Independent sets and independent dominating sets in the strong product of paths and cycles

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Abstract. *In this paper we will consider independent sets and independent dominating sets in the strong product of two paths, two cycles and a path and a cycle.*

Key words: *path, cycle, strong product, independent dominating set, independent set*

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1. Introduction

For any graph G we denote the vertex-set and the edge-set of G by $V(G)$ and $E(G)$, respectively. A subset $S \subset V(G)$ is independent if every two vertices from S are not adjacent in G.

Independence number $\beta(G)$ of graph G is the maximum cardinality of an independent set of vertices.

A subset $D \subset V(G)$ is called a *dominating set*, if for every vertex y not in D, there exists at least one vertex $x \in D$, such that $d(x,y) \leq 1$. For convenience we also say that D *dominates* G. *Dominating number* $\gamma(G)$ of graph G is the maximum cardinality of a dominating set of vertices.

A set D of vertices in a graph G is called an *independent dominating set* of G if D is both an independent and a dominating set of G. This set is also called a *stable set* or a *kernel* of the graph.

Independent dominating sets were introduced into the theory of games by Neumann and Morgenstern in 1944 (see [18]). The *independent domination number* $i(G)$ is the cardinality of the smallest independent dominating set.

The *strong product* of two graphs is a graph with $V(G \cdot H) = V(G) \cdot V(H)$ and $((g_1, h_1), (g_2, h_2)) \in E(G \cdot H)$, if one of the following holds:

a) $(g_1, g_2) \in E(G)$ and $(h_1, h_2) \in E(H)$,

b) $g_1 = g_2$ and $(h_1, h_2) \in E(H)$,

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c) $(g_1, g_2) \in E(G)$ and $h_1 = h_2$.

Independent sets were introduced into the communication theory on noisy channels (see [19]). The *noisy channel* consists of transmission alphabet T and receiving alphabet R, and the information about which letters of T can be received as which letters of R. The noisy channel can be represented by a bipartite digraph, which in one partition has vertices from T, and in the other partition vertices from R. Vertex $x \in T$ is adjacent to vertex $y \in R$, if letter x can be received as letter y.

A corresponding *confusion graph* C has vertices which are elements of T, and two vertices are adjacent if and only if they can be received as the same letter (see [19]).

Assume a transmitter can emit four signals a, b, c and d, and a receiver receives signals α,β,γ and δ. Because of noise, a can be received as either α or β, b can be received as either β or γ, c can be received as either γ or δ, and d can be received as either δ or α (see *Figure 1*).

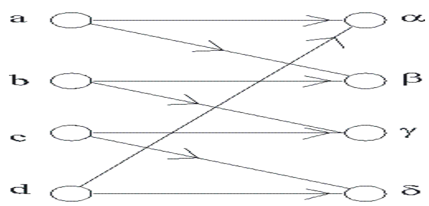

Figure 1.

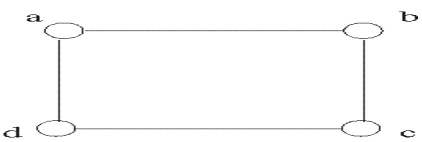

Figure 2. *A confusion graph*

Given a noisy channel, we would like to make errors impossible, so that no signal in $T' \subset T$ is confusable with another signal in T'. This corresponds to choosing an independent set of vertices in the confusion graph C. Given a fixed noisy channel, we might ask if it is possible to find a larger unambiguous code alphabet. For example, suppose we consider all ordered pairs of elements from T. We can draw a new confusion graph whose vertices are strings of length two from T. This graph has the following property: strings xy and zv can be confused if and only if one of the following holds:

a) x and z can be confused and y and v can be confused

b) $x = z$ and y and v can be confused

c) x and z can be confused and $y = v$.

In terms of the original confusion graph C, the new confusion graph is the strong product $C \cdot C$.

Also, if we have two noisy channels with confusion graphs C_1 and C_2 and consider strings of length two with the first element coming from the first alphabet and the second from the second alphabet, then the new confusion graph is the strong product $C_1 \cdot C_2$.

In [13],[14],[15] and [16] we investigated dominating sets and k-dominating sets on the Cartesian and the cardinal product of two graphs. In [9] Jha and Klavžar considered independance in the direct product (another name for the cardinal product) of graphs.

In the rest of the paper we will consider an independent set and an independent dominating set on the strong product of two paths, two cycles and a path and a cycle.

By P_n we will denote a path with n vertices, and by C_n a cycle with n vertices. For a fixed m, $1 \le m \le n$, the set $(P_k)_m := P_k \cdot m$ is called a *column* of $P_k \cdot P_n$; the set $(P_n)^r := r \cdot P_n$ is called a *row* of $P_k \cdot P_n$.

2. Bounds for $i(G \cdot H)$ and $\beta(G \cdot H)$

Lemma 1. *For any two graphs G and H the following properties hold*

a) $i(G \cdot H) \le i(G) \cdot i(H)$

b) $\beta(G \cdot H) \ge \beta(G) \cdot \beta(H)$.

Proof. a) Let D_1 and D_2 be minimal independent dominating sets for G and H, and $D = D_1 \cdot D_2 \subset V(G) \cdot V(H)$. Let (u, v) be an arbitrary vertex of $G \cdot H$. Then there are vertices $u' \in D_1$ and $v' \in D_2$ such that $(u, u') \in E(G)$ and $(v, v') \in E(H)$. Therefore $(u', v') \in D$, and (u, v) and (u', v') are adjacent. Then D is a dominating set on $G \cdot H$. From definitions of D and the strong product it follows that D is independent.

b) Let D_1 and D_2 be maximal independent sets for G and H. An argument as above in a) implies that $D_1 \cdot D_2$ is an independent set on $G \cdot H$. □

3. Independent sets and independent dominating sets on $P_m \cdot P_n$

Observation 1. *For each path P_n, $n \ge 1$, independence number $\beta(P_n)$ is equal to $\lceil \frac{n}{2} \rceil$.*
Independent dominating number $i(P_n)$ is equal to $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.
Also, $P_n \cdot P_1 = P_n$ holds.

Theorem 1. *For every two paths P_m and P_n*

$$\beta(P_m \cdot P_n) = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil = \beta(P_m) \cdot \beta(P_n).$$

Proof. Let $V(P_m) = \{1, 2, ..., m\}$, $V(P_n) = \{1, 2, ..., n\}$ and $S = \{(1+2l, 1+2k) | l = 0, 1, ..., \lceil \frac{m}{2} \rceil - 1; k = 0, 1, ..., \lceil \frac{n}{2} \rceil - 1\}$. It is easy to see that $|S| = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ and that S is an independent set on $V(P_m \cdot P_n)$.

Proof of the maximality: Let I be an independent set on $G = P_m \cdot P_n$. From the first column in G we can take at most each odd vertex in I. If each odd vertex from $(P_m)_1$ is in I (such vertices are $\lceil \frac{m}{2} \rceil$), all vertices from $(P_m)_2$ are adjacent to at least one vertex from $(P_m)_1$. Then no vertex from $(P_m)_2$ is in I. If $|I \cap (P_m)_2| = \emptyset$, at most $\lceil \frac{m}{2} \rceil$ vertices from $(P_m)_3$ are in I. If $|I \cap (P_m)_3| = \lceil \frac{m}{2} \rceil$, then $|I \cap (P_m)_4| = \emptyset$ holds, and so on.

It follows that then we can take vertices from at most $\lceil \frac{n}{2} \rceil$ columns, and from each column at most $\lceil \frac{m}{2} \rceil$ vertices. Then at most $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ vertices are independent. □

Lemma 2.
$$i(P_2 \cdot P_n) = \lceil \frac{n}{3} \rceil = i(P_n).$$

Proof. For $D = \{(2, 2+3k) | k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 1\}$ D is an independent dominating set for $n \equiv 0, 2 (mod\ 3)$.

If $n \equiv 1 (mod\ 3)$, then $D = \{(2, 2+3k) | k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 2\} \cup (2, n)$. $|D| = \lceil \frac{n}{3} \rceil$. Minimality follows from the fact that

$$\lceil \frac{n}{3} \rceil \geq i(P_2 \cdot P_n) \geq i(P_n) = \lceil \frac{n}{3} \rceil.$$

□

Theorem 2. *Let $m, n \geq 2$. Then,*

$$i(P_m \cdot P_n) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil = i(P_m) \cdot i(P_n).$$

Proof. Let $D = \{(2+3l, 2+3k) | l = 0, 1, ..., \lceil \frac{m}{3} \rceil - 1; k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 1\}$. D is an independent dominating set for $m \equiv 0, 2 (mod\ 3)$ and $n \equiv 0, 2 (mod\ 3)$.

If $m \equiv 0, 2 (mod\ 3)$ and $n \equiv 1 (mod\ 3)$, then

$$\begin{aligned} D = \{&(2+3l, 2+3k) | l = 0, 1, ..., \lceil \frac{m}{3} \rceil - 1; k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 2\} \\ &\cup \{(2+3l, n) | l = 0, 1, ..., \lceil \frac{m}{3} \rceil - 1\}. \end{aligned}$$

If $m \equiv 1 (mod\ 3)$ and $n \equiv 0, 2 (mod\ 3)$, then

$$\begin{aligned} D = \{&(2+3l, 2+3k) | l = 0, 1, ..., \lceil \frac{m}{3} \rceil - 2; k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 1\} \\ &\cup \{(m, 2+3k) | k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 1\}. \end{aligned}$$

If $m \equiv 1 (mod\ 3)$ and $n \equiv 1 (mod\ 3)$, then

$$\begin{aligned} D = \{&(2+3l, 2+3k) | l = 0, 1, ..., \lceil \frac{m}{3} \rceil - 2; k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 2\} \\ &\cup \{(2+3l, n) | l = 0, 1, ..., \lceil \frac{m}{3} \rceil - 2; (m, 2+3k) | k = 0, 1, ..., \lceil \frac{n}{3} \rceil - 1\}. \end{aligned}$$

D is an independent dominating set on $P_m \cdot P_n$ and it holds $|D| = \lceil\frac{m}{3}\rceil\lceil\frac{n}{3}\rceil$.

Proof of the minimality:

a) If $m, n \equiv 0(mod\ 3)$, the proof is obvious, because each vertex is dominated by exactly one vertex and each dominating vertex dominates nine vertices, which is maximal.

b) If $m \equiv 0(mod\ 3)$ but $n \equiv k(mod\ 3)$, $k \neq 0$, then on $P_m \cdot P_{n-k}$ each vertex is dominated by only one vertex, and there are $\lceil\frac{m}{3}\rceil\lfloor\frac{n}{3}\rfloor$ vertices.

From the construction of set D it follows that$|D \cap (P_m)_{n-k}| = 0$, and then no vertex on the rest of the graph can be dominated by one of the previous dominating vertices.

To dominate the rest of the graph, which is $P_m \cdot P_k$ $(1 \leq k \leq 2)$, we need $\lceil\frac{m}{3}\rceil$ vertices (*Observation 1* and *Lemma 1*). Then on the whole graph we have at least

$$\lceil\frac{m}{3}\rceil(\lfloor\frac{n}{3}\rfloor + 1) = \lceil\frac{m}{3}\rceil\lceil\frac{n}{3}\rceil$$

vertices.

c) If $n \equiv 0(mod\ 3)$ but $m \equiv k(mod\ 3)$, $k \neq 0$, the proof is the same as in b).

d) If $m \equiv l(mod\ 3)$, $l \neq 0$ and $n \equiv k(mod\ 3)$, $k \neq 0$, then on $P_{m-l} \cdot P_{n-k}$ we have a perfect dominating set which is independent, and it has $\lfloor\frac{m}{3}\rfloor\lfloor\frac{n}{3}\rfloor$ vertices. To dominate the remaining vertices (on the blocks $P_l \cdot P_n$ and $P_{m-l} \cdot P_k$) we need at least $\lceil\frac{n}{3}\rceil + \lceil\frac{m-l}{3}\rceil = \lceil\frac{n}{3}\rceil + \lfloor\frac{m}{3}\rfloor$ vertices and then the result follows.

$$\lfloor\frac{m}{3}\rfloor\lfloor\frac{n}{3}\rfloor + \lceil\frac{n}{3}\rceil + \lfloor\frac{m}{3}\rfloor = \lceil\frac{m}{3}\rceil\lceil\frac{n}{3}\rceil.$$

□

4. Independent sets and independent dominating sets on $C_m \cdot C_n$ and $P_m \cdot C_n$

Observation 2. *For each cycle C_n, $n \geq 3$, independence number*

$$\beta(C_n) = \begin{cases} \lceil\frac{n}{2}\rceil - 1, & n\ odd \\ \lceil\frac{n}{2}\rceil, & n\ even \end{cases}$$

and independent dominating number $i(C_n) = \lceil\frac{n}{3}\rceil = \gamma(C_n)$.

Theorem 3. *For every two cycles C_m and C_n*

$$\beta(C_m \cdot C_n) = \beta(C_m) \cdot \beta(C_n).$$

Proof. If m and n are even, then from *Lemma 1* and *Observation 2* it follows

$$\lceil\frac{m}{2}\rceil \cdot \lceil\frac{n}{2}\rceil = \beta(C_m) \cdot \beta(C_n) \leq \beta(C_m \cdot C_n) \leq \beta(P_m \cdot P_n) = \lceil\frac{m}{2}\rceil \cdot \lceil\frac{n}{2}\rceil.$$

If m is odd and n is even, then it follows that

$$\beta(C_m \cdot C_n) \geq \beta(C_m) \cdot \beta(C_n) = (\lceil\frac{m}{2}\rceil - 1) \cdot \lceil\frac{n}{2}\rceil.$$

Proof that $\beta(C_m \cdot C_n) \leq (\lceil \frac{m}{2} \rceil - 1) \cdot \lceil \frac{n}{2} \rceil$.

Because the structure on $C_m \cdot C_n$ is very similar to the one on $P_m \cdot P_n$ we will also say that for a fixed m and r, $1 \leq m \leq n$, $1 \leq r \leq m$, the set $(C_k)_m := C_k \cdot m$ is a column of $C_k \cdot C_n$ and the set $(C_n)^r := r \cdot C_n$ is called a row. Also we consider the same set S as on $P_m \cdot P_n$.

If m is odd and n is even in the last column on $C_m \cdot C_n$ there are no vertex from S. But in the last row there are $(m, 1), (m, 3), ..., (m, n-1)$ from S which are adjacent to the vertices from the first row $(1, 1), (1, 3), ..., (1, n-1)$ which are also in S. Then vertices from at least one of these two rows cannot be in S. For n odd and m even is the same.

For m, n both odd, vertices from the first and the last row, and from the first and the last column are in S. It follows that at least vertices from one column and one row we must omit.

The maximality follows from the maximality of the set S on $P_m \cdot P_n$. □

Theorem 4.

$$i(C_m \cdot C_n) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil = i(C_m) \cdot i(C_n) = i(P_m \cdot P_n).$$

Proof. It is easy to see that the same set D, which is a minimal independent dominating set on $P_m \cdot P_n$ (*Theorem 2*) is also an indepedent dominating set on $C_m \cdot C_n$. Also, from *Theorem 1* it follows that $i(C_m \cdot C_n) \leq i(C_m) \cdot i(C_n)$. Proof of the minimality:

a) If $m, n \equiv 0 (mod\ 3)$, it is obvious, because each vertex is dominated by exactly one vertex, and each dominating vertex dominates maximal number vertices.

b) If at least one of m and n is not $\equiv 0 (mod\ 3)$ we have a similar situation as in *Theorem 2*, only we have edges between $(C_m)_1$ and $(C_m)_n$, and between $(C_n)^1$ and $(C_n)^m$. From construction of D it follows that in $(C_m)_1$ and $(C_n)^1$ there are no dominating vertices. Then these new edges do not make any influence on minimal independent dominating set. □

Corollary 1.

$$\beta(P_m \cdot C_n) = \beta(P_m) \cdot \beta(C_n)$$

Proof. When both m, n are even, is obvious, because $\beta(C_m \cdot C_n) \leq \beta(P_m \cdot C_n) \leq \beta(P_m \cdot P_n)$, and for this case it holds $\beta(C_m \cdot C_n) = \beta(P_m \cdot P_n)$.

When m is odd and n is even we can take the same independent set as the set S in *Theorem 1* (on $P_m \cdot P_n$). $|S| = \lceil \frac{m}{2} \rceil \cdot \lceil \frac{n}{2} \rceil$. From this and the fact that for such n

$$\beta(P_m \cdot C_n) \leq \beta(P_m) \cdot \beta(P_n) = \beta(P_m) \cdot \beta(C_n)$$

there follows the proof.

When n is odd (m arbitrary), vertices from $(P_m)_1$ and $(P_m)_n$ are adjacent. So, at most vertices from one of these columns can be in indepedent set. It follows that at most $\lceil \frac{n}{2} \rceil - 1$ columns can have vertices from S. In each column there can be at most $\lceil \frac{m}{2} \rceil$ independent vertices. Then from previous two facts it holds that for n odd $\beta(P_m \cdot C_n) \leq \lceil \frac{m}{2} \rceil \cdot (\lceil \frac{n}{2} \rceil - 1) = \beta(P_m) \cdot \beta(C_n)$. □

Corollary 2.

$$i(P_m \cdot C_n) = \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil.$$

Proof. It follows from *Theorem 4* and the fact that $i(C_m \cdot C_n) \leq i(P_m \cdot C_n) \leq i(P_m \cdot P_n)$. □

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