The fixed point property for arc component preserving mappings of non-metric tree-like continua

IVAN LONČAR*

Abstract. *The main purpose of this paper is to study the fixed point property of non-metric tree-like continua. Using the inverse systems method, it is proved that if X is a non-metric tree-like continuum and if $f : X \to X$ is a mapping which sends each arc component into itself, then f has the fixed point property.*

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1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by $\text{card}(A)$. We shall use the notion of an inverse system as in [2, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

Let A be a partially ordered directed set. We say that a subset $A_1 \subset A$ *majorates* [1, p. 9] another subset $A_2 \subset A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if its every two elements are comparable. The symbol $\sup B$, where $B \subset A$, denotes the lower upper bound of B (if such an element exists in A). Let $\tau \geq \aleph_0$ be a cardinal number. A subset B of A is said to be τ-*closed* in A if for each chain $C \subset B$, with $\text{card}(B) \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain C of A of elements of A with $\text{card}(C) \leq \tau$, there exists an element $\sup C$ in A.

Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system X into the system* $\mathbf{Y}$ [1, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$

*Faculty of Organization and Informatics Varaždin, University of Zagreb, Pavlinska 2, HR-42 000 Varaždin, e-mail: `ivan.loncar1@vz.tel.hr`

consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array} \tag{1}$$

diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y} \tag{2}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [1, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1. *[1, Corollary 1.3.2, p. 18]. If $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be *τ-continuous* [1, p. 19] if for each chain B in A with $\text{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be a *τ-system* [1, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

A *σ-system* is a τ-system, where $\tau = \aleph_0$. The following theorem is called the *Spectral Theorem* [1, p. 19].

Theorem 1. *[1, Theorem 1.3.4, p. 19]. If a τ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a,\ q_{ab},\ A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of the limit projections of systems in *Theorem 1* is essential [1, p. 21].

2. The fixed point property of non-metric continua

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every mapping $f : X \to X$ has a fixed point.

The key theorem is the following.

Theorem 2. *Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be a* σ*-system of compact spaces with limit* X *and onto projections* $p_a : X \to X_a$. *Let* $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ *be a morphism. Then the induced mapping* $f = \lim\{f_a\} : X \to X$ *has a fixed point if and only if each mapping* $f_a : X_a \to X_a$, $a \in A$, *has a fixed point.*

Proof. *The if part.* Let $F_a, a \in A$, be the set of fixed points of the mapping f_a.

Claim 1. *Every set* F_a *is closed.* This is a consequence of the following theorem [2, Theorem 1.5.4., p. 59]: *For any pair f,g of mappings of a space X into a Hausdorff space Y, the set*

$$\{x \in X : f(x) = g(x)\} \tag{3}$$

is closed in X.

It suffices to set $g(x) = x$ and $Y = X$.

Claim 2. *If* $a \leq b$, *then* $p_{ab}(F_b) \subset F_a$. Let x_b be any point of F_b. From the commutativity of diagram (1) it follows $p_{ab}(f_b(x)) = f_a(p_{ab}(x_b))$. We have $p_{ab}(x_b) = f_a(p_{ab}(x_b))$ since $f_b(x) = x_b$. This means that for the point $y = p_{ab}(x_b) \in X_a$ we have $y = f_a(y)$, i.e., $y \in F_a$. We infer that $p_{ab}(x_b) \in F_a$ and $p_{ab}(F_b) \subset F_a$.

Claim 3. $\mathbf{F} = \{F_a,\ p_{ab}|F_b,\ A\}$ *is an inverse system of compact spaces with a non-empty limit* F.

Claim 4. *The set* $F \subset X$ *is the set of fixed points of the mapping* f. Let $x \in F$ and let $x_a = p_a(x)$, $a \in A$. Now, $f_a(x_a) = x_a$ since $x_a \in F_a$. We infer that $f(x) = x$ since the morphism $\{f_a : a \in A\}$ induces f. The proof of the "if" part is complete.

The only if part. Suppose that the induced mapping f has a fixed point x. Let us prove that every mapping $f_a, a \in A$, has a fixed point. Now we have $f_a p_a(x) = p_a f(x)$. From $f(x) = (x)$ it follows $f_a p_a(x) = p_a(x)$. We infer that $p_a(x)$ is a fixed point for f_a. □

As an immediate consequence of this theorem and the Spectral theorem 1 we have the following result.

Theorem 3. *Let a non-metric continuum* X *be the inverse limit of an inverse* σ*-system* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *such that each* X_a *has the fixed point property and each bonding mapping* p_{ab} *is onto. Then* X *has the fixed point property.*

3. The fixed point property of the inverse limit space of tree-like continua

A continuum X with precisely two non-separating points is called a *generalized arc.*

A *simple n-od* is the union of n generalized arcs $A_1O, A_2O, ..., A_\alpha O$, each two of which have only the point O in common. The point O is called the *vertex* or the *top* of the n-od.

By a *branch point* of a compact space X we mean a point p of X which is the vertex of a simple triod lying in X. A point $x \in X$ is said to be the *end point* of

X if for each neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and $\mathrm{card}(Bd(V)) = 1$.

Let S be the set of all end points and all branch points of a continuum X. An arc pq in X is called a *free arc* in X if $pq \cap S = \{p, q\}$.

A continuum is a *graph* if it is the union of a finite number of metric free arcs. A *tree* is an acyclic graph.

A continuum X is *tree-like* (*arc-like*) if for each open cover $\mathcal{U}$ of X, there is a tree (arc) $X_{\mathcal{U}}$ and a $\mathcal{U}$-mapping $f_{\mathcal{U}} : X \to X_{\mathcal{U}}$ (the inverse image of each point is contained in a member of $\mathcal{U}$).

Every tree-like continuum is hereditarily unicoherent. Every arc-like continuum is tree-like.

Let Y^X be the set of all mappings of X to Y. If Y is a metric space with a metric d, then on the set Y^X one can define a metric $\widehat{d}$ by letting

$$\widehat{d}(f,g) = \sup_{x \in X} d\left(f(x), g(x)\right). \tag{4}$$

Proposition 1. *Let X be any tree-like continuum, let P be a polyhedron with a given metric d, $r > 0$ a real number and $f : X \to P$ a mapping. Then there exist a tree Q, a mapping $g : X \to Q$ and a mapping $p : Q \to P$ such that $g(X) = Q$ and $\widehat{d}(f, pg) \leq r$.*

Proof. Let K be a triangulation of P of mesh not greater than $r/2$. Let a_i be the vertices of K, and let St a_i be the open star of K around the vertex a_i. Hence, $\{\mathrm{St}\ a_i\}$ is an open covering for P, and so is $\mathcal{U} = \{f^{-1}(\mathrm{St}\ a_i)\}$ for X. There exist a tree Q and a mapping $g : X \to Q$ such that g is a $\mathcal{U}$-mapping and $g(X) = Q$. There exists a triangulation L of Q with vertices b_j such that the cover $\mathcal{V} = \{g^{-1}(\mathrm{St}\ b_j)\}$ refines the cover $\mathcal{U}$. Let x be a point of X and let s be a simplex of Q with vertices $b_{j_1}, ..., b_{j_k}$ containing $g(x)$. This means that $\left\{g^{-1}(\mathrm{St}\ b_{j_1}), ..., g^{-1}(\mathrm{St}\ b_{j_k})\right\}$ is a collection of some $g^{-1}(\mathrm{St}\ b_j)$ containing x. It follows that $g^{-1}(\mathrm{St}\ b_{j_1}) \cap ... \cap g^{-1}(\mathrm{St}\ b_{j_k}) \neq \emptyset$. We infer that $\mathrm{St}\ b_{j_1} \cap ... \cap \mathrm{St}\ b_{j_k} \neq \emptyset$. Let $p : Q \to P$ be a simplicial mapping sending each vertex b_j of Q into a vertex a_i having the property that $g^{-1}(\mathrm{St}\ b_i) \subset f^{-1}(\mathrm{St}\ a_i)$. It remains to prove that $d(f, pg) \leq r$. Now, for each $g^{-1}(\mathrm{St}\ b_{i_j})$ we have some $f^{-1}(\mathrm{St}\ a_{i_j})$ with $g^{-1}(\mathrm{St}\ b_{i_j}) \subset f^{-1}(\mathrm{St}\ a_{i_j})$. From $g^{-1}(\mathrm{St}\ b_{j_1}) \cap ... \cap g^{-1}(\mathrm{St}\ b_{j_k}) \neq \emptyset$ it follows that $f^{-1}(\mathrm{St}\ b_{j_1}) \cap ... \cap f^{-1}(\mathrm{St}\ b_{j_k}) \neq \emptyset$, i.e., that there exists a simplex σ of K with vertices $b_{j_1}, ..., b_{j_k}$ such that $f(x) \in \mathrm{St}\ \sigma$. Clearly, $pg(x) \in \mathrm{St}\ \sigma$. Finally, $\widehat{d}(f, pg) \leq r$. □

Proposition 2. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of tree-like continua and if p_{ab} are onto mappings, then the limit $X = \lim \mathbf{X}$ is a tree-like continuum.*

Proof. Let $\mathcal{U} = \{U_1, ..., U_n\}$ be an open covering of X. There exist an $a \in A$ and an open covering $\mathcal{U}_a = \{U_{1a}, ..., U_{ka}\}$ such that $\left\{p_a^{-1}(U_{1a}), ..., p_a^{-1}(U_{ka})\right\}$ refines the covering $\mathcal{U}$. There exist a tree T_a and a $\mathcal{U}_a$-mapping $f_{\mathcal{U}_a} : X_a \to T_a$ since X_a is tree-like. It is clear that $f_{\mathcal{U}_a} p_a : X \to T_a$ is a $\mathcal{U}$-mapping. Hence, X is tree-like. □

Proposition 3. *If X is a tree-like continuum, Q a tree and $f : X \to Q$ a mapping, then $f(X)$ is also a tree.*

Proof. This follows from the fact that a subcontinuum of a tree is a tree. □

Now we shall prove an expanding theorem of tree-like continua into inverse σ-systems of metric tree-like continua.

Theorem 4. *If X is a non-metric tree-like continuum, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric tree-like continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$.*

Proof. Let us observe that *Propositions 1-3* are conditions (A)-(C) in [4, p. 220]. Then from Mardešić's General Expansion Theorem [4, Theorem 2] it follows that there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and onto bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. It remains to prove that there exists such σ-system. The proof is broken down into several steps.

Step 1. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y) : x, y \in \Delta_n\}$, where $m(x,y)$ is a member of A such that $x, y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\text{card}(\Delta) = \text{card}(\Delta_0)$. Moreover, Δ is directed by $\leq$. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}. \tag{5}$$

Step 2. *If A is a directed set, then A_σ is σ-directed and σ-complete.* Let $\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ be a countable subset of A_σ. Then $\Delta_0 = \cup\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ is a countable subset of A_σ. Define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n \colon n \in \mathbb{N}\}$. It is clear that $\text{card}(\Delta) = \text{card}(\Delta_0)$. This means that Δ is countable. Moreover, $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence, A_σ is σ-directed. Let us prove that A_σ is σ-complete. Let $\Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ...$ be a countable chain in A_σ. Then $\Delta = \cup\{\Delta^i : i \in \mathbb{N}\}$ is a countable and directed subset of A, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that A_σ is σ-complete.

Step 3. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b, p_{bb'}, \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $P_{\Delta\Gamma}$: $X_\Gamma \to X_\Delta$ denote the map induced by the projections $p^\Gamma_\delta : X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system $\mathbf{X}^\Gamma$.

Step 4. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ is a σ-directed and σ-complete inverse system such that $\lim \mathbf{X}$ and $\lim \mathbf{X}_\sigma$ are homeomorphic.* Each thread $x = (x_a : a \in A)$ induces a thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $x_\Delta \in X_\Delta$. This means that we have a mapping $H : \lim \mathbf{X} \to \lim \mathbf{X}_\sigma$ such that $H(x) = (x_\Delta : \Delta \in A_\sigma)$. It is obvious that H is continuous and 1-1. The mapping H is onto since the collection of threads $\{x_\Delta : \Delta \in A_\sigma\}$ induces the thread in $\mathbf{X}$. We infer that H is a homeomorphism since $\lim \mathbf{X}$ is compact.

Step 5. *Every X_Δ is a metric tree-like continuum.* Apply *Proposition 2.*

Step 6. *Every projection $P_\Delta : \lim \mathbf{X}_\sigma \to X_\Delta$ is onto.* This follows from the assumption that the bonding mappings p_{ab} are surjective.

Finally, $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ is the desired σ-system. □

By a similar proof we obtain the following theorem.

Theorem 5. *If X is a non-metric arc-like continuum, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric arc-like continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$.*

From [4, Theorem 1], [4, Theorem 2] and [4, Corollary 1] we obtain the following well known result [5, Theorem 2.13, p. 24].

Theorem 6. *Each metrizable tree-like (arc-like) continuum is homeomorphic with the inverse limit of an inverse sequence of trees (arcs).*

Now we shall investigate the fixed point property of non-metric tree-like continua. Let us recall the following known result.

Theorem 7. *[3, Theorem 1.2]. Suppose f is a map of a tree-like metric continuum M that sends each arc component of M into itself. Then f has a fixed point.*

A map $f : X \to X$ is a *deformation* if there exists a map $H : X \times [0,1] \to X$ onto X such that $H(p,0) = p$ and $H(p,1) = f(p)$ for each point $p \in X$. We say that a map $f : X \to X$ is a *generalized deformation* if there exist a generalized arc L (with end points 0 and 1) and a map $H : X \times L \to X$ onto X such that $H(p,0) = p$ and $H(p,1) = f(p)$ for each point $p \in X$. Since (generalized) deformations send arc-components into themselves, we have [3, Corollary 1.3].

Corollary 1. *Every metric tree-like continuum has the fixed point property for deformations.*

For non-metric tree-like continua we shall prove the following theorem.

Theorem 8. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse σ-system of metric tree-like continua with onto bonding mappings and with limit X. If $f : X \to X$ is a mapping which sends each arc component into itself, then f has a fixed point.*

Proof. By virtue of *Theorem 1* there exist a cofinal subset $B(f)$ of A and mappings $f_b : X_b \to Y_b$, where $b \in B(f)$, such that the mapping f is induced by a collection $\{f_b : b \in B(f)\}$. From *Theorems 2* and *7* it follows that it suffices to prove that each f_b sends each arc component into itself. Let $x_b \in X_b$. We have to prove that x_b and $f_b(x_b)$ lie in some arc component of X_b, i.e., there is an arc in X_b with end points x_b and $f_b(x_b)$. There exists a point $x \in X$ such that $x_b = p_b(x)$. There exists a generalized arc L in X with end points x and $f(x)$ since f sends each arc component into itself. This means that $f_b(L)$ contains the points $p_b(x) = x_b$ and $p_b f(x) = f_b(p_b(x)) = f_b(x_b)$. We infer that there is an arc with end points x_b and $f_b(x_b)$ since $f_b(L)$ is arcwise connected [6, p. 201, Theorem 9]. The proof is complete. □

The non-metric analogue of *Theorem 7* is the following result.

Theorem 9. *Let X be a non-metric continuum tree-like. If $f : X \to X$ is a mapping which sends each arc component into itself, then f has a fixed point.*

Proof. Apply *Theorems 4, 7* and *8.* □

Corollary 2. *Every non-metric tree-like continuum has the fixed point property for generalized deformations.*

Corollary 3. *Let X be a non-metric tree-like arcwise connected continuum. If $f : X \to X$, then f has a fixed point.*

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