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Two- and Three-dimensional Tilings Based on a Model of the Six-dimensional Cube

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ABSTRACT

A central-symmetric three-dimensional model of the six-dimensional cube can give us the idea of filling the space with mosaics of zonotopes. This model yields also plane tilings by its intersections. Using the parts of the model the mosaic and the tiling can further be dissected by projections, associations and Boolean operations. Further constructions are also indicated in the paper.

Key words: 3-dimensional models of the hypercube, plane-tiling, space-filling

MSC 2000: 51M20, 68U07

2-dimenzionalno i 3-dimenzionalno popločavanje zasnovano na modelu 6-dimenzionalne kocke

SAŽETAK

Centralno simetrični 3-dimenzionalni model 6-dimenzionalne kocke može nam dati ideju kako prostor ispuniti s mozaicima zonotopa. Pomoću presjeka, ovaj model vodi također i ka ravninskom popločavanju. Koristeći dijelove modela, mozaik i popločavanje mogu biti razdijeljeni projekcijama, asocijacijama i Boolovim operacijama. U članku se također navode i daljnje konstrukcije.

Ključne riječi: 3-dimenzionalni model hiperkocke, ravninsko popločavanje, prostorno popunjavanje

Lifting the vertices of a k sided regular polygon from their plane perpendicularly by the same height and joining them with the centre of the polygon, we get the k edges of the hypercube (k-cube) modelled in the three-dimensional space (3-model). From these the 3-models or their polyhedral surface (Fig. 1) can be generated as well in different procedures [3, 4, 5]. Each polyhedron from these will be a so called zonotope [6], i.e. a "translational sum" (Minkowski-sum) of some segments.

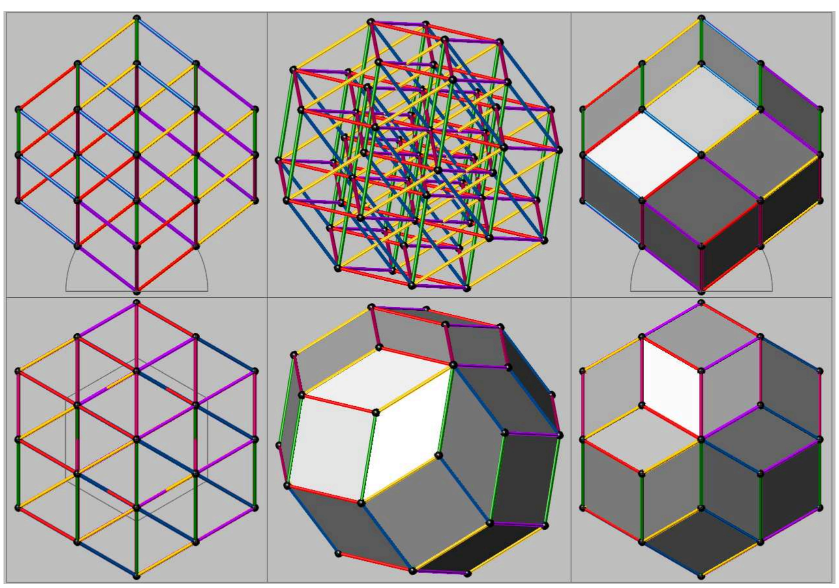

Figure 1

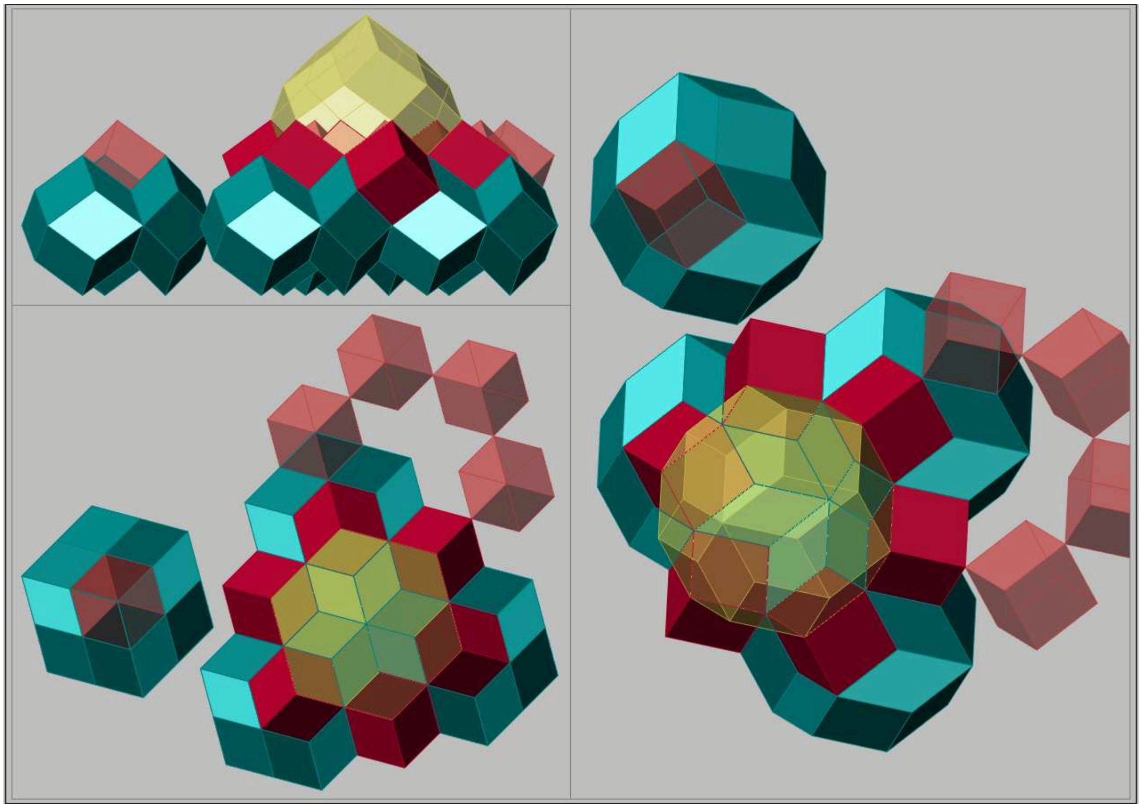

Figure 2

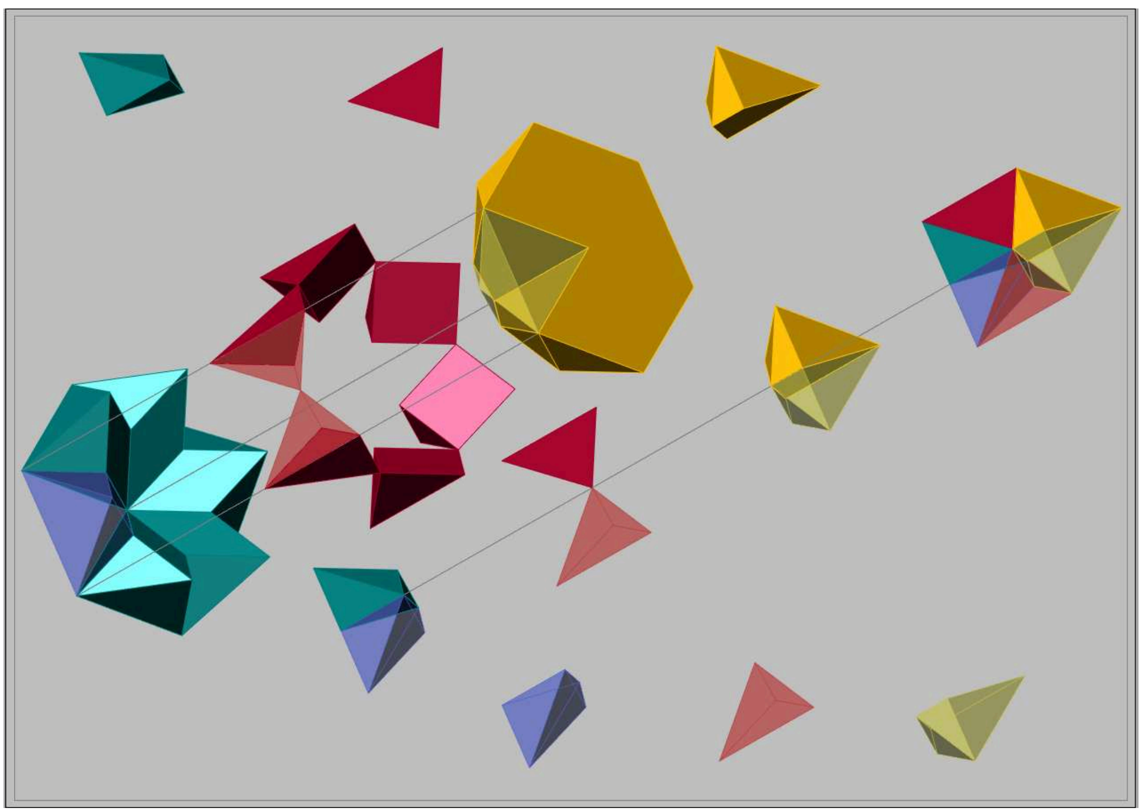

Figure 3

The 2-dimensional ortogonal projection of these 3-models indicates the idea how to construct space-filling with this model. However our 3-model of the 6-cube for example does not fill the space. The projected grid of the 3-cube joins our grid above and the cube fills the space well known. The edges of the cube can be selected from the conveniant lifted edges of the 6-cube's 3-model. With the selected four edges of the grid we can build the 3-model of the 4-cube. The shell of this is a rhombic dodekahedron which fills the space but this arrangement has not any rotational symmetry without additional assumptions. We can however replace a cube in the hole of the rotational-symmetrically arranged rhombic dodekahedra and continue the filling in a sixfold polar array with a rhombic triacontahedron which contains our 3-model of the 6-cube (Fig. 2).

It can be seen, that we can fill the space with these solids. The basic stones are to cut from a honeycomb by symmetry-planes. If the cutting process has been completed, we have the basic stones from the three starting solids (Fig. 3).

Another possibility is to rearrange our space-filling, assembling the 3-models of the k- and j-cubes from lower-dimensional cube 3-models. From the given 6 edges we can combine the 3-models of $2 < j < k$ cubes: 4 of the 3-cubes, 3 of the 4-cubes and 1 of the 5-cubes. Their additions (Fig. 4) can replace the 3-models of the above k- and j-cubes in our mosaic.

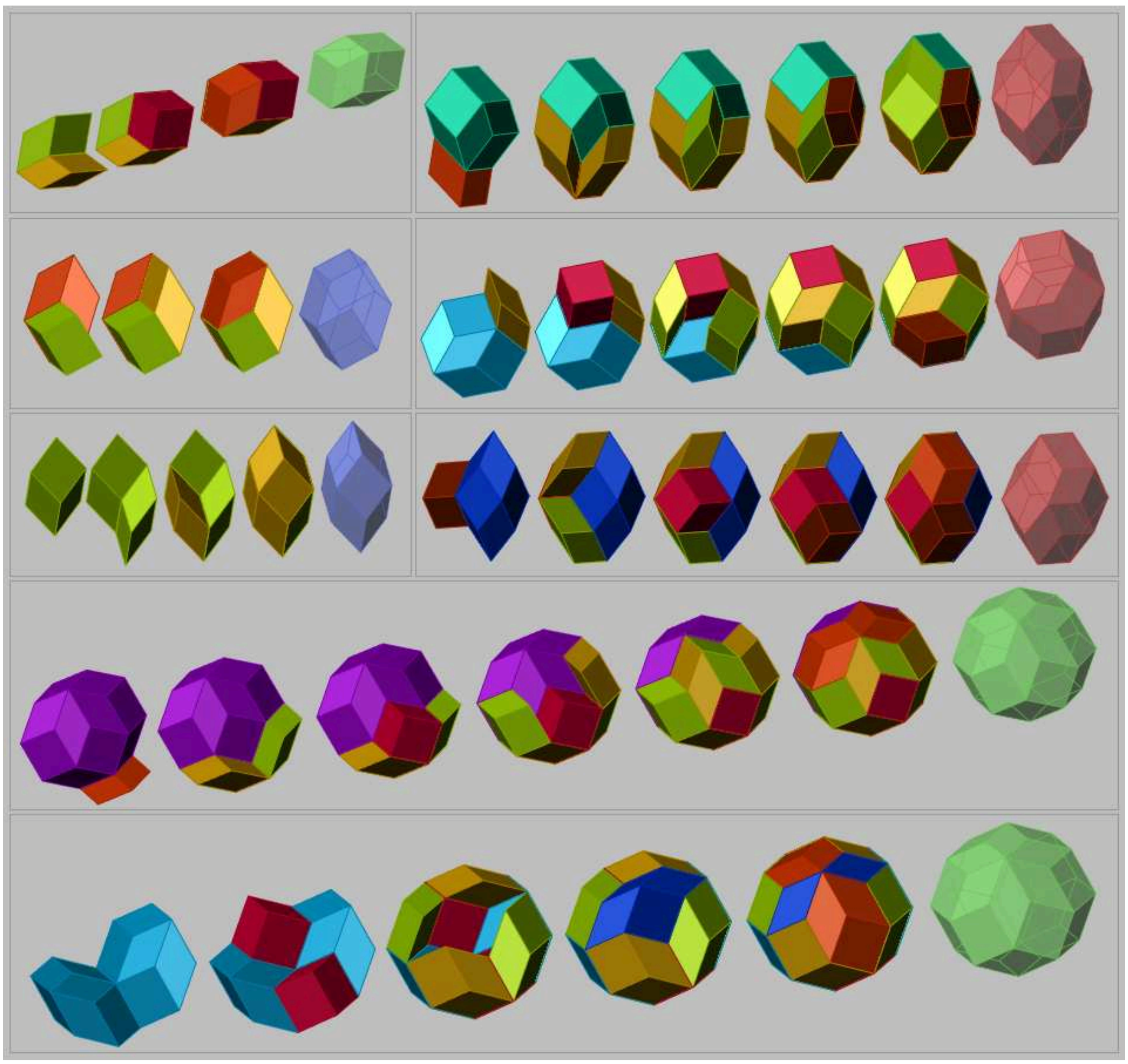

Figure 4

Interpreting the starting construction of the k-cube 3-model as a sequence of dispositions, the increasing dimensional inner $2 < j < k$ cube 3-models can "easily" be separated. The edges of the $0,1,\ldots,k$ cube model-sequence are parallel to the k-segment chain approaching a starting helix, and the disposition vectors are joining each other along this segment chain. The model $0,1,\ldots,k-1$ parts can also be interpreted as intersections of two full models so that the equal dimensional parts are positioned around the main diagonal of a full model, symmetrically to its centre point. More on this full model (3-model of the n-cube) can be read in [4], [5], [7], and on periodic and aperiodic tilings, based on d-dimensional crystallographic space groups, you find references in [1]. A further related topic might be: To what extent are these 3-models certain axonometric pictures of higher-dimensional cubes, created by a sequence of parallel projections? The Pohlke-theorem has surely limited validity in higher dimensions [2].

As it follows from our construction, the vertices lie in planes parallel to the basic plane of the construction, therefore a plane-tiling appears on these horizontal intersections of our space-filling solid-mosaics based on the 3-model of the 6-cube (Fig. 5). This has rotational symmetries but the diagonal intersections can be identical with the longitudinal and cross-intersections (Fig. 6).

We can see in Fig. 7 the horizontal intersections alternating one another $(0,1,2,3,2,1,0,1,\ldots)$ in the space-filling mosaic based on the 3-model of the 6-cube. The tiling of the intersections can further be dissected by the perpendicular projected edges of the intersected solids (Fig. 8). A similar phenomenon could be seen in the projection of the inner edges of the j-cube 3-models.

Projecting the combination of the intersection grids, the tiling can further be dissected (Fig. 9). The coloring here is kept to one intersection and the grid of another one is projected into this plane.

In Fig. 10 we have combined the grids of three and finally of all four horizontal intersections. This is further dissected by the projected edges of the intersected solids.

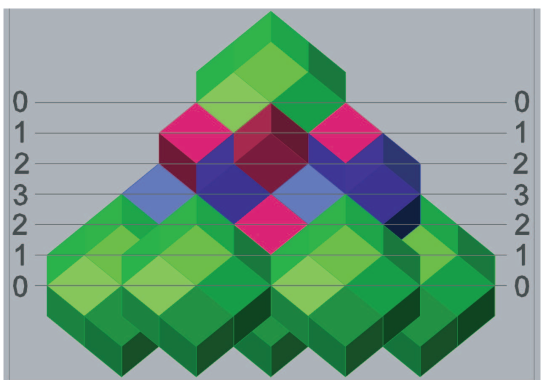

Figure 5

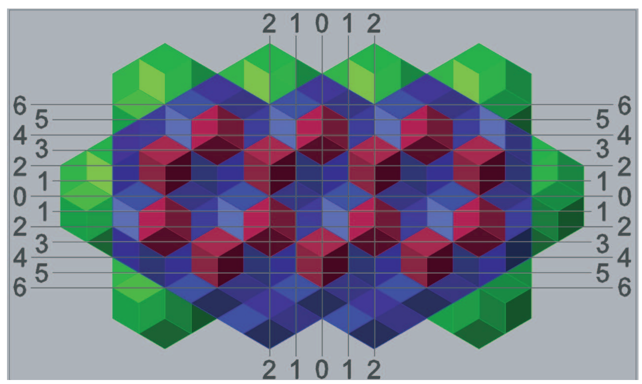

Figure 6

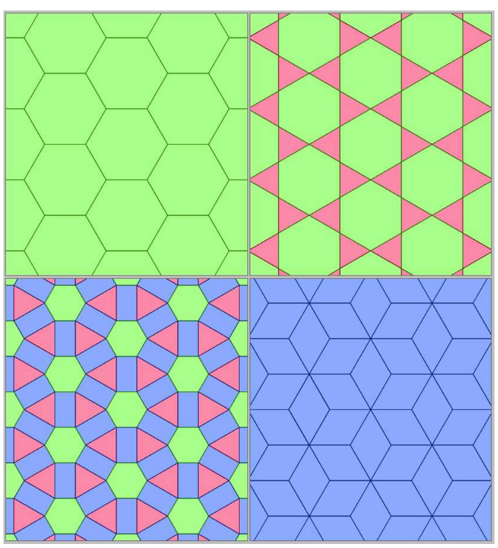

Figure 7

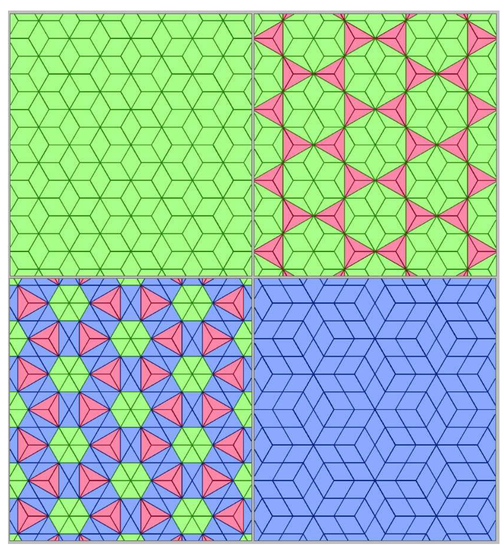

Figure 8

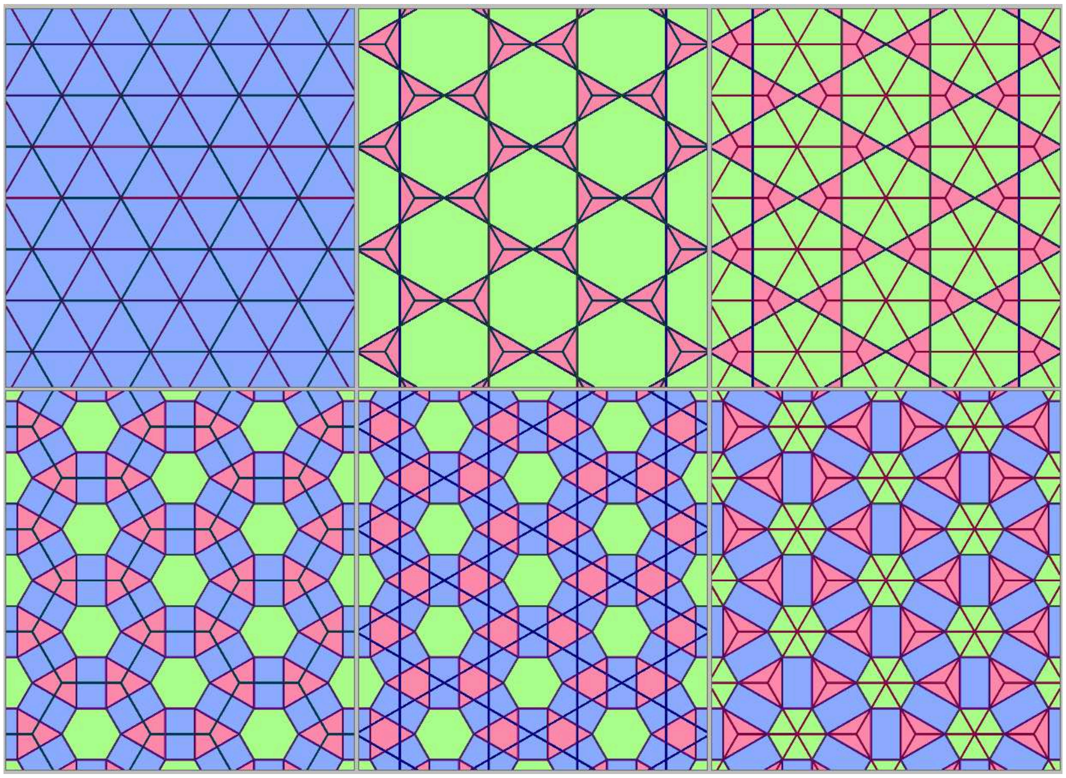

Figure 9

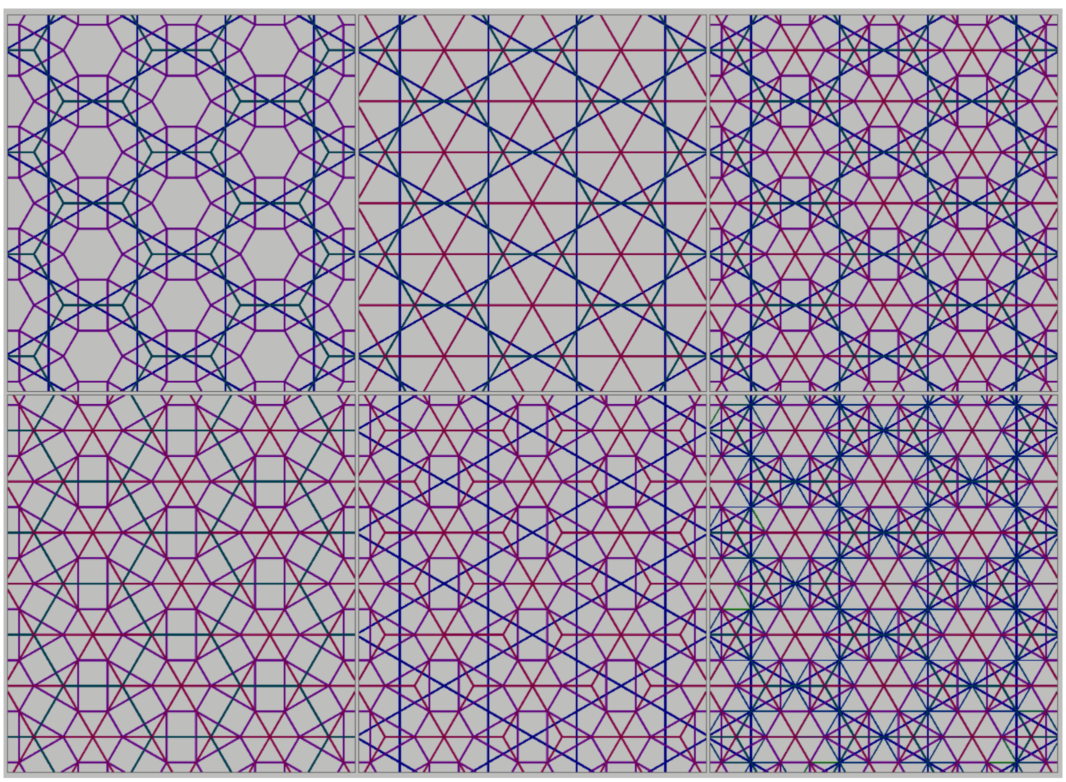

Figure 10

We can see in Fig. 11 the cross-intersections alternating one another $(0,1,2,1,0,1,\ldots)$ in the space-filling mosaic based on the 3-model of the 6-cube. In the bottom row are the intersections supplemented by the projected edges of the intersected solids.

The alternating $(0,1,2,3,4,5,6,5,\ldots)$ longitudinal intersections of our mosaic are descripted in Fig. 12. The methods of the further dissections could be applied here similarly to the horizontal intersections.

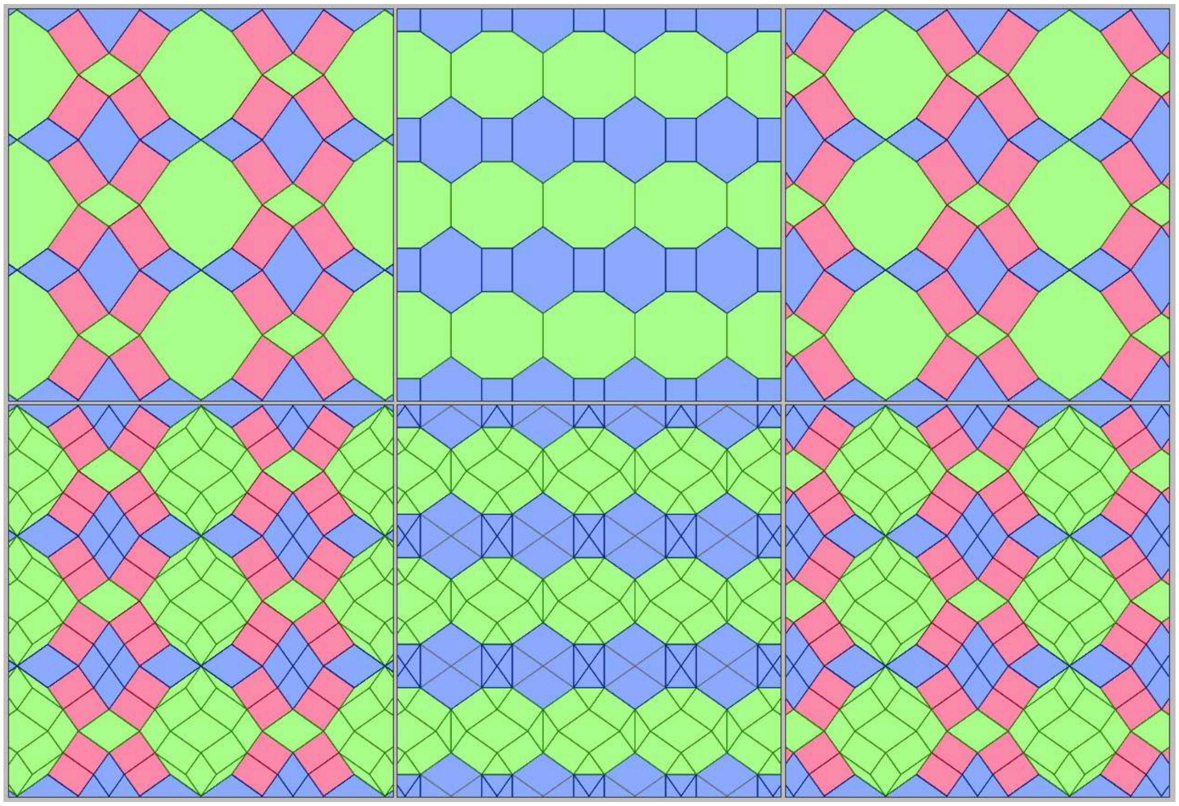

Figure 11

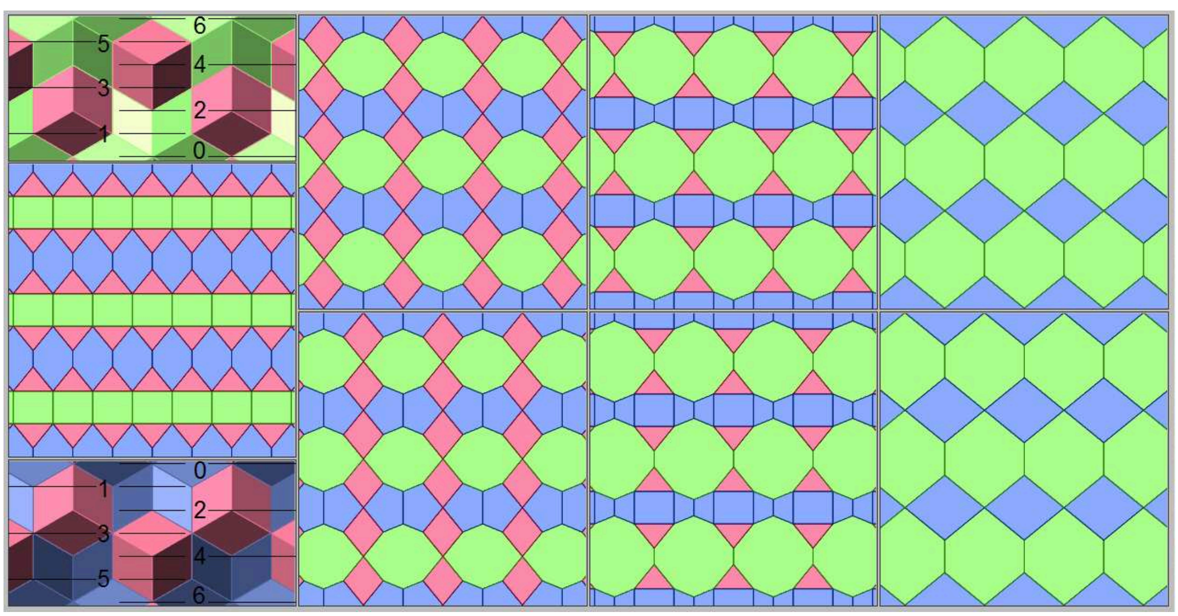

Figure 12

With the above methods two- and three-dimensional tilings based on the 3-models of k-cubes, can surely be made up to $k = 10$ and probably furthermore, too. These cases are just examined but not displayed yet in all details by the author.

The creation of the constructions and figures required for the paper was aided by the AutoCAD program and the Autolisp routines developed by the author.

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