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THE HYPERSPACES 2^X AND $C_n(X)$ FOR A NON-METRIC CONTINUUM X

In this paper we shall investigate some properties of the hyperspaces 2^X and $Cn(X)$, where X is a non-metric continuum.

1. Preliminaries

The main purpose of this paper is to establish some properties for the hyperspaces 2^X and $C_n(X)$, where X is a non-metric continuum. To do this we shall use the inverse systems method and we shall need some special properties of the inverse systems since each non-metrizable continuum is the limit of an inverse system of metric continua. These properties are explained in Appendix.

All spaces in this paper are compact Hausdorff and all mappings are continuous. We shall use the notion of inverse system as in [1, pp. 135-142]. An inverse system by X = $\{X_a; p_{ab}; A\}$ is denoted. An inverse system X = $\{X_a, p_{ab}, A\}$ is said to be a well-ordered inverse system if A is a well-ordered set.

Let X be a compact space. We define its hyperspaces as the following sets:

$2^X = \{F \subseteq X : F$ is closed and nonempty$\}$,

$C(X) = \{F \in 2^X : F$ is closed and connected$\}$,

$C_n(X) = \{F \in 2^X : F$ closed and has at most n components$\}$; $n \in N$,

$F_n(X) = \{F \in 2^X :$ F has at most n points$\}$; $n \in N$.

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The topology on 2^X is the Vietoris topology and $C(X)$; $C_n(X)$, and $F_n(X)$ are subspaces of 2^X. We agree that $C(X) = C_1(X)$. For each $n \in N$ we have

$$F_n(X) \subset C_n(X),$$
$$C_n(X) \subset C_{n+1}(X),$$
$$F_n(X) \subset F_{n+1}(X).$$

Let X and Y be compact spaces and let $f : X \to Y$ be a continuous mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ *for* $F \in 2^X$. By [9,5.10], 2^f is continuous and $2^f(C(X)) \subset C(Y)$ and $2^f(F_n(X)) \subset F_n(Y)$.

The restriction $2^f | C(X)$ is denoted by $C(f)$. Similarly, $2^f(C_n(X)) \subset C_n(Y)$. The restriction $2^f | C_n(X)$ is denoted by $C_n(f)$.

Let $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim X \to X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{Xa}, 2^{Pab}, A\}$; $C(\mathbf{X}) = \{C(X_a), C(p_{ab}); A\}$ and $F_n(\mathbf{X}) = \{F_n(X_a); 2^{Pab} | F_n(X_b); A\}$ form inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim X$, $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{Pa} : 2^{\lim \mathbf{X}} \to 2^{Xa}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$ since $\{p_a(F) : \mathrm{a} \in \mathrm{A}\}$ is a thread of the system $2^{\mathbf{X}}$. The mapping M is continuous and 1-1. It is also an onto mapping since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$; the set $F' = \cap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Thus, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{Xa}$, $a \in A$, are the projections, then $P_aM = 2^{Pa}$. Identifying F by $M(F)$ we have $P_a = 2^{Pa}$.

Lemma 1.1. [3, Lemma 2.]. *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ *and* $F_n(X) = \lim F_n(\mathbf{X})$.

Arguing as above one can obtain the following lemma:

Lemma 1.2. [6, Corollary 4.5]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *and* $X = \lim \mathbf{X}$. *Then for each* $n \in N$ *the space* $C_n(X)$ *is homeomorphic to* $\lim C_n(\mathbf{X}) = \lim\{C_n(X_a), C_n(p_{ab}), A\}$.

2. The fixed point property for hyperspaces of locally connected continua

A fixed point of a mapping $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every surjection $f : X \to X$ has a fixed point.

We start with following theorem

Theorem 2.1. *Let a non-metric continuum* X *be the inverse limit of an inverse* σ*-system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *such that each* X_a *has the fixed point prop-*

erty and each bonding mapping p_{ab} is onto. Then X has the fixed point property.

Proof. Let $f : X \to X$ be a function. By virtue of Theorem 4.6 there exists a cofinal subset $B(f)$ of A and the mappings $f_b : X_b \to X_b$, $b \in B(f)$, such that each diagram

(2.1)
$$\begin{tikzcd} X_b & X_c \arrow[l, "p_{bc}"'] \\ X_b \arrow[from=u, "f_b"] & X_c \arrow[l, "p_{bc}"'] \arrow[from=u, "f_c"] \end{tikzcd}$$

commutes and the mapping f is induced by the collection $\{f_b : b \in B(f)\}$, i.e., each diagram

(2.2)
$$\begin{tikzcd} X_b & X \arrow[l, "p_b"'] \\ X_b \arrow[from=u, "f_b"] & X \arrow[l, "p_b"'] \arrow[from=u, "f"] \end{tikzcd}$$

commutes. Let F_b, $b \in B(f)$; be a set of fixed points of the mapping f_b.

Claim 1. *Every set F_b is closed.* This is a consequence of the following theorem [1, p. 59, Theorem 1.5.4.]. *For any pair f,g of continuous mappings of a space X into a Hausdor¤ space Y, the set*

$$\{x \in X : f(x) = g(x)\}$$

is closed in X.

It suffices to set $g(x) = x$ and $Y = X$.

Claim 2. *If $b \leq c$ in $B(f)$, then $p_{bc}(F_c) \subset F_b$.* Let x_c be any point of F_c. From the commutativity of the first of the above diagrams it follows $p_{bc}(f_c(x)) = f_b(p_{bc}(x))$. We have $p_{bc}(x) = f_b(p_{bc}(x))$ since $f_c(x) = x$. This means that for the point $y = p_{bc}(x) \in X_b$ we have $y = f_b(y)$, *i.e.*, $y \in F_b$. We infer that $p_{bc}(x) \in F_b$ *and* $p_{bc}(F_c) \subset F_b$.

Claim 3. $\mathbf{F} = \{F_b, p_{bc}|F_c, B(f)\}$ *is an inverse system of compact space with non-empty limit F.*

Claim 4. *The set $F \subset X$ is the set of fixed points of the mapping f.* Let $x \in F$ and let $x_b = p_b(x)$, $b \in B(f)$. Now, $f_b(x_b) = x_b$ since $x_b \in F_b$. We infer that $f(x) = x$ since the collection $\{f_b : b \in B(f)\}$ induces f. The proof is complete.

Theorem 2.2. *If X is a locally connected continuum, then 2^X and $C_n(X)$ have the fixed point property.*

Proof. We give the proof for $C(X)$ only, since the proof for 2^X is similar.

If X is metrizable, then apply Theorem 22.1 of [2]. Assume that X is non-metrizable. Theorem 4.10 states that for every locally connected continuum X

there exists an inverse σ-system X = $\{X_a, p_{ab}, A\}$ such that X is homeomorphic to lim **X**; each X_a is a metric space and each mapping $C(p_{ab}) : C(X_b) \to C(X_a)$ is a surjection. By virtue of [6, Theorem] each $C_n(X_a)$ is an AR. Hence, each $C(X_a)$ has the fixed point property. This means that we have the system $C_n(\mathbf{X}) = \{C(X_a), C(p_{ab}); A\}$ which satisfies the assumption of Theorem 2.1. Thus, $C(X)$ has the fixed point property since $C(X)$ is homeomorphic to lim C(**X**) (Lemma 1.1).

3. The fixed point property for hyperspaces of chainable continua

Theorem 3.1. [12, Theorem 3, p. 247]. *Let X be a metric chainable continuum. Then C(X) has the fixed point property.*

We generalize Theorem 3.1 as follows.

Theorem 3.2. *If X is a chainable continuum, then C(X) has the fixed point property.*

Proof. If X is a metric chainable continuum, then apply Theorem 3.1. Suppose that X is non-metrizable. Theorem 4.14 states that for every chainable continuum that there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to lim **X**; each X_a is a metric space and each mapping $C(p_{ab}) : C(X_b) \to C(X_a)$ is a surjection. By virtue of Theorem 3.1 each $C(X_a)$ has the fixed point property. This means that we have the system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ which satisfies the assumption of Theorem 2.1. Thus, $C(X)$ has the fixed point property since $C(X)$ is homeomorphic to lim $C(\mathbf{X})$ (Lemma 1.1).

4. Appendix

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system; an element $\{x_a\}$ of the Cartesian product $\Pi\{X_a : a \in A\}$ is called a *thread* of **X** if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \le b$. The subspace of $\Pi\{X_a : a \in A\}$ consisting of all threads of **X** is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by lim**X** or by lim $\{X_a, p_{ab}, A\}$ [1, p.135].

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ-*directed* if for each countable subset $B = \{a_1, a_2, ..., a_k, ...\}$ of the members of A there is an $a \in A$ such that $a \ge a_k$ for each $k \in N$.

Theorem 4.1. [4, Theorem 1.1] *Let* X = $\{X_a; p_{ab}; A\}$ *be* σ-*directed inverse system of compact spaces with surjective bonding mappings and with a limit X. Let Y be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \ge a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.*

Let τ be an infinite cardinal number. We say that a directed set A is τ-*complete* if for each chain $a_1 < a_2 < ... < a_\alpha ..., \alpha < \tau ; a_\alpha \in A$, there exists sup $a_\alpha \in A$.

Let Ω be a well-ordered set of all ordinals $\alpha < \omega$, where ω is an ordinal number. A well-ordered inverse system $\{X_\alpha, p_{\alpha\beta}, \Omega\}$ is continuous if for each limit ordinal γ, $0 < \gamma < \Omega$, the maps $p_{\alpha\gamma} : X_\gamma \to X_\alpha$ induce a homeomorphism of the spaces X_γ onto a lim $\{X_\alpha, p_{\alpha\beta}, \alpha < \beta < \gamma\}$*: An* inverse system $\mathbf{X} = \{X_a, p_{ab}, B\}$ is *continuous* if for each chain $B \subset A$ with sup $B = \gamma$ the maps $p_{\alpha\gamma} : X_\gamma \to X_\alpha$ induce a homeomorphism of the spaces X_γ and lim $\{X_a, p_{ab}, B\}$.

Definition 4.1. *An inverse system $\{X_a, p_{ab}, A\}$ is said to be inverse τ-complete system if $\{X_a, p_{ab}, A\}$ is continuous and A is τ-complete. An inverse system is said to be an inverse τ-system if it is τ-complete and $w(X_a) \leq \tau$, $a \in A$* [13, p. 9]. *A directed set A is σ-complete if A is $\aleph_0$-complete. An inverse system is said to be an inverse σ-system if it is σ-complete and $w(X_a) \leq \aleph_0$, $a \in A$.*

In the sequel we shall use the following theorems.

Theorem 4.2. [5, Theorem 2]. *For each Tychonoff cube I^m, $m \geq \aleph_1$, there exists an inverse σ-system* $\mathbf{I} = \{I^a, p_{ab}, A\}$ *of the Hilbert cubes I^a such that I^m is homeomorphic to* lim I.

By the similar method of proof we have the following theorem.

Theorem 4.3. [5, Theorem 3]. *For each uncountable Cartesian product $\Pi\{X_a : a \in A\}$ of compact spaces X_a there exists an inverse σ-system* $\mathbf{X} = \{X^b, p_{ab}, B\}$ *of countable infinite Cartesian products X^b and monotone mappings P_{ab} such that $\Pi\{X_a : a \in A\}$ is homeomorphic to* limX:

Theorem 4.4. [5, Theorem 4]. *Let X be compact Hausdorff space such that $w(X) \geq \aleph_1$. There exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to lim$\mathbf{X}$.*

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is said to be factorizable* [13, p.24] if for each continuous real-valued function f : lim $\mathbf{X} \to I = [0, 1]$ there exists an $a \in A$ such that for $b \geq a$ there exists a continuous function $f_b : X_b \to I$ such that $f = f_b p_b$.

By virtue of Theorem 4.1 we have the following lemma.

Lemma 4.5. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then $\mathbf{X}$ is factorizable.*

Theorem 4.6. [13, Theorem 40.]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{X_a, q_{ab}, A\}$ are factorizable inverse τ-systems with surjective bonding mappings, then for each mapping f : lim$\mathbf{X} \to$ lim$\mathbf{Y}$ there exists a cofinal subset $B(f)$ of A and the mappings $f_b : X_b \to Y_b$; $b \in B(f)$, such that each diagram*

$$(4.1)\qquad \begin{array}{ccc} X_b & \overset{p_{bc}}{\leftarrow} & X_c \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\leftarrow} & Y_c \end{array}$$

commutes and the mapping f is induced by the collections $\{f_b : b \in B(f)\}$, *i.e., each diagram*

$$(4.2)\qquad \begin{array}{ccc} X_b & \overset{p_b}{\leftarrow} & lim\, \mathbf{X} \\ \downarrow f_b & & \downarrow f \\ Y_b & \overset{q_b}{\leftarrow} & lim\, \mathbf{X} \end{array}$$

commutes. If $f : \lim \mathbf{X} \to \lim \mathbf{Y}$ *is a homeomorphism, then each* f_b *is a homeomorphism.*

Now, we shall discuss the necessary and sufficient conditions for surjectivity of the bonding mappings of the inverse σ-system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit (by Lemma 1.1) is $C(\lim \mathbf{X})$. We adopt the notion of hyper-onto representation ([10, p. 183, Definition (1.186)], [2, p. 439]) as follows.

A continuum X *is said to have a hyper onto representation* provided that there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that: (i) X is homeomorphic to $\lim \mathbf{X}$; (ii) each X_a is a metric space and (iii) each mapping $C(p_{ab}) : C(X_b) \to C(X_a)$ is a surjection.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ satisfying (i) through (iii) is called a *hyper-onto representation for* X.

A continuous mapping $f : X \to Y$ is said to be *weakly confuent* [10, p. 22] if for each subcontinuum Q of Y there exists a component K of $f^{1}(Q)$ such that $f(K) = Q$. Every monotone surjection is weakly confluent. If $f : X \to Y$ is a mapping onto a continuum Y, then $C(f) : C(X) \to C(Y)$ is a surjection if and only if f is weakly confluent [10, p. 24, Theorem (0.49.1)].

It is clear that, as in [10, p. 186, Theorem (190)], the following theorem holds.

Theorem 4.7. *A continuum X has a hyper-onto representation if and only if there exists an inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *satisfying (i) and (ii) and such that every bonding mapping* $p_{ab} : X_b \to X_a$ *is weakly confluent.*

A space X is said to be *rim-metrizable* if it has a basis β such that $Bd(U))$ is metrizable for each $U \in \beta$. Equivalently, a space X is rimme-trizable if and only if for each pair F, G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

The following theorem was proved in [5, Theorem 9]:

Theorem 4.8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of compact spaces and surjective bonding mappings* p_{ab}. *Then*:

1) *There exists an inverse system* $M(X) = \{M_a, m_{ab}, A\}$ *of compact spaces such that* m_{ab} *are monotone surjections and* $\lim X$ *is homeomorphic to* $\lim M(X)$,

2) If X is σ*-directed, then* $M(X)$ *is* σ*-directed,*

3) If X is σ*-complete, then* $M(X)$ *is* σ*-complete,*

4) If every X_a *is a metric space and* $\lim X$ *is locally connected (a rim-metrizable continuum), then every* M_a *is metrizable.*

As an immediate consequence of the theorem above we have the following result:

Theorem 4.9. [5, Theorem 10]. *Let* $\mathbf{X} = \{X_a, pab, A\}$ *be a* σ*-directed inverse system of compact spaces and surjective bonding mappings* p_{ab}. *If* $\lim \mathbf{X}$ *is a locally connected space (rim-metrizable continuum), then there exists an* $a \in A$ *such that the projection* p_b *is monotone, for every* $b \geq a$.

Theorem 4.10. [5, Theorem 11]. *If X is a locally connected or rim-metrizable continuum, then X has a hyper-onto representation.*

A *chain* $\{U_1,..., U_n\}$ is a finite collection of sets U_i such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A continuum X is said to be *chainable* or if each open covering of X can be refined by an open covering $u = \{U_1, ..., U_n\}$ such that $\{U_1,..., U_n\}$ is a chain.

Theorem 4.11. [8, Theorem 2*]. *Every chainable continuum X is homeomorphic with the inverse limit of an inverse system* $\{Q_a, q_{ab}, A\}$ of metric chainable continua Q_a.

Remark 4.12. *One can assume that* q_{ab} *are onto mappings since a closed connected subset C of chainable continuum is chainable* [7, Lemma 12].

Theorem 4.13. [11, Theorem 4.]. *If* $f: X \to Y$ *is a mapping of the metric continuum X onto a chainable continuum Y, then f is weakly confluent.*

Representation in Theorem 4.11 is not hyper-onto since A is not σ- directed, but in [5, Theorem 14] the following theorem was established.

Theorem 4.14. *If X is a chainable continuum, then X has the hyperonto representation* $\mathbf{Q}_\sigma = \{Q_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ *such that each* Q_Δ *is a metric chainable continuum and each* $p_{\Delta\Gamma}$ *is a weakly confluent surjection.*

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