One-dimensional flow of a compressible viscous micropolar fluid: The Cauchy problem

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Abstract. *The Cauchy problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodynamically perfect and polytropic. A corresponding initial-boundary value problem has a unique strong solution on* $]0,1[\times]0,T[$*, for each* $T>0$*. By using this result we construct a sequence of approximate solutions which converges to a solution of the Cauchy problem.*

Key words: *micropolar fluid, the Cauchy problem, strong solution, weak convergence*

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1. Statement of the problem and the main result

In this paper we consider nonstationary 1-D flow of a compressible and heat-conducting micropolar fluid. The equations of motion for this fluid are derived from the integral form of conservation laws for polar fluids, under a number of supplementary assumptions such as politropy, Fourier's law, Boyle's law and selection of constitutive equations (see [7]). A corresponding initial-boundary value problem has a unique strong solution on $]0,1[\times]0,T[$, for each $T>0$ ([8]). By using this result we prove a global-in-time existence theorem for the Cauchy problem. In our proof we follow some ideas of S.N.Antontsev, A.V.Kazhykhov and V.N.Monakhov, applied to the case of a classical fluid ([1]).

Let ρ, v, ω and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangean description. Governing equations of the flow under consideration are as follows ([7]):

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0 \ , \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\rho \frac{\partial v}{\partial x}\right) - K \frac{\partial}{\partial x}(\rho\theta) \ , \tag{1.2}$$

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$$\rho\frac{\partial\omega}{\partial t}=A\left[\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial\omega}{\partial x}\right)-\omega\right]\ , \tag{1.3}$$

$$\rho\frac{\partial\theta}{\partial t}=-K\rho^2\theta\frac{\partial v}{\partial x}+\rho^2\left(\frac{\partial v}{\partial x}\right)^2+\rho^2\left(\frac{\partial\omega}{\partial x}\right)^2+\omega^2+D\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial\theta}{\partial x}\right) \tag{1.4}$$

in $\mathbb{R}\times\mathbb{R}^+$, where K,A and D are positive constants. Equations (1.1)-(1.4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. We take the following non-homogeneous initial conditions :

$$\rho(x,0)=\rho_0(x)\ , \tag{1.5}$$

$$v(x,0)=v_0(x)\ , \tag{1.6}$$

$$\omega(x,0)=\omega_0(x)\ , \tag{1.7}$$

$$\theta(x,0)=\theta_0(x) \tag{1.8}$$

for $x\in\mathbb{R}$, where ρ_0,v_0,ω_0 and θ_0 are given functions. We assume that there exist $m,M\in\mathbb{R}^+$, such that

$$0<m\le\rho_0(x)\le M,\quad m\le\theta_0(x)\le M,\quad x\in\mathbb{R}\ . \tag{1.9}$$

The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let the initial functions satisfy conditions (1.9) and*

$$\rho_0-1,v_0,\omega_0,\theta_0-1\in H^1(R). \tag{1.10}$$

Then for each $T\in\mathbb{R}^+$ there exists a state function

$$S(x,t)=(\rho,v,\omega,\theta)(x,t)\quad (x,t)\in\Pi=\mathbb{R}\times]0,T[\ , \tag{1.11}$$

with the properties:

$$\rho-1\in L^\infty(0,T;H^1(R))\cap H^1(\Pi)\ , \tag{1.12}$$

$$v,\omega,\theta-1\in L^\infty(0,T;H^1(R))\cap H^1(\Pi)\cap L^2(0,T;H^2(R)) \tag{1.13}$$

which satisfies equations (1.1)-(1.4) in the sense of distributions in Π and conditions (1.5)-(1.8) in the sense of traces.

We denote by $B^k(R), k\in\mathbf{N_0}$, the Banach space

$$B^k(R)=\{u\in C^k(R):\lim_{|x|\to\infty}|D^nu(x)|=0,\ 0\le n\le k\}, \tag{1.14}$$

where D^n is the n-th derivative; the norm is defined by

$$\|u\|_{B^k(R)}=\sup_{n\le k}\{\sup_{x\in R}|D^nu(x)|\}. \tag{1.15}$$

Remark 1.1. *From Sobolev's embedding theorem ([3, Chapter IV])and theory of vector-valued distributions ([4, pp. 467-480]) one can conclude that from (1.12) and (1.13) it follows:*

$$\rho - 1 \in L^{\infty}(0,T;B^0(R)) \cap C([0,T];L^2(R)), \tag{1.16}$$

$$v,\omega,\theta - 1 \in L^2(0,T;B^1(R)) \cap C([0,T];H^1(R)) \cap L^{\infty}(0,T;B^0(R)) \tag{1.17}$$

and hence

$$v,\omega \in C([0,T];B^0(R)), \quad \rho,\theta \in L^{\infty}(\Pi). \tag{1.18}$$

The state function S and its distributional derivatives that occur in (1.1)-(1.4) are locally integrable functions in Π and system (1.1)-(1.4) is satisfied a. e. in Π. In other words, state function (1.11) is a strong solution of our system (1.1)-(1.4).

In the proof of *Theorem 1.1* we construct a sequence of approximations $S_n = (\rho_n, v_n, \omega_n, \theta_n)_{n\in N}$ of the state function in Π ; we establish some estimates of approximations S_n which show that $\{S_n\}_{n\in N}$ belongs to a fixed ball (i.e. independent of n) of a certain normed space. Using the results of weak compactness of a unit ball in a Hilbert space (resp. a Banach space or resp. the dual of a normed space) from $\{S_n\}_{n\in N}$ we extract a subsequence which has limit in the some weak sense. Finally, we show that this limit is the solution of our problem.

2. Approximate solutions and a priori estimates

First we introduce the restrictions of the initial functions ρ_0 and θ_0 to $]-n,n[$. For $n \in \mathbf{N}$ let

$$\rho_{0n} = \rho_0 \ \text{ on } \ ]-n,n[, \tag{2.1}$$

$$\theta_{0n} = \theta_0 \ \text{ on } \ ]-n,n[. \tag{2.2}$$

We can easily verify that

$$\rho_{0n}\ ,\ \theta_{0n}\ \in H^1(]-n,n[). \tag{2.3}$$

Since $\mathcal{D}(R)$ is dense in $H^1(R)$, there exist the sequences $\{v_{0n}\}$ and $\{\omega_{0n}\}$ of approximations of the initial functions v_0 and ω_0 with the following properties:

$$\text{(i)} \quad v_{0n},\omega_{0n} \in H_0^1(]-n,n[), \ \ v_{0n}=0, \ \omega_{0n}=0 \ \text{ on } \ \mathbb{R}\backslash]-n,n[, \tag{2.4}$$

$$\text{(ii)} \quad v_{0n} \to v_0, \ \ \omega_{0n} \to \omega_0 \ \text{ strongly in } H^1(R). \tag{2.5}$$

Let us consider equations (1.1) - (1.4) on $]-n,n[\times\mathbb{R}^+$, with the boundary conditions

$$v(-n,t) = v(n,t) = 0, \ \ \omega(-n,t) = \omega(n,t) = 0, \tag{2.6}$$

$$\frac{\partial\theta}{\partial x}(-n,t) = \frac{\partial\theta}{\partial x}(n,t) = 0 \tag{2.7}$$

for $t > 0$ and with $(\rho_{0n}, v_{0n}, \omega_{0n}, \theta_{0n})$ as the initial data on $]-n, n[$. Functions $\rho_{0n}, v_{0n}, \omega_{0n}$ and θ_{0n} satisfy the conditions taken for the initial functions in Theorem 1.1 of [8] and we conclude that, for each $n \in \mathbf{N}$ and $T > 0$, problem (1.1)-(1.4), (2.6)-(2.7) has a unique strong solution

$$S_n(x,t) = (\rho_n, v_n, \omega_n, \theta_n)(x,t)\ ,\ (x,t) \in Q_{nT} =]-n,n[\times]0,T[\tag{2.8}$$

with the properties:

$$\rho_n \in L^\infty(0,T; H^1(]-n,n[)) \cap H^1(Q_{nT})\ , \tag{2.9}$$

$$\begin{aligned} v_n, \omega_n, \theta_n \in L^\infty(0,T; H^1(]-n,n[)) \cap H^1(Q_{nT}) \\ \cap L^2(0,T; H^2(]-n,n[))\ , \end{aligned} \tag{2.10}$$

$$\rho_n > 0\ ,\ \theta_n > 0 \quad \text{on } \bar{Q}_{nT}, \tag{2.11}$$

that in [7] and [8] is named a generalised solution. From embedding theorem ([6, Chapter II, Theorem 2.2.1]), theories of vector-valued distributions and interpolations ([4, pp. 467-480]) we observe that from (2.9) and (2.10) it follows:

$$\rho_n \in L^\infty(0,T; C([-n,n])) \cap C([0,T]; L^2(]-n,n[), \tag{2.12}$$

$$v_n, \omega_n, \theta_n \in L^2(0,T; C^1([-n,n])) \cap C([0,T]; H^1(]-n,n[)), \tag{2.13}$$

$$v_n, \omega_n, \theta_n \in C(\bar{Q}_{nT}). \tag{2.14}$$

From the properties of the function ρ_n (see [1, pp. 44-45]) we get

$$\rho_n \in C(\bar{Q}_{nT}). \tag{2.15}$$

Next we prove uniform (in $n \in \mathbf{N}$) a priori estimates for S_n in Q_{nT}. By $C \in \mathbb{R}^+$ we denote a generic constant, independent of $n \in \mathbf{N}$.

We introduce non-negative functions U_n and V_n defined on $]0, T[$ by

$$\begin{aligned} U_n(t) = \int_{-n}^{n} \Big[\frac{1}{2K} v_n^2 + \frac{1}{2AK} \omega_n^2 + \frac{1}{\rho_n} (\rho_n \ln \rho_n - \rho_n + 1) \\ + \frac{1}{K} (\theta_n - \ln \theta_n - 1) \Big] dx, \end{aligned} \tag{2.16}$$

$$\begin{aligned} V_n(t) = \frac{1}{K} \int_{-n}^{n} \Big[\frac{\rho_n}{\theta_n} \left(\frac{\partial v_n}{\partial x} \right)^2 + \frac{\rho_n}{\theta_n} \left(\frac{\partial \omega_n}{\partial x} \right)^2 + \frac{\omega_n^2}{\rho_n \theta_n} \\ + D \frac{\rho_n}{\theta_n^2} \left(\frac{\partial \theta_n}{\partial x} \right)^2 \Big] dx. \end{aligned} \tag{2.17}$$

Using the inequality $\ln x \le x - 1$ for $U_n(0)$ we have

$$U_n(0) \le \int_R \left[\frac{1}{2K} v_{0n}^2 + \frac{1}{2AK} \omega_{0n}^2 + \frac{(\rho_0 - 1)^2}{\rho_0} + \frac{1}{K} \frac{(\theta_0 - 1)^2}{\theta_0} \right] dx \tag{2.18}$$

and taking into account (1.9), (1.10) and (2.5) we immediately get

$$U_n(0) \leq C. \tag{2.19}$$

Now, multiply (1.1), (1.2), (1.3) and (1.4) by $\rho^{-1}(1-\rho^{-1})$, $K^{-1}v$, $A^{-1}K^{-1}\omega\rho^{-1}$ and $K^{-1}(1-\theta^{-1})\rho^{-1}$, respectively, and integrate over $]-n,n[$ and over $]0,t[, t \in]0,T[$. After addition of the obtained equations we find that

$$U_n(t) + \int_0^t V_n(\tau)d\tau = U_n(0) \leq C. \tag{2.20}$$

Lemma 2.1. *For* $t \in]0,T[$,

$$\|v_n(t)\|_{L^2(]-n,n[)} \leq C \ , \tag{2.21}$$

$$\|\omega_n(t)\|_{L^2(]-n,n[)} \leq C \ . \tag{2.22}$$

Proof. These estimates follow from (2.20) and the inequalities

$$\frac{1}{2K}\|v_n(t)\|^2_{L^2(]-n,n[)} \leq U_n(t), \ \frac{1}{2AK}\|\omega_n(t)\|^2_{L^2(]-n,n[)} \leq U_n(t). \tag{2.23}$$

□

Like in [1, pp. 68-75] we can conclude that for each subset $]m,m+1[$, $m \in \{-n,-n+1,...,n-1\}$, of $]-n,n[$ there exists $a_m(t) \in]m,m+1[$ such that the restriction of ρ_n to $Q'_{mT} =]m,m+1[\times]0,T[$ has the form

$$\rho_n(x,t) = \frac{\rho_{0n}(x)Y_{nm}(t)B_{nm}(x,t)}{1+K\rho_{0n}(x)\int_0^t Y_{nm}(\tau)B_{nm}(x,\tau)\theta_n(x,\tau)d\tau}, \tag{2.24}$$

where

$$Y_{nm}(t) = \frac{1}{\rho_{0n}(a_m(t))} exp\left\{K\int_0^t \rho_n(a_m(t),\tau)\theta_n(a_m(t),\tau)d\tau\right\} \ , \tag{2.25}$$

$$B_{nm}(x,t) = \rho_n(a_m(t),t) exp\left\{\int_{a_m(t)}^x [v_{0n}(\xi) - v_n(\xi,t)]d\xi\right\}. \tag{2.26}$$

Also, there exist constants $C_i (i = 1,..,5)$ (independent of m and n) such that the estimates

$$C_1 \leq \int_m^{m+1} \theta_n(x,t)dx \leq C_2 \ , \tag{2.27}$$

$$C_3^{-1} \leq B_{nm}(x,t) \leq C_3 \ \ , \ \ C_4 \leq Y_{nm}(t) \leq C_5 \tag{2.28}$$

are satisfied for $t \in]0,T[$ and $(x,t) \in Q'_{mT}$.
Because of (2.14) and (2.15) there exist positive functions

$$m_{\rho_n}(t) = \inf_{x\in]-n,n[} \rho_n(x,t) \ , \ m_{\theta_n}(t) = \inf_{x\in]-n,n[} \theta_n(x,t), \tag{2.29}$$

$$M_{\rho_n}(t) = \sup_{x\in]-n,n[} \rho_n(x,t) \ , \ M_{\theta_n}(t) = \sup_{x\in]-n,n[} \theta_n(x,t) \tag{2.30}$$

defined on $]0,T[$ and we have the following results.

Lemma 2.2. *For $t \in]0,T[$,*

$$M_{\rho_n}(t) \le C \ , \tag{2.31}$$

$$m_{\rho_n}(t) \ge C \left(1 + \int_0^t M_{\theta_n}(\tau) d\tau\right)^{-1} . \tag{2.32}$$

Proof. Using (1.9), (2.11), (2.29), (2.30) and estimates (2.28) from (2.24) we get (2.31) and (2.32). □

We define non-negative functions I_{1n} and I_{2n} in $]0,T[$ by

$$I_{1n}(t) = \int_{-n}^{n} \rho_n(x,t) \left(\frac{\partial \theta_n}{\partial x}(x,t)\right)^2 dx \ , \tag{2.33}$$

$$I_{2n}(t) = \int_0^t I_{1n}(\tau) d\tau \ . \tag{2.34}$$

Obviously, I_{1n} and I_{2n} belong to $L^1(]0,T[)$.

Lemma 2.3. *For $\varepsilon > 0$ sufficiently small, there exists a constant $C_\varepsilon \in \mathbb{R}^+$ such that, for $t \in]0,T[$, the inequality*

$$M^2_{\theta_n}(t) \le \varepsilon I_{1n}(t) + C_\varepsilon (1 + I_{2n}(t)) \tag{2.35}$$

holds true.

Proof. We introduce the function ψ_{nm} on Q'_{mT} by

$$\psi_{nm}(x,t) = \theta_n(x,t) - \int_m^{m+1} \theta_n(x,t) dx \ . \tag{2.36}$$

There exists $x_m(t) \in]m, m+1[$ such that $\psi_{nm}(x_m(t),t) = 0$. By means of the Hölder inequality we find that

$$|\psi_{nm}(x,t)|^{\frac{3}{2}} \le \int_{x_m(t)}^{x} \frac{\partial}{\partial \xi} |\psi_{nm}(\xi,t)|^{\frac{3}{2}} d\xi \le$$

$$\frac{3}{2}\left(\int_m^{m+1} \rho_n^{-1}(\xi,t) |\psi_{nm}(\xi,t)| d\xi\right)^{\frac{1}{2}} \left(\int_m^{m+1} \rho_n(\xi,t) \left(\frac{\partial \psi_{nm}}{\partial \xi}(\xi,t)\right)^2 d\xi\right)^{\frac{1}{2}}. \tag{2.37}$$

Because of (2.27) we have $\int_m^{m+1} |\psi_{nm}(\xi,t)| d\xi \le C$ (independently of m and n). Taking into account (2.32), (2.27), (2.33) and $\frac{\partial \psi_{nm}}{\partial \xi}(\xi,t) = \frac{\partial \theta_n}{\partial \xi}(\xi,t)$ from (2.37) we obtain

$$M^2_{\theta_n}(t) \le C \left[\left(1 + \int_0^t M_{\theta_n}(\tau) d\tau\right)^{\frac{2}{3}} (I_{1n}(t))^{\frac{2}{3}} + 1\right] \ , \ t \in]0,T[. \tag{2.38}$$

Applying the Young inequality with parameter $\varepsilon > 0$, from (2.38) it follows

$$M_{\theta_n}^2(t) \le \varepsilon I_{1n}(t) + C_\varepsilon \left(1 + \int_0^t M_{\theta_n}^2(\tau) d\tau\right) \tag{2.39}$$

and by means of the Gronwall's inequality ([1, p.25]) we get (2.35). □

Now, we introduce the function

$$\Phi_n = \frac{1}{2} v_n^2 + \frac{1}{2A} \omega_n^2 + (\theta_n - 1) \text{ on } Q_{nT} . \tag{2.40}$$

Multiply equations (1.2), (1.3) and (1.4) by $v_n \Phi_n$, $A^{-1}\rho_n^{-1}\omega_n\Phi_n$ and $\rho_n^{-1}\Phi_n$, respectively, and integrate over $]-n, n[$ using (2.6) and (2.7). After addition of the obtained equations, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{-n}^n \Phi_n^2 dx + \int_{-n}^n \rho_n \left(\frac{\partial \Phi_n}{\partial x}\right)^2 + (1 - A^{-1}) \int_{-n}^n \rho_n \frac{\partial \omega_n}{\partial x} \omega_n \frac{\partial \Phi_n}{\partial x} dx \\ &+ (D-1)\int_{-n}^n \rho_n \frac{\partial \theta_n}{\partial x}\frac{\partial \Phi_n}{\partial x} dx - K \int_{-n}^n \rho_n \theta_n v_n \frac{\partial \Phi_n}{\partial x} dx = 0 \end{aligned} \tag{2.41}$$

on $]0, T[$. Taking into account (2.21), (2.31) and (2.35), in the same way as in [8, Lemma 2.4], we conclude that the inequality

$$\begin{aligned} &\frac{d}{dt}\left(\int_{-n}^n (\Phi_n^2 + C_1 v_n^4 + C_2 \omega_n^4) dx + D I_{2n}\right) \\ &\le C(1 + \int_{-n}^n (\Phi_n^2 + C_1 v_n^4 + C_2 \omega_n^4) dx + D I_{2n}) \end{aligned} \tag{2.42}$$

holds true. Using the embedding $H^1(R) \subset B^0(R)$, (2.2), (1.10) and (2.5) we obtain that $\|\Phi_n(0)\|_{L^2(]-n,n[)}^2 \le C$ and after integration (2.42) over $]0, t[$, $t \in]0, T[$, we find that

$$\int_{-n}^n (\Phi_n^2 + C_1 v_n^4 + C_2 \omega_n^4) dx + D I_{2n} \le C \text{ on }]0T[. \tag{2.43}$$

Lemma 2.4. *For $t \in]0, T[$,*

$$\|(\theta_n - 1)(t)\|_{L^2(]-n,n[)} \le C , \tag{2.44}$$

$$\int_0^t M_{\theta_n}^2(\tau) d\tau \le C , \tag{2.45}$$

$$m_{\rho_n}(t) \ge C , \tag{2.46}$$

$$\int_0^t \left\|\frac{\partial \theta_n}{\partial x}(\tau)\right\|_{L^2(]-n,n[)}^2 d\tau \le C . \tag{2.47}$$

Proof. Estimate (2.44) follows from (2.43) and the inequality $\|\theta_n - 1\|_{L^2(]-n,n[)}^2 \le \int_{-n}^n \Phi_n^2 dx$. Integrating (2.35) over $]0, t[$ and taking into account (2.33), (2.34) and

(2.43) we get (2.45). From (2.32) and (2.45) we obtain (2.46). At last, using (2.46) and the estimate for I_{2n} from (2.43) we conclude that (2.47) holds. □

Differentiating equality (2.24) with respect to x we get

$$\frac{\partial \rho_n}{\partial x} = \rho_n \varphi_n - \rho_n^2 Y_{nm}^{-1} B_{nm}^{-1} \left[\frac{d}{dx}\left(\frac{1}{\rho_{0n}}\right) + K \int_0^t B_{nm} Y_{nm} \left(\frac{\partial \theta_n}{\partial x} + \theta_n \varphi_n\right) d\tau \right], \tag{2.48}$$

where $\varphi_n(x,t) = v_{0n}(x) - v_n(x,t)$. Using (1.9), (1.10), (2.28) and (2.31) from (2.48) we obtain

$$\begin{aligned} \left\| \frac{\partial \rho_n}{\partial x}(t) \right\|^2_{L^2(]-n,n[)} &\leq C \left(\|v_{0n}\|^2_{L^2(]-n,n[)} + \|v_n(t)\|^2_{L^2(]-n,n[)} \right) \\ &+ C \left[1 + \int_0^t \left\| \frac{\partial \theta_n}{\partial x}(\tau) \right\|^2_{L^2(]-n,n[)} d\tau + \int_0^t M^2_{\theta_n}(\tau) \left(\|v_{0n}\|^2_{L^2(]-n,n[)} \right.\right. \\ &+ \left.\left. \|v_n(\tau)\|^2_{L^2(]-n,n[)} \right) d\tau \right], \quad t \in]0,T[. \end{aligned} \tag{2.49}$$

Lemma 2.5. *For $t \in]0,T[$,*

$$\left\| \frac{\partial \rho_n}{\partial x}(t) \right\|_{L^2(]-n,n[)} \leq C. \tag{2.50}$$

Proof. By means of estimates (2.21), (2.45), (2.47) and (2.5) the result follows directly from (2.49). □

Multiplying (1.2) and (1.3) by v_n and $\rho_n^{-1}\omega_n$, respectively, integrating over $]-n,n[$ and using (2.21) and (2.50) in the first equation, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-n}^n v_n^2 dx + \int_{-n}^n \rho_n \left(\frac{\partial v_n}{\partial x}\right)^2 dx &\leq K M_{\theta_n} \left\| \frac{\partial \rho_n}{\partial x} \right\|_{L^2(]-n,n[)} \|v_n\|_{L^2(]-n,n[)} \\ + C \left\| \frac{\partial \theta_n}{\partial x} \right\|_{L^2(]-n,n[)} \|v_n\|_{L^2(]-n,n[)} &\leq C \left(M_{\theta_n} + \left\| \frac{\partial \theta_n}{\partial x} \right\|_{L^2(]-n,n[)} \right), \end{aligned} \tag{2.51}$$

$$\frac{1}{2}\frac{d}{dt}\int_{-n}^n \omega_n^2 dx + A \int_{-n}^n \rho_n \left(\frac{\partial \omega_n}{\partial x}\right)^2 dx + A \int_{-n}^n \frac{\omega_n^2}{\rho_n} dx = 0 \text{ on }]0,T[. \tag{2.52}$$

Lemma 2.6. *For $t \in]0,T[$,*

$$\int_0^t \left\| \frac{\partial v_n}{\partial x}(\tau) \right\|^2_{L^2(]-n,n[)} d\tau \leq C, \tag{2.53}$$

$$\int_0^t \left\| \frac{\partial \omega_n}{\partial x}(\tau) \right\|^2_{L^2(]-n,n[)} d\tau \leq C. \tag{2.54}$$

Proof. Integrating (2.51) and (2.52) over $]0,t[, t \in]0,T[$, and applying (2.45)-(2.47), (2.5) and (2.31) we get (2.53) and (2.54). □

Now, we write equation (1.1) in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho_n}\right)=\frac{\partial v_n}{\partial x}. \tag{2.55}$$

Integrating over $]0,t[, t\in]0,T[$, squaring and integrating again over $]-n,n[$, we obtain the inequality

$$\int_{-n}^{n}\left(\frac{1-\rho_n}{\rho_n}\right)^2\leq C\left[\int_{-n}^{n}\left(\frac{1-\rho_{0n}}{\rho_{0n}}\right)^2dx+\int_0^t\int_{-n}^{n}\left(\frac{\partial v_n}{\partial x}\right)^2dxd\tau\right]. \tag{2.56}$$

Lemma 2.7. *For* $t\in]0,T[$,

$$\|(\rho_n-1)(t)\|^2_{L^2(]-n,n[)}\leq C. \tag{2.57}$$

Proof. Using (1.9), (1.10), (2.31) and (2.53), from (2.56) we easily get (2.57). □

Lemma 2.8. *For* $t\in]0,T[$,

$$\left\|\frac{\partial v_n}{\partial x}(t)\right\|^2_{L^2(]-n,n[)}+\int_0^t\left\|\frac{\partial^2 v_n}{\partial x^2}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C, \tag{2.58}$$

$$\left\|\frac{\partial \omega_n}{\partial x}(t)\right\|^2_{L^2(]-n,n[)}+\int_0^t\left\|\frac{\partial^2 \omega_n}{\partial x^2}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C, \tag{2.59}$$

$$\left\|\frac{\partial \theta_n}{\partial x}(t)\right\|^2_{L^2(]-n,n[)}+\int_0^t\left\|\frac{\partial^2 \theta_n}{\partial x^2}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C. \tag{2.60}$$

Proof. After multiplying (1.2) by $\partial^2 v_n/\partial x^2$ and integrating by parts over $]-n,n[$ and over $]0,t[$, in the same way as in [1, pp.53-54], we obtain (2.58). Multiplying (1.3) and (1.4) by $A^{-1}\rho_n^{-1}\partial^2\omega_n/\partial x^2$ and $\rho_n^{-1}\partial^2\theta_n/\partial x^2$, respectively, and integrating by parts over $]-n,n[$ and over $]0,t[$, in the same way as in [8, Lemmas 2.7, 2.8] we get estimates (2.59) and (2.60). □

Lemma 2.9. *For* $t\in]0,T[$,

$$\int_0^t\left\|\frac{\partial \rho_n}{\partial t}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C, \tag{2.61}$$

$$\int_0^t\left\|\frac{\partial v_n}{\partial t}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C, \tag{2.62}$$

$$\int_0^t\left\|\frac{\partial \omega_n}{\partial t}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C, \tag{2.63}$$

$$\int_0^t\left\|\frac{\partial \theta_n}{\partial t}(\tau)\right\|^2_{L^2(]-n,n[)}d\tau\leq C. \tag{2.64}$$

Proof. We square equations (1.1) and (1.2), integrate over $]-n,n[$ and $]0,t[$. Then in the same way as in [1, pp.53-54] we get (2.61) and (2.62). Also, squaring equations (1.3) and (1.4), integrating over $]-n,n[$ and $]0,t[$ in the same way as in [8, Lemmas 2.7, 2.8] we obtain (2.63) and (2.64). □

3. Proof of Theorem 1.1

Let us denote again by ρ_n and θ_n the extensions of ρ_n and θ_n by 1 from Q_{nT} to Π and by v_n and ω_n the extensions of v_n and ω_n by zero outside of Q_{nT}.

We can find a function $\varphi \in \mathcal{D}(R)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases} \tag{3.1}$$

and then we define φ_n by

$$\varphi_n(x) = \varphi(\frac{2x}{n}), \ \ n \in N. \tag{3.2}$$

For v_n and ω_n we put

$$\bar{v}_n = v_n \varphi_n, \ \ \bar{\omega}_n = \omega_n \varphi_n \tag{3.3}$$

and for ρ_n and θ_n we introduce

$$\bar{\rho}_n = (\rho_n - 1)\varphi_n + 1, \ \ \bar{\theta}_n = (\theta_n - 1)\varphi_n + 1. \tag{3.4}$$

One can easily conclude that the function $\bar{S}_n = (\bar{\rho}_n, \bar{v}_n, \bar{\omega}_n, \bar{\theta}_n)$ satisfies system (1.1)-(1.4) a.e. in $]-\frac{n}{2}, \frac{n}{2}[\times]0, T[$ and initial data (2.1), (2.2) and (2.4) a.e. in $]-\frac{n}{2}, \frac{n}{2}[$. Using the properties of ρ_n, v_n, ω_n and θ_n from (3.2)-(3.4) we observe that

$$\bar{\rho}_{n0} - 1 \rightarrow \rho_0 - 1, \ \ \bar{\theta}_{n0} - 1 \rightarrow \theta_0 - 1 \ \text{ strongly in } L^2(R) \tag{3.5}$$

and

$$\bar{v}_{n0} \rightarrow v_0, \ \ \bar{\omega}_{n0} \rightarrow \omega_0 \ \text{ strongly in } L^2(R), \tag{3.6}$$

where

$$\begin{gathered} \bar{\rho}_{n0} = \bar{\rho}_n(x, 0), \ \bar{\theta}_{n0} = \bar{\theta}_n(x, 0), \\ \bar{v}_{n0} = \bar{v}_n(x, 0), \ \bar{\omega}_{n0} = \bar{\omega}_n(x, 0), \ x \in \mathbb{R}. \end{gathered} \tag{3.7}$$

In order to simplify a notation in what follows we write ρ_n instead $\bar{\rho}_n$, etc..

From *Lemmas 2.5, 2.7* and *2.9* we conclude that

$$\{\rho_n - 1\} \text{ is bounded in } \ L^\infty(0, T; H^1(R)) \text{ and } H^1(\Pi). \tag{3.8}$$

Moreover, taking into account (2.31) and (2.46), from (3.4) we obtain that

$$\{\rho_n\} \text{ is bounded in } \ L^\infty(\Pi) \ . \tag{3.9}$$

By means of *Lemmas 2.1, 2.4, 2.6, 2.8* and *2.9* from (3.3) and (3.4) we get that

$$\begin{gathered} \{v_n\}, \{\omega_n\}, \{\theta_n - 1\} \text{ are bounded in } L^\infty(0, T; H^1(R)), H^1(\Pi) \\ \text{and } L^2(0, T; H^2(R)) \ . \end{gathered} \tag{3.10}$$

Lemma 3.1. *There exists a function*

$$\rho - 1 \in H^1(\Pi) \cap L^\infty(0, T; H^1(R)) \tag{3.11}$$

and a subsequence of $\{\rho_n - 1\}$ *(for simplicity denoted again as* $\{\rho_n - 1\}$*) such that*

$$\rho_n - 1 \to \rho - 1 \ \textit{weakly}^* \ \textit{in} \ L^\infty(0,T;H^1(R)), \tag{3.12}$$

$$\rho_n - 1 \to \rho - 1 \ \textit{weakly in} \ H^1(\Pi). \tag{3.13}$$

The function ρ *belongs to* $L^\infty(\Pi)$ *and has the properties:*

$$\rho(x,0) = \rho_0(x) \ \ \textit{a.e. in} \ \mathbb{R}, \tag{3.14}$$

$$m_1 \le \rho \le M_1 \ \ \textit{a.e. in} \ \Pi, \tag{3.15}$$

where $m_1, M_1 \in \mathbb{R}^+$.

Proof. Since the sequence $\{\rho_n - 1\}$ is bounded in $L^\infty(0,T;H^1(R))$ (dual of $L^1(0,T;H^{-1}(R))$), it is possible to extract a subsequence (denoted again as $\{\rho_n-1\}$) such that $\rho_n - 1 \to \rho - 1$ weakly* in $L^\infty(0,T;H^1(R))$ (see [4, pp.498-503]). It means that for $g \in L^1(0,T;H^{-1}(R)), (g(t) = (g_1(t), g_2(t)) \in L^2(R) \times L^2(R))$ we have

$$\int_\Pi (\rho_n - 1) g_1 dxdt + \int_\Pi \frac{\partial \rho_n}{\partial x} g_2 dxdt \to \int_\Pi (\rho - 1) g_1 dxdt + \int_\Pi \frac{\partial \rho}{\partial x} g_2 dxdt \ . \tag{3.16}$$

Specially, for all $\varphi \in \mathcal{D}(\Pi)$ from (3.16) we obtain

$$\int_\Pi (\rho_n - 1)\varphi dxdt \to \int_\Pi (\rho - 1)\varphi dxdt \ , \tag{3.17}$$

$$\int_\Pi \frac{\partial \rho_n}{\partial x} \varphi dxdt \to \int_\Pi \frac{\partial \rho}{\partial x} \varphi dxdt \ . \tag{3.18}$$

Also, $\{\rho_n\}$ is bounded in $L^\infty(\Pi)$ and therefore there exists a subsequence (denoted by $\{\rho_n\}$) such that $\rho_n \to \rho$ weakly* in $L^\infty(\Pi)$. Specially, for all $\varphi \in \mathcal{D}(\Pi)$ we get

$$\int_\Pi \rho_n(x,t)\varphi(x,t) dxdt \to \int_\Pi \rho(x,t)\varphi(x,t) dxdt \ . \tag{3.19}$$

Because of (3.8) we can take a further subsequence of $\{\rho_n - 1\}$ such that $\rho_n - 1 \to \rho - 1$ weakly in $H^1(\Pi)$. From this convergence we find out that for $\varphi \in \mathcal{D}(\Pi)$, it holds

$$\int_\Pi \frac{\partial \rho_n}{\partial t}(x,t)\varphi(x,t) dxdt \to \int_\Pi \frac{\partial \rho}{\partial t}(x,t)\varphi(x,t) dxdt \ . \tag{3.20}$$

Statement (3.11) is a consequence of the above convergences.

Taking into account (2.31), (2.46), (3.1), (3.2), (3.4) and (3.19) we conclude that there exist $m_1, M_1 \in \mathbb{R}^+$ such that (3.15) holds. From the embedding theorem (see [4, p.473]) we observe that functions $\rho_n - 1, \rho - 1$ belong to $C([0,T]; L^2(R))$ being equipped with the norm of uniform convergence. Now we may speak of the traces $\rho_n(x,0) - 1$ and $\rho(x,0) - 1$.
Let $\psi \in C^\infty([0,T]), \psi(0) \neq 0$ and ψ vanishes in a neighbourhood of T. Applying Green's formula ([4, p.477]) we obtain

$$\int_0^T \int_R \frac{\partial \rho_n}{\partial t}(x,t)u(x)\psi(t)dxdt + \int_0^T \int_R (\rho_n - 1)(x,t)u(x)\frac{d\psi}{dt}(t)dxdt = -\psi(0)\int_R (\rho_{n0} - 1)u(x)dx\ , \tag{3.21}$$

$$\int_0^T \int_R \frac{\partial \rho}{\partial t}(x,t)u(x)\psi(t)dxdt + \int_0^T \int_R (\rho - 1)(x,t)u(x)\frac{d\psi}{dt}(t)dxdt = -\psi(0)\int_R (\rho(x,0) - 1)u(x)dx\ , \tag{3.22}$$

for all $u \in \mathcal{D}(R)$. Comparing (3.21) and (3.22) (when $n \to \infty$) and using (3.17), (3.20) and (3.5) we find that $\rho(x,0) = \rho_0(x)$ in the sense of distributions in $\mathbb{R}$. □

Lemma 3.2. *There exist functions*

$$v, \omega, \theta - 1 \in L^\infty(0,T;H^1(R)) \cap H^1(\Pi) \cap L^2(0,T;H^2(R)) \tag{3.23}$$

and a subsequence of $\{v_n, \omega_n, \theta_n - 1\}$ *(denoted again as* $\{v_n, \omega_n, \theta_n - 1\}$*) such that*

$$\{v_n, \omega_n, \theta_n - 1\} \to \{v, \omega, \theta - 1\} \text{ weakly }^* \text{ in } (L^\infty(0,T;H^1(R))^3, \tag{3.24}$$

$$\{v_n, \omega_n, \theta_n - 1\} \to \{v, \omega, \theta - 1\} \text{ weakly in } (H^1(R))^3\ , \tag{3.25}$$

$$\{v_n, \omega_n, \theta_n - 1\} \to \{v, \omega, \theta - 1\} \text{ weakly in } (L^2(0,T;H^2(R))^3. \tag{3.26}$$

Functions v, ω *and* θ *have the properties:*

$$v(x,0) = v_0(x)\ ,\ \omega(x,0) = \omega_0(x)\ ,\ \theta(x,0) = \theta_0(x) \text{ a.e. in } \mathbf{R}. \tag{3.27}$$

Proof. Conclusions (3.23)-(3.26) follow immediately from (3.10). From the weak convergences we conclude that for $\varphi \in \mathcal{D}(\Pi)$, it follows

$$\int_\Pi v_n(x,t)\varphi(x,t)dxdt \to \int_\Pi v(x,t)\varphi(x,t)dxdt, \tag{3.28}$$

$$\int_\Pi \frac{\partial v_n}{\partial x}(x,t)\varphi(x,t)dxdt \to \int_\Pi \frac{\partial v}{\partial x}(x,t)\varphi(x,t)dxdt, \tag{3.29}$$

$$\int_\Pi \frac{\partial v_n}{\partial t}(x,t)\varphi(x,t)dxdt \to \int_\Pi \frac{\partial v}{\partial t}(x,t)\varphi(x,t)dxdt, \tag{3.30}$$

$$\int_\Pi \frac{\partial^2 v_n}{\partial^2 x}(x,t)\varphi(x,t)dxdt \to \int_\Pi \frac{\partial^2 v}{\partial^2 x}(x,t)\varphi(x,t)dxdt \tag{3.31}$$

(when $n \to \infty$), which is true for $\{\omega_n\}$ and $\{\theta_n - 1\}$ also. By means of Green's formula we get properties (3.27) in the same way as (3.14). □

Lemma 3.3. *Functions* ρ, v, ω *and* θ*, defined by* Lemma 3.1 *and* Lemma 3.2 *satisfy equations (1.1)-(1.4) a.e. in* Π.

Proof. Let $\{S_n = (\rho_n, v_n, \omega_n, \theta_n) : n \in \mathbf{N}\}$ be the subsequence defined by *Lemmas 3.1* and *3.2*. By means of (3.9) and (3.15) we obtain the inequalities

$$\begin{aligned}\left|\int_\Pi (\rho_n^2 \frac{\partial v_n}{\partial x} - \rho^2 \frac{\partial v}{\partial x})\varphi dxdt\right| &\leq \left|\int_\Pi \rho_n^2 (\frac{\partial v_n}{\partial x} - \frac{\partial v}{\partial x})\varphi dxdt\right| \\ &+ \left|\int_\Pi \frac{\partial v}{\partial x}(\rho_n - \rho)(\rho_n + \rho)\varphi dxdt\right| \\ &\leq C\left|\int_\Pi (\frac{\partial v_n}{\partial x} - \frac{\partial v}{\partial x})\varphi dxdt\right| \\ &+ C\left|\int_\Pi \frac{\partial v}{\partial x}(\rho_n - \rho)\varphi dxdt\right| , \end{aligned} \tag{3.32}$$

for all $\varphi \in \mathcal{D}(\Pi)$ and after integrating by parts we get

$$\begin{aligned}\left|\int_\Pi (\rho_n^2 \frac{\partial v_n}{\partial x} - \rho^2 \frac{\partial v}{\partial x})\varphi dxdt\right| &\leq C\left|\int_\Pi (\frac{\partial v_n}{\partial x} - \frac{\partial v}{\partial x})\varphi dxdt\right| \\ &+ C\left|\int_\Pi v(\frac{\partial \rho_n}{\partial x} - \frac{\partial \rho}{\partial x})\varphi dxdt\right| \\ &+ C\left|\int_\Pi v(\rho_n - \rho)\frac{\partial \varphi}{\partial x} dxdt\right| . \end{aligned} \tag{3.33}$$

Taking into account (3.29), (3.19), (3.18) and (1.18), we conclude that for all $\varphi \in \mathcal{D}(\Pi)$, from (3.33) it follows

$$\int_\Pi \rho_n^2 \frac{\partial v_n}{\partial x}\varphi dxdt \to \int_\Pi \rho^2 \frac{\partial v}{\partial x}\varphi dxdt. \tag{3.34}$$

For $\varphi \in \mathcal{D}(\Pi)$ there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ functions ρ_n and v_n satisfy (1.1) in the sense of distributions in Π and therefore we have

$$\int_\Pi \left(\frac{\partial \rho_n}{\partial t} + \rho_n^2 \frac{\partial v_n}{\partial x}\right)\varphi dxdt = 0. \tag{3.35}$$

Applying (3.20) and (3.34), from (3.35) we get

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0 \;\text{ a.e in } \Pi. \tag{3.36}$$

In the same way one can prove that equations (1.2), (1.3) and (1.4) are satisfied. □

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