can be determined on the base of Euler theorem ([11], [2], [4]). Mean and Gauss curvature values can be computed from the curvature tensor, i.e. from the Weingarten matrix or from its symmetric extension by eigen-decomposition ([12], [18]). The Gauss-Bonnet theorem gives a direct method for the computation of Gauss curvature. It has two different discrete forms which provide good approximations for special triangulations of surfaces [20].

In this paper we define normal curvatures on each face of the triangle mesh in order to estimate the principal directions and to characterize elliptical, umbilical, flat and hyperbolic regions. The proposed new method is presented in Chapter 2. In the examples (Chapter 3) we show the proposed method on ‘synthetic’ and real triangulated surfaces.

2 Curvatures defined on faces

2.1 Geodesic circle of a triangle

Instead of computing surface properties at vertices in vertex neighborhoods we define curvature values ordered to faces. The center of the defined region is the barycentric center of the given triangle. We intersect the mesh with normal planes passing through the face normal of the triangle, then we measure a given geodesic radius along the polygonal lines of intersection in both directions from the center point. In this way we get a number of curved diameters of the geosedic circle bent on the mesh around the face. We call this geodesic neighborhood “splat” after Kobbelt [8] and the set of the polygonal diameters “spider” after Simari [16] (Fig. 3). Then we compute the chord lengths of the constructed diameters in order to estimate normal curvature values.

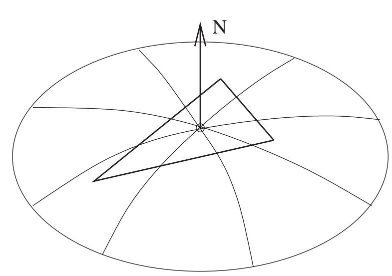

Figure3: *Geodesic circle of a face*

2.2 Osculating circle and normal curvature

We define in each normal plane an osculating circle to the face determined by the endpoints of the bent diameter and the face normal (Fig. 4).

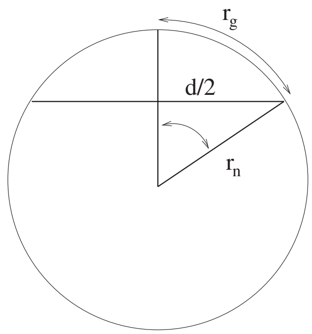

Figure 4: *Osculating circle in a normal section*

Denote r_g the given geodesic radius, d the chord length between the endpoints of the curved diameter, 2α the unknown central angle and r_n the required radius of the osculating circle. From the equations

$$r_n\alpha = r_g \qquad \text{and} \qquad r_n \sin\alpha = \frac{d}{2}$$

we get for α

$$\frac{d}{2r_g} = \frac{\sin\alpha}{\alpha}.$$

We apply the approximation

$$\sin\alpha \approx \alpha - \frac{\alpha^3}{6}, \quad 0 < \alpha << 1$$

and get

$$\alpha \approx \sqrt{(1 - \frac{d}{2r_g})6},$$

consequently,

$$r_n \approx \frac{r_g}{\alpha} \qquad \text{if} \qquad \alpha \neq 0 \qquad \text{and} \qquad \kappa_n \approx \frac{\alpha}{r_g}$$

is the radius of the osculating circle and the normal curvature, respectively.

2.3 Principal directions and principal curvatures

Repeating this computation for a set of normal sections in the geodesic neighborhood of the given triangle we obtain normal curvature values $\kappa_{n,i}$, $i = 1,\ldots k$. If the mesh is a dense triangulation of a regular surface then the normal planes belonging to the minimal and maximal normal curvatures are orthogonal to each other. They determine the principal directions.

We select the maximal curvature and define it as first principal curvature κ_1 and the corresponding direction T_1 in the plane of the current triangle as first principal direction. This direction is fairly stable, even if we compute with smaller geodesic circles. The second principal direction T_2 is orthogonal to it. In the case of properly defined geodesic circle and nearly regular triangulation it is the direction belonging to the maximal chord length (Fig. 5).

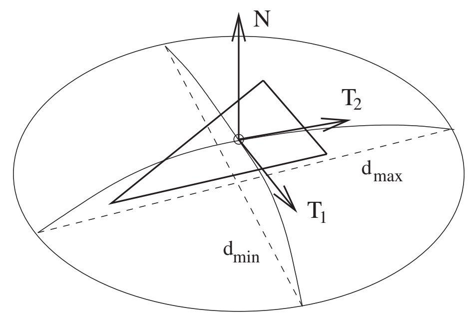

Figure 5: *Principal directions*

3 Examples

In order to compute a geodesic disk around a triangle face in a mesh we have to construct a flexible polyhedral data structure on the mesh which differentiates inner and boundary edges, moreover "feature edges" along sharp ridges. Then we have to implement an algorithm for computing the lines of intersection of the mesh and the defined normal planes [17]. The normal sections on the mesh are polygonal lines, and the arc length on the approximated surface is measured along these polygonal lines.

The triangulated cylindrical meshes in Fig. 6 and 7 are generated from the analytical description of a cylindrical surface. The vertices of such a "synthetic" mesh are lying exactly on the surface approximated by the mesh. The geodesic radius in Fig 6 is 3.5 times the average size of the triangles. The number of the computed diameters is 24. In Fig 7 the same cylindrical surface is shown with different triangulation. The geodesic radius is 0.6 times the average size of the triangles in the mesh. The largest curvature computed by the proposed method are 0.0402 and 0.0405, respectively, instead of 0.0400. Hence the relative errors in the curvature estimation are 0.5% and 1.2%, respectively. The corresponding principal directions are in both cases the most perpendicular ones to the axis of the cylinder among the 24 tangent directions. These directions are shown in the figures with longer segments.

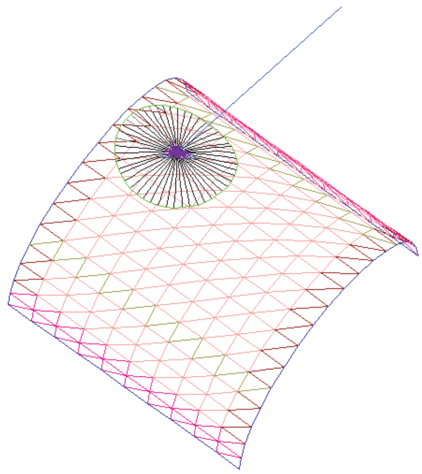

Figure 6: *Splat on a cylindrical surface*

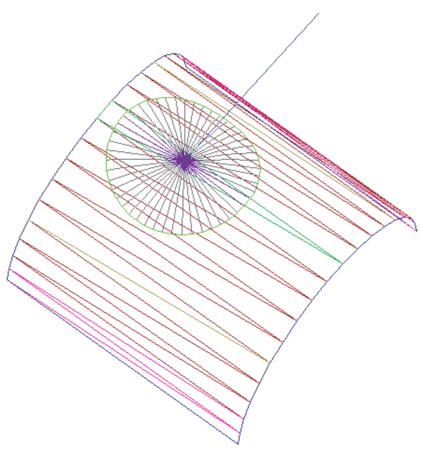

Figure 7: *The same cylindrical surface*

The second example shows that our curvature estimation works also in the case, where one-ring neighborhoods cannot be applied.

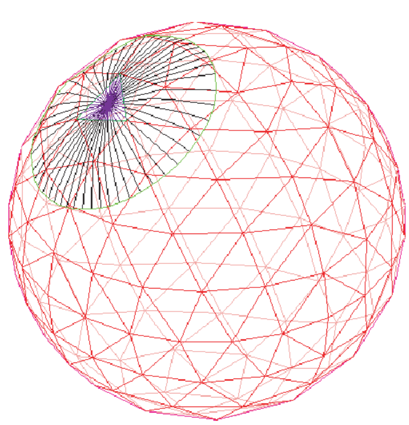

Figure 8: *Splat on the sphere*

In Fig. 8 a real triangle mesh approximating a sphere is shown. In this mesh there is a noise in the vertex data, and in the size of the triangles. The chosen geodesic radius is three times the average triangle size. The difference in the computed normal curvatures is about 3%, and this relative error does not change when the geodesic radius is varying between 2.5 and 7 times the average triangle size. This accuracy in the computation is comparable to the published results in the literature analysing curvature estimation methods [5].

In Fig. 9 and 10 a synthetic mesh of one eighth of a torus is shown. In Fig 9 the geodesic radius is three times the average triangle size, and the results in the principal directions are very good, considering the relative coarse approximation with 24 directions in the disk. In Fig 10 the geodesic disk is apparently too big to get reliable approximations (5 times the average triangle size) for the specified triangle face. In this picture only boundary and silhouette edges are shown besides the 24 diameters of the geodesic disk.

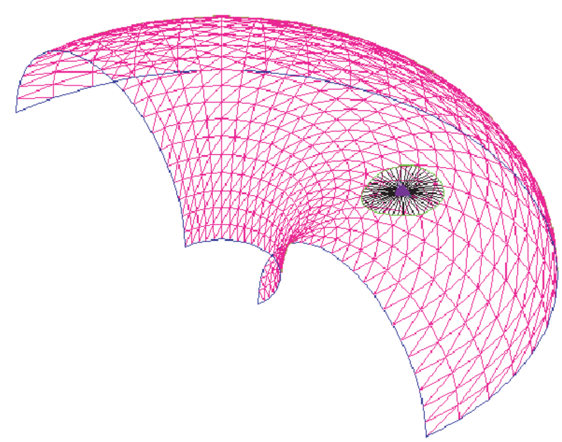

Figure 9: *Splat on the torus*

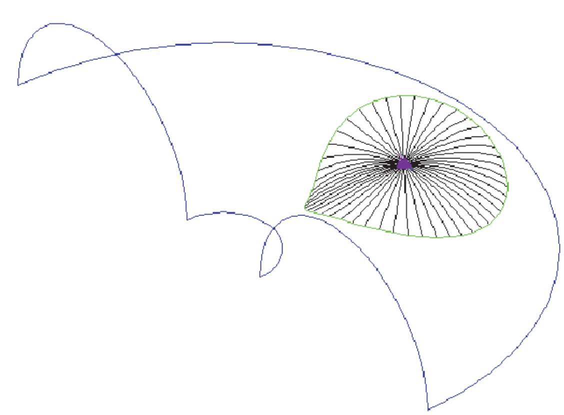

Figure 10: *The same facette with a bigger splat*

According to our experience the best measurement of the radius of a geodesic disk is between 2.5 and 3.5 times the average triangle size in a dense mesh. Of course, the mesh should provide a proper surface approximation. The examples have shown that our face-based curvature estimation works better than vertex-oriented methods using a one-ring neighborhood, especially in the case when the mesh contains long narrow triangles.

Some more investigations have to be done in the future, e.g., in the analysis of relative errors and in further applications.

4 Conclusion

We have introduced curvature values ordered to faces in triangle meshes by laying a flexible circular disk with user-specified radius onto each face of the mesh. From the chords of such a bent disk and from the face normal we have defined normal curvatures of the current face. The examples have shown that the obtained principal curvature values and the corresponding principal directions are quite reliable if the radius of the disk achieves an optimal size. Our method provides a good classification of elliptic, parabolic, flat and hyperbolic regions of the mesh.

We have implemented the algorithm for constructing geodesic disks on triangle meshes and for estimating normal curvatures and principal directions in Java.

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