A generalized q-numerical range

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Abstract. *For a given $q \in \mathbf{C}$ with $|q| \leq 1$, we study the C-numerical range of a Hilbert space operator where C is an operator of the form*

$$\begin{pmatrix} qI_n & \sqrt{1-|q|^2}I_n \\ 0_n & 0_n \end{pmatrix} \oplus 0.$$

Some known results on the q-numerical range are extended to this set.

Key words: *C^*-algebra, Hilbert C^*-module, adjointable operator, C-numerical range of an operator, q-numerical range of an operator*

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1. Introduction

Throughout this paper H will denote a complex Hilbert space with an inner product $(\cdot\,|\,\cdot)$. The algebra of all bounded linear operators on H and the ideal of all compact operators on H will be denoted by $B(H)$ and $K(H)$, respectively.

The *q-numerical range* of $A \in B(H)$ is, by definition, the set

$$W_q(A) = \{(Ax|y) : (x|x) = (y|y) = 1, (x|y) = q\}$$

where $q \in \mathbf{C}$, $|q| \leq 1$. When $q = 1$, this set reduces to the classical numerical range. The q-numerical range is a useful tool for studying matrices and operators and it has been investigated extensively (see [10], [12] or [18]).

For a fixed trace class operator $C \in B(H)$, the *C-numerical range* of $A \in B(H)$, denoted by $W_C(A)$, is defined as the set of all complex numbers of the form $\mathrm{tr}(CU^*AU)$, where U ranges over all unitary operators in $B(H)$. In the case of the finite dimensional space H, the set $W_C(A)$ was introduced by M. Goldberg and E. G. Straus [5] and it is a further generalization of the classical numerical range. It is also a generalization of the q-numerical range of a Hilbert space operator which can be obtained when C is chosen to be rank one operator of the form

$$\begin{pmatrix} q & \sqrt{1-|q|^2} \\ 0 & 0 \end{pmatrix} \oplus 0.$$

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The C-numerical range is well-studied in the case of some special classes of operators such as normal, block-shift or rank one operators, but less is known for general C. For general properties of the C-numerical range the reader is referred to [3], [6], [8], [9], [11] or [20].

The aim of this paper is to study the C-numerical range of a Hilbert space operator for $C = C_q \oplus 0$ where

$$C_q = \begin{pmatrix} qI_n & \sqrt{1-|q|^2} I_n \\ 0_n & 0_n \end{pmatrix}$$

and I_n denotes the identity operator on the n-dimensional subspace of H. Since $W_{C_q \oplus 0}(A)$ is a natural generalization of the q-numerical range, it is not surprising that some good results on the q-numerical range can be extended to this set. This is done in *Section 3*. *Section 4* is devoted to the set $W_{C_0 \oplus 0}(A)$.

It is interesting to note that the concept of the q-numerical range can also be considered in a more general context than the Hilbert space. In *Section 2* we introduce it in the context of Hilbert C^*-modules over a C^*-subalgebra $\mathcal{A}$ of $B(H)$ containing $K(H)$. It turns out that such a set can be interpreted as the $C_q \oplus 0$-numerical range of some Hilbert space operator. Thus, it was the motivation for introducing the C-numerical range for such particular C.

2. The set $W_q^n(A)$

In this section we introduce the concept of the q-numerical range in the context of Hilbert C^*-modules.

Recall that a (left) Hilbert C^*-module X over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module X equipped with an $\mathcal{A}$-valued inner product $\langle \cdot\,, \cdot \rangle$ on $X \times X$ which is linear over $\mathcal{A}$ in the first and conjugate-linear in the second variable, such that X is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. X is said to be a full Hilbert $\mathcal{A}$-module if the closure of the linear span of the set $\{\langle x, y\rangle : x, y \in X\}$ coincides with $\mathcal{A}$. By $B(X)$ we denote the C^*-algebra of all adjointable operators on X. (The basic theory of Hilbert C^*-modules can be found in [7] and [19].)

Although our results are proved for a Hilbert C^*-module over an arbitrary C^*-subalgebra $\mathcal{A}$ of $B(H)$ containing $K(H)$, for technical simplicity we first consider the case of a Hilbert C^*-module over the C^*-algebra $K(H)$.

Hence, in the sequel X will denote a full left Hilbert C^*-module over the C^*-algebra $K(H)$. One significant property of such modules is presence of an orthonormal basis ([1, Theorem 2]). (An orthonormal basis for X is by definition an orthogonal system (x_λ) that generates a dense submodule of X such that x_λ are basic vectors in the sense that $\langle x_\lambda, x_\lambda\rangle$ are orthogonal projections in $K(H)$ of rank 1). The orthogonal dimension of X (i.e., the cardinal number of any of its orthonormal bases) will be designated by $\dim_{K(H)} X$. Furthermore, X contains a Hilbert space X_e with respect to the inner product $(\cdot\,, \cdot) = \mathrm{tr}(\langle \cdot\,, \cdot \rangle)$ where 'tr' means the trace. More precisely, for a fixed orthogonal projection e in $K(H)$ of rank 1, X_e is given as the set of all ex, $x \in X$. Also, for all $x, y \in X_e$ we obtain that $\langle x, y\rangle = (x, y)e$ ([1, Remark 4(c)]). It is known that X and the Hilbert space X_e have the same dimension ([1, Remark 4(e)]). Moreover, X_e is an invariant subspace for each A

in $B(X)$ and the map $A \mapsto A|X_e$ establishes an isomorphism between C^*-algebras $B(X)$ and $B(X_e)$ where $B(X_e)$ denotes the algebra of all bounded operators on X_e ([1, Remark 4(b), Theorem 5]).

Before stating the results we establish some more notations as follows. First, a positive integer n is fixed and it is supposed that H has dimension greater than or equal to n. Then let us fix an n-dimensional orthogonal projection p in $K(H)$. H_n will designate the n-dimensional range of p. We now choose an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ for H_n which is to be held fixed for the rest of this section. For $\xi, \eta \in H$, $e_{\xi,\eta}$ in $B(H)$ is defined by $e_{\xi,\eta}(\nu) = (\nu|\eta)\xi$. From now on we denote $e_i = e_{\xi_i,\xi_i}$ for $i = 1, \dots, n$. Evidently, $p = e_1 + \cdots + e_n$. In the rest of the section let us also fix a unit vector ξ in H and denote by e the orthogonal projection $e_{\xi,\xi}$ to the one-dimensional subspace spanned by ξ.

Finally, we denote by S^- and $[S]$ the topological closure and the linear span of a set S, respectively.

Definition 1. *For any complex number q with $|q| \leq 1$ and $A \in B(X)$ we define the set*

$$_pW_q^n(A) = \{ \operatorname{tr}\langle Ax, y\rangle : x, y \in X, \langle x, x\rangle = \langle y, y\rangle = p, \langle x, y\rangle = qp\}.$$

Remark 1. *Suppose that vectors $x, y \in X$ satisfy $\langle x, x\rangle = \langle y, y\rangle = p$, $\langle x, y\rangle = qp$, where $q \in \mathbf{C}$, $|q| < 1$. Let us put $x_i = e_{\xi,\xi_i}x$, $y_i = e_{\xi,\xi_i}y$ for $i = 1, \dots, n$. We claim that $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ is a linearly independent set of a Hilbert space X_e. Namely, for $i, j = 1, \dots, n$ we have*

$$\langle x_i, x_j\rangle = e_{\xi,\xi_i}\langle x, x\rangle e_{\xi_j,\xi} = e_{\xi,\xi_i} p e_{\xi_j,\xi} = \delta_{i,j} e$$

and analogously $\langle y_i, y_j\rangle = \delta_{i,j}e$. Also, the condition $\langle x, y\rangle = qp$ implies that

$$\langle x_i, y_j\rangle = e_{\xi,\xi_i}\langle x, y\rangle e_{\xi_j,\xi} = e_{\xi,\xi_i} qp e_{\xi_j,\xi} = q\delta_{i,j} e.$$

From this we deduce that $x_i, y_i \in X_e$ and it holds that

$$(x_i, x_j) = (y_i, y_j) = \delta_{i,j}, \ \ (x_i, y_j) = q\delta_{i,j} \tag{1}$$

for $i, j = 1, \dots, n$. Let us now suppose that $\sum_{i=1}^n \alpha_i x_i + \sum_{i=1}^n \beta_i y_i = 0$ for some $\alpha_i, \beta_i \in \mathbf{C}$. Multiplying this equality on its right-hand side by x_i and then by y_i we get (by using (1)) $\alpha_i + \beta_i\bar{q} = 0$ and $\alpha_i q + \beta_i = 0$ from which it follows that $\alpha_i(1 - |q|^2) = 0$, $i = 1, \dots, n$. Hence, $\alpha_i = 0$ and thus $\beta_i = 0$, $i = 1, \dots, n$. Therefore, $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ is a linearly independent set in X_e. Hence, to avoid the trivial case $_pW_q^n(A) = \emptyset$, the dimension of the Hilbert space X_e must be greater than or equal to $2n$.

So, in what follows we shall assume that $\dim_{K(H)} X = k \geq 2n$.

Remark 2. *Observe that when $n = 1$ the condition $\langle x, x\rangle = \langle y, y\rangle = e_1$, $\langle x, y\rangle = qe_1$ is equivalent to the fact that x and y are unit vectors of Hilbert space X_{e_1} such that $(x, y) = q$. Moreover, $\operatorname{tr}\langle Ax, y\rangle = (Ax, y)$, so the set $_pW_q^1(A)$ is the q-numerical range $W_q(A|X_{e_1})$ of an operator $A|X_{e_1} \in B(X_{e_1})$.*

Remark 3. *The condition $\langle x, x\rangle = \langle y, y\rangle = p$ obviously implies $\langle x-px, x-px\rangle = \langle y - py, y - py\rangle = 0$, i.e., $x = px$ and $y = py$, so we have $\langle Ax, y\rangle = p\langle Ax, y\rangle p$.*

Furthermore, for every $\eta \perp H_n$ we get $\langle Ax, y\rangle\eta = p\langle Ax, y\rangle p\eta = 0$. Therefore, $\langle Ax, y\rangle$ can be regarded as an operator acting on the n-dimensional space H_n.

Next we show that ${}_pW_q^n(A)$ is non-empty.

Lemma 1. *For any $q \in \mathbf{C}$ with $|q| \le 1$ there exist $x, y \in X$ such that*

$$\langle x, x\rangle = \langle y, y\rangle = p, \qquad \langle x, y\rangle = qp.$$

Proof. Let $\{u_1, \dots, u_{2n}\}$ be an orthonormal set of the Hilbert space X_e. We define $v_i = \overline{q}u_i + \sqrt{1-|q|^2}u_{n+i}$ for $i = 1, \dots, n$. Then we get

$$\begin{aligned}(v_i, v_j) &= \left(\overline{q}u_i + \sqrt{1-|q|^2}u_{n+i}, \overline{q}u_j + \sqrt{1-|q|^2}u_{n+j}\right)\\ &= \overline{q}q(u_i, u_j) + \overline{q}\sqrt{1-|q|^2}(u_i, u_{n+j}) + \sqrt{1-|q|^2}q(u_{n+i}, u_j)\\ &\quad +(1-|q|^2)(u_{n+i}, u_{n+j})\\ &= |q|^2\delta_{i,j} + (1-|q|^2)\delta_{i,j} = \delta_{i,j}\end{aligned}$$

for $i, j = 1, \dots, n$. Also, we have

$$(u_i, v_j) = \left(u_i, \overline{q}u_j + \sqrt{1-|q|^2}u_{n+j}\right) = q(u_i, u_j) + \sqrt{1-|q|^2}(u_i, u_{n+j}) = q\delta_{i,j}$$

for $i, j = 1, \dots, n$. Let us put $x_i = e_{\xi_i,\xi}u_i$, $y_i = e_{\xi_i,\xi}v_i$ for $i = 1, \dots, n$. Thus we obtain

$$\langle x_i, x_j\rangle = e_{\xi_i,\xi}\langle u_i, u_j\rangle e_{\xi,\xi_j} = e_{\xi_i,\xi}(u_i, u_j)ee_{\xi,\xi_j} = \delta_{i,j}e_i$$

and analogously $\langle y_i, y_j\rangle = \delta_{i,j}e_i$ for $i, j = 1, \dots, n$. Moreover,

$$\langle x_i, y_j\rangle = e_{\xi_i,\xi}\langle u_i, v_j\rangle e_{\xi,\xi_j} = e_{\xi_i,\xi}(u_i, v_j)ee_{\xi,\xi_j} = q\delta_{i,j}e_i$$

for $i, j = 1, \dots, n$. Then $x = x_1 + \dots + x_n$ and $y = y_1 + \dots + y_n$ are desired vectors. □

Thus we have the following

Corollary 1. *The set ${}_pW_q^n(A)$ is non-empty for all $A \in B(X)$.*

Remark 4. *The definition of the set ${}_pW_q^n(A)$ does not depend on the choice of the rank n projection $p \in K(H)$. Indeed, if $p' \in K(H)$ is an arbitrary n-dimensional projection and if $\{\xi_1, \dots, \xi_n\}$ and $\{\eta_1, \dots, \eta_n\}$ are orthonormal bases for the ranges of p and p' respectively, then similarly as in the proof of Proposition 2.5 of [15], it can be shown that the map $\Phi :_p W_q^n(A) \to_{p'} W_q^n(A)$ defined by*

$$\Phi(\mathrm{tr}\langle Ax, y\rangle) = \mathrm{tr}\left\langle A\left(\sum_{i=1}^n e_{\eta_i,\xi_i}x\right), \sum_{i=1}^n e_{\eta_i,\xi_i}y\right\rangle$$

is a bijection.

So, in the sequel we shall write $W_q^n(A)$ instead of ${}_pW_q^n(A)$.

In the following lemma we collect some basic properties of $W_q^n(A)$ which are obvious consequences of *Definition 1.*

Lemma 2. *Let $A, B \in B(X)$. Then we have the following properties*:

(a) $W_q^n(U^*AU) = W_q^n(A)$ *whenever $U \in B(X)$ is unitary.*

(b) $W_q^n(\alpha A + \beta I) = \alpha W_q^n(A) + \beta nq$ *for all $\alpha, \beta \in \mathbf{C}$.*

(c) $W_q^n(A + B) \subseteq W_q^n(A) + W_q^n(B)$.

(d) $W_q^n(A^*) = \overline{W_q^n(A)} := \left\{\overline{\operatorname{tr}\langle Ax, y\rangle} : x, y \in X, \langle x, x\rangle = \langle y, y\rangle = p, \langle x, y\rangle = qp\right\}$ *for $0 \le q \le 1$.*

(e) $W_{\mu q}^n(A) = \mu W_q^n(A)$ *for $\mu \in \mathbf{C}$, $|\mu| = 1$.*

In the sequel we denote by $\{z_i\}$ an orthonormal basis of the Hilbert space X_e. Let Y_m, $1 \le m \le 2n$, be an m-dimensional subspace of X_e spanned by vectors $z_1, \dots, z_m$.

In our next theorem we give an alternative description of the set $W_q^n(A)$.

Theorem 1. *Let A be an operator in $B(X)$. Then*

$$W_q^n(A) = \{\operatorname{tr}((C_q \oplus 0_{k-2n})U^*A|X_eU) : U : X_e \to X_e \text{ is unitary}\},$$

where $C_q : Y_{2n} \to Y_{2n}$ is a linear operator given by its action on the basis $\{z_1, \dots, z_{2n}\}$:

$$\begin{aligned} C_q z_i &= q z_i, & i &= 1, \dots, n, \\ C_q z_{n+i} &= \sqrt{1-|q|^2} z_i, & i &= 1, \dots, n. \end{aligned}$$

Proof. Given a unitary operator $U : X_e \to X_e$, we define

$$\begin{aligned} x_i &= e_{\xi_i,\xi} U z_i \\ y_i &= \overline{q} e_{\xi_i,\xi} U z_i + \sqrt{1-|q|^2} e_{\xi_i,\xi} U z_{n+i} \end{aligned}$$

for $i = 1, \dots, n$. Since $\{Uz_1, \dots, Uz_{2n}\}$ is an orthonormal set in X_e, arguing as in the proof of *Lemma 1* we obtain that $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = \delta_{i,j} e_i$, $\langle x_i, y_j\rangle = q\delta_{i,j} e_i$ for all $i, j = 1, \dots, n$. Let us put $x = x_1 + \dots + x_n$ and $y = y_1 + \dots + y_n$. Then we have $\langle x, x\rangle = \langle y, y\rangle = p$, $\langle x, y\rangle = qp$. Also, it holds that

$$e_{\xi,\xi_i} x = e_{\xi,\xi_i}(x_1 + \dots + x_n) = e_{\xi,\xi_i} x_i = e_{\xi,\xi_i} e_{\xi_i,\xi} U z_i = eUz_i = Uz_i, \tag{2}$$

$$\begin{aligned} e_{\xi,\xi_i} y &= e_{\xi,\xi_i}(y_1 + \dots + y_n) = e_{\xi,\xi_i} y_i \\ &= e_{\xi,\xi_i}\left(\overline{q} e_{\xi_i,\xi} U z_i + \sqrt{1-|q|^2} e_{\xi_i,\xi} U z_{n+i}\right) \\ &= \overline{q} eUz_i + \sqrt{1-|q|^2} eUz_{n+i} = \overline{q} Uz_i + \sqrt{1-|q|^2} Uz_{n+i} \end{aligned} \tag{3}$$

for $i = 1, \dots, n$. In particular, $e_{\xi,\xi_i} x$, $e_{\xi,\xi_i} y \in X_e$, so we get

$$\langle Ae_{\xi,\xi_i} x, e_{\xi,\xi_i} y\rangle = (Ae_{\xi,\xi_i} x, e_{\xi,\xi_i} y)e.$$

On the other hand, we have

$$\begin{aligned}\langle Ae_{\xi,\xi_i}x, e_{\xi,\xi_i}y\rangle &= e_{\xi,\xi_i}\langle Ax,y\rangle e_{\xi_i,\xi} = e_{\xi,\xi_i}e_{\xi_i,\xi_i}\langle Ax,y\rangle e_{\xi_i,\xi_i}e_{\xi_i,\xi}\\ &= e_{\xi,\xi_i}\langle Ae_ix, e_iy\rangle e_{\xi_i,\xi} = e_{\xi,\xi_i}(Ae_ix,e_iy)e_ie_{\xi_i,\xi}\\ &= (Ae_ix,e_iy)e.\end{aligned}$$

Therefore,

$$(Ae_{\xi,\xi_i}x, e_{\xi,\xi_i}y) = (Ae_ix, e_iy) \tag{4}$$

for $i = 1,\dots,n$. Notice that

$$\begin{aligned}C_q^*z_i &= \overline{q}z_i + \sqrt{1-|q|^2}z_{n+i}, \quad i = 1,\dots,n,\\ C_q^*z_i &= 0, \qquad\qquad\qquad\qquad i = n+1,\dots,2n,\end{aligned}$$

so by (2), (3) and (4) we obtain that

$$\begin{aligned}\operatorname{tr}((C_q\oplus 0_{k-2n})U^*A|X_eU) &= \sum_{i=1}^n (U^*A|X_eUz_i, C_q^*z_i) = \sum_{i=1}^n (A|X_eUz_i, UC_q^*z_i)\\ &= \sum_{i=1}^n \left(A|X_eUz_i, U\left(\overline{q}z_i + \sqrt{1-|q|^2}z_{n+i}\right)\right)\\ &= q\sum_{i=1}^n (A|X_eUz_i, Uz_i) + \sum_{i=1}^n \left(A|X_eUz_i, \sqrt{1-|q|^2}Uz_{n+i}\right)\\ &= q\sum_{i=1}^n (Ae_{\xi,\xi_i}x, e_{\xi,\xi_i}x) + \sum_{i=1}^n (Ae_{\xi,\xi_i}x, e_{\xi,\xi_i}y - \overline{q}e_{\xi,\xi_i}x)\\ &= \sum_{i=1}^n (Ae_{\xi,\xi_i}x, e_{\xi,\xi_i}y) = \sum_{i=1}^n (Ae_ix, e_iy) = \operatorname{tr}\left(\sum_{i=1}^n (Ae_ix,e_iy)e_i\right)\\ &= \operatorname{tr}\left(\sum_{i=1}^n \langle Ae_ix, e_iy\rangle\right) = \operatorname{tr}\left(\sum_{i=1}^n e_i\langle Ax,y\rangle e_i\right) = \operatorname{tr}(p\langle Ax,y\rangle p)\\ &= \operatorname{tr}\langle Ax,y\rangle.\end{aligned}$$

Conversely, let $x, y \in X$ satisfy $\langle x,x\rangle = \langle y,y\rangle = p$ and $\langle x,y\rangle = qp$. Suppose first that $|q| < 1$. Define a linear operator $U : Y_{2n} \to X_e$ by its action on the basis $\{z_1,\dots,z_{2n}\}$:

$$\begin{aligned}Uz_i &= e_{\xi,\xi_i}x,\\ Uz_{n+i} &= \frac{1}{\sqrt{1-|q|^2}}(e_{\xi,\xi_i}y - \overline{q}e_{\xi,\xi_i}x)\end{aligned}$$

for $i = 1,\dots,n$. It is easy to check that the operator U is a well-defined isometry. Namely, for $i,j = 1,\dots,n$ we have

$$\langle Uz_i, Uz_j\rangle = e_{\xi,\xi_i}\langle x,x\rangle e_{\xi_j,\xi} = e_{\xi,\xi_i}pe_{\xi_j,\xi} = \delta_{i,j}e,$$

which implies $Uz_i \in X_e$ and $(Uz_i, Uz_j) = \delta_{i,j}$. Also, for $i, j = 1, \dots, n$ it holds that

$$\begin{aligned}
\langle Uz_{n+i}, Uz_{n+j}\rangle &= \frac{1}{1-|q|^2}\langle e_{\xi,\xi_i}y - \overline{q}e_{\xi,\xi_i}x, e_{\xi,\xi_j}y - \overline{q}e_{\xi,\xi_j}x\rangle \\
&= \frac{1}{1-|q|^2}\big(e_{\xi,\xi_i}\langle y,y\rangle e_{\xi_j,\xi} - qe_{\xi,\xi_i}\langle y,x\rangle e_{\xi_j,\xi} - \overline{q}e_{\xi,\xi_i}\langle x,y\rangle e_{\xi_j,\xi} \\
&\qquad + |q|^2 e_{\xi,\xi_i}\langle x,x\rangle e_{\xi_j,\xi}\big) \\
&= \frac{1}{1-|q|^2}\big(e_{\xi,\xi_i}pe_{\xi_j,\xi} - q\overline{q}e_{\xi,\xi_i}pe_{\xi_j,\xi} - \overline{q}qe_{\xi,\xi_i}pe_{\xi_j,\xi} + |q|^2 e_{\xi,\xi_i}pe_{\xi_j,\xi}\big) \\
&= \frac{1}{1-|q|^2}\delta_{i,j}(e - |q|^2e - |q|^2e + |q|^2e) = \delta_{i,j}e,
\end{aligned}$$

so $Uz_{n+i} \in X_e$ and $(Uz_{n+i}, Uz_{n+j}) = \delta_{i,j}$. Furthermore,

$$\begin{aligned}
\langle Uz_i, Uz_{n+j}\rangle &= \frac{1}{\sqrt{1-|q|^2}}e_{\xi,\xi_i}\langle x, y - \overline{q}x\rangle e_{\xi_j,\xi} = \frac{1}{\sqrt{1-|q|^2}}e_{\xi,\xi_i}(\langle x,y\rangle - q\langle x,x\rangle)e_{\xi_j,\xi} \\
&= \frac{1}{\sqrt{1-|q|^2}}e_{\xi,\xi_i}(qp - qp)e_{\xi_j,\xi} = 0
\end{aligned}$$

for $i, j = 1, \dots, n$. Hence, $(Uz_i, Uz_{n+j}) = 0$ for $i, j = 1, \dots, n$. Therefore, U is an isometry and can be extended to a unitary operator $U : X_e \to X_e$. Finally, since (2), (3) and (4) are also valid, the same calculation as before shows that $\operatorname{tr}((C_q \oplus 0_{k-2n})U^*A|X_eU) = \operatorname{tr}\langle Ax, y\rangle$.

Now, suppose that $|q| = 1$. Then we have $C_q^* z_i = \overline{q}z_i$ for $i = 1, \dots, n$ and $C_q^* z_i = 0$ for $i = n+1, \dots, 2n$. Define a linear operator $U : Y_n \to X_e$ on the orthonormal basis $\{z_1, \dots, z_n\}$ by putting $Uz_i = e_{\xi,\xi_i}x$ for $i = 1, \dots, n$. It is clear that U is a well-defined isometry and can be extended to a unitary operator $U : X_e \to X_e$. Let us put $x_i = e_i x$, $y_i = e_i y$ for $i = 1, \dots, n$. Thus we have $\langle x_i, x_i\rangle = e_i\langle x,x\rangle e_i = e_ipe_i = e_i$ and analogously $\langle y_i, y_i\rangle = e_i$ for $i = 1, \dots, n$. Moreover, $\langle x_i, y_i\rangle = e_i\langle x,y\rangle e_i = e_iqpe_i = qe_i$ for $i = 1, \dots, n$. So we deduce that x_i, y_i are unit vectors of the Hilbert space X_{e_i} such that $(x_i, y_i) = q$, $i = 1, \dots, n$. Also, since $|(x_i, y_i)| = |q| = 1 = |(x_i, x_i)|^{\frac{1}{2}} \cdot |(y_i, y_i)|^{\frac{1}{2}}$, it follows that $y_i = \alpha_i x_i$ for some $\alpha_i \in \mathbf{C}$, $i = 1, \dots, n$. But, $q = (x_i, y_i) = (x_i, \alpha_i x_i) = \overline{\alpha_i}(x_i, x_i) = \overline{\alpha_i}$ from which it follows that $y = py = y_1 + \dots + y_n = \overline{q}(x_1 + \dots + x_n) = \overline{q}px = \overline{q}x$. Thus we get

$$\begin{aligned}
\operatorname{tr}((C_q \oplus 0_{k-2n})U^*A|X_eU) &= \sum_{i=1}^n (U^*A|X_eUz_i, C_q^*z_i) = \sum_{i=1}^n (A|X_eUz_i, U(\overline{q}z_i)) \\
&= \sum_{i=1}^n (Ae_{\xi,\xi_i}x, \overline{q}e_{\xi,\xi_i}x) = \sum_{i=1}^n (Ae_{\xi,\xi_i}x, e_{\xi,\xi_i}y) \\
&= (4) = \sum_{i=1}^n (Ae_ix, e_iy) \\
&= \operatorname{tr}\langle Ax, y\rangle,
\end{aligned}$$

which completes the proof. □

Observe that *Theorem 1* can be reformulated in

Corollary 2. *Let A be an operator in $B(X)$. Then*

$$W_q^n(A) = W_C(A|X_e),$$

where

$$\begin{pmatrix} qI_n & \sqrt{1-|q|^2}I_n \\ 0_n & 0_n \end{pmatrix} \oplus 0_{k-2n}$$

is the matrix representation of $C \in B(X_e)$ with respect to some fixed orthonormal basis of X_e.

Finally, we discuss the case when X is a full (left) Hilbert C^*-module over a C^*-subalgebra $\mathcal{A}$ of $B(H)$ which contains $K(H)$. The associated ideal submodule $X_{K(H)}$ is defined by

$$X_{K(H)} = [\{ax : a \in K(H), x \in X\}]^-.$$

Clearly, $X_{K(H)}$ can be regarded as a Hilbert $K(H)$-module. Furthermore, $X_{K(H)}$ is a full Hilbert C^*-module over $K(H)$, since X is a full Hilbert $\mathcal{A}$-module ([2, Proposition 1.3]). After applying Hewitt-Cohen factorization ([2, Proposition 1.2 and Proposition 1.3]) we also have

$$X_{K(H)} = \{ax : a \in K(H), x \in X\} = \{x \in X : \langle x, x\rangle \in K(H)\}.$$

We assume that $2n \leq \dim_{K(H)} X_{K(H)} \leq \infty$.

Let $A \in B(X)$ be an arbitrary operator. Observe that $X_{K(H)}$ is invariant for A and also $(A|X_{K(H)})^* = A^*|X_{K(H)}$, so $A|X_{K(H)} \in B(X_{K(H)})$. Furthermore, the map $\alpha : B(X) \to B(X_{K(H)})$, $\alpha(T) = T|X_{K(H)}$, is a well-defined injective morphism of C^*-algebras ([2, Theorem 1.12]). Hence, its restriction $\alpha|C^*(A) : C^*(A) \to C^*(A|X_{K(H)})$ is an isomorphism of C^*-algebras.

Given a fixed rank n projection $p \in K(H)$ we can define the set $W_q^n(A)$ as it was done before in *Definition 1*. It is obvious that $W_q^n(A) = W_q^n(A|X_{K(H)})$ and all our results remain true for the set $W_q^n(A)$.

3. Some properties of $W_{\widetilde{C}_q}(A)$

From now on we suppose that H is a Hilbert space of dimension $2 \leq k \leq \infty$. Denote by $\{e_i\}$ a fixed orthonormal basis of H. For $q \in \mathbf{C}$, $|q| \leq 1$, and $n \in \mathbf{N}$, $2n \leq k$, let us fix an operator $\widetilde{C}_q \in B(H)$ of the form $C_q \oplus 0_{k-2n}$ where

$$\begin{pmatrix} qI_n & \sqrt{1-|q|^2}I_n \\ 0_n & 0_n \end{pmatrix}$$

is the matrix representation of C_q with respect to the basis $\{e_1, \dots, e_{2n}\}$.

In this section we study some properties of the set $W_{\widetilde{C}_q}(A)$ for a Hilbert space operator $A \in B(H)$.

First, observe that this set can also be described in the following way.

Lemma 3. *For $A \in B(H)$ we have*

$$W_{\widetilde{C}_q}(A) = \{\sum_{i=1}^{n}(Ax_i|y_i) : (x_i), (y_i) \text{ are orthonormal sequences in } H,$$

$$(x_i|y_j) = q\delta_{i,j},\ i, j = 1, \dots, n\}.$$

Proof. Given $t \in W_{\widetilde{C}_q}(A)$ there is a unitary $U \in B(H)$ such that $t = \mathrm{tr}(\widetilde{C}_q U^* AU)$. Let us put $x_i = Ue_i$, $y_i = U\widetilde{C}_q^* e_i$, $i, j = 1, \dots, n$. Then we have $t = \sum_{i=1}^{n}(Ax_i|y_i)$ where $(x_i|x_j) = (y_i|y_j) = \delta_{i,j}$, $(x_i|y_j) = q\delta_{i,j}$. Conversely, suppose that $t = \sum_{i=1}^{n}(Ax_i|y_i)$ where $(x_i|x_j) = (y_i|y_j) = \delta_{i,j}$, $(x_i|y_j) = q\delta_{i,j}$. If $|q| < 1$ define $Ue_i = x_i$, $Ue_{n+i} = \frac{1}{\sqrt{1-|q|^2}}(y_i - \overline{q}x_i)$, $i = 1, \dots, n$. In the case $|q| = 1$ let us put $Ue_i = x_i$, $i = 1, \dots, n$. Then U is a well-defined isometry on the subspace of H and can be extended to a unitary operator $U \in B(H)$. Thereby, $t = \mathrm{tr}(\widetilde{C}_q U^* AU)$. □

In what follows we list some known results on the set $W_{\widetilde{C}_q}(A)$.

Corollary 3. *If $\dim H < \infty$ and $A \in B(H)$, then*

(a) *$W_{\widetilde{C}_q}(A)$ is a compact set,*

(b) *$W_{\widetilde{C}_q}(A)$ is star-shaped with respect to star-center $\frac{1}{k}nq\mathrm{tr}A$,*

(c) *$W_{\widetilde{C}_q}(A) = \{\lambda\}$ if and only if $A = \mu I$ such that $\mu nq = \lambda$.*

Proof. Since $W_{\widetilde{C}_q}(A)$ is a continuous image of the compact set of all unitary operators in $B(H)$, it must be a compact set, so (a) follows. Statement (b) is a consequence of Theorem 4 in [3]. Statement (c) follows from Theorem 2.5 in [8] since $\widetilde{C}_q$ is not a scalar matrix. □

Remark 5. *It is easy to see that statement (c) from the above corollary holds in the infinite dimensional case as well. Indeed, one direction is trivial. To prove the other, assume that $W_{\widetilde{C}_q}(A) = \{\lambda\}$. For arbitrary e_i, e_j, $i \neq j$, denote by $P_{i,j} \in B(H)$ the orthogonal projection onto the $l_{i,j}$-dimensional subspace $M_{i,j}$ of H spanned by $\{e_1, \dots, e_{2n}, e_i, e_j\}$. Then we have*

$$W_{C_q \oplus 0_{l_{i,j}-2n}}((P_{i,j}AP_{i,j})|M_{i,j}) \subseteq W_{\widetilde{C}_q}(A) = \{\lambda\},$$

so by Theorem 2.5 in [8] it follows that $P_{i,j}AP_{i,j} = \mu_{i,j}P_{i,j}$ where $\mu_{i,j}nq = \lambda$. From this we get

$$(Ae_i|e_j) = (P_{i,j}AP_{i,j}e_i|e_j) = (\mu_{i,j}P_{i,j}e_i|e_j) = \mu_{i,j}(e_i|e_j) = 0$$

and

$$(Ae_i|e_i) = (P_{i,j}AP_{i,j}e_i|e_i) = \mu_{i,j} = (P_{i,j}AP_{i,j}e_j|e_j) = (Ae_j|e_j),$$

so we deduce that all $\mu_{i,j}$ are equal and $A = \mu I$ where $\mu = \mu_{i,j}$.

Also, the infinite dimensional analogue of statement (b) is a consequence of Jones' result [6] (see also [3]):

Corollary 4. *If* $\dim H = \infty$ *and* $A \in B(H)$*, then the closure of the set* $W_{\widetilde{C}_q}(A)$ *is star-shaped with respect to the set* $nqW_e(A)$*, where* $W_e(A)$ *denotes the essential numerical range of* A *(i.e., the set of all complex numbers of the form* $\varphi(A)$ *where* φ *runs over all states of* $B(H)$ *vanishing on compact operators in* $B(H)$*).*

We do not know whether $W_{\widetilde{C}_q}(A)$ is always a convex set. Our next results give sufficient conditions for its convexity.

Corollary 5. *Let* $\dim H < \infty$ *and let* $A \in B(H)$*. Then* $W_{\widetilde{C}_q}(A)$ *is a convex set if one of the following conditions holds.*

(a) *There exist* $\alpha, \beta \in \mathbf{C}$ *with* $\alpha \neq 0$ *such that* $\alpha A + \beta I$ *is hermitian, i.e.,* A *is a normal operator and the eigenvalues of* A *are collinear on the complex plane.*

(b) *There exists* $\alpha \in \mathbf{C}$ *such that* $A - \alpha I$ *is unitarily similar to* $M = [M_{ij}]_{1 \le i,j \le m}$ *in block form, where* M_{ii} *are square matrices and* $M_{ij} = 0$ *if* $i \neq j+1$*. In this case* $W_{\widetilde{C}_q}(A)$ *is a circular disc on the complex plane centered at* αnq*.*

(c) *There exists* $\alpha \in \mathbf{C}$ *such that* $A - \alpha I$ *has rank one.*

Proof. Statement (a) is a slight extension of the result of [20], using the fact that $W_{\widetilde{C}_q}(\alpha A + \beta I) = \alpha W_{\widetilde{C}_q}(A) + \beta nq$. Statement (b) follows by [11, Theorem 2.1 and Corollary 2.2] and statement (c) by [18, Theorem 2]. □

Corollary 6. *Let* $\dim H = \infty$ *and let* $A \in B(H)$*. Then* $W_{\widetilde{C}_q}(A)$ *is a convex set if one of the following conditions holds.*

(a) A *is hermitian.*

(b) *There exists* $\alpha \in \mathbf{C}$ *such that* $A - \alpha I$ *has rank one.*

Proof. Suppose that any of the conditions (a) or (b) holds. Let us take arbitrary $t, s \in W_{\widetilde{C}_q}(A)$, $0 \le \lambda \le 1$. Then we have $t = \operatorname{tr}(\widetilde{C}_q U^* A U)$, $s = \operatorname{tr}(\widetilde{C}_q V^* A V)$ for some unitary operators $U, V \in B(H)$. Denote by K the l-dimensional subspace of H spanned by vectors $e_1, \ldots, e_{2n}, Ue_1, \ldots, Ue_{2n}, Ve_1, \ldots, Ve_{2n}$. Let $P \in B(H)$ be the orthogonal projection from H onto K. Then we have $t \in W_{C_q \oplus 0_{l-2n}}(PAP|K)$ and $s \in W_{C_q \oplus 0_{l-2n}}(PAP|K)$. However, by *Corollary 5* $W_{C_q \oplus 0_{l-2n}}(PAP|K)$ is convex. Thus, we have $\lambda t + (1-\lambda)s \in W_{C_q \oplus 0_{l-2n}}(PAP|K) \subseteq W_{\widetilde{C}_q}(A)$. □

It is known (see (4.1) of [9]) that for a matrix $A \in M_n(\mathbf{C})$, unitarily similar to $A_1 \oplus A_2$, it holds that

$$W_C(A) = \cup \left\{ W_{C_1}(A_1) + W_{C_2}(A_2) : \begin{pmatrix} C_1 & X \\ Y & C_2 \end{pmatrix} \in \mathcal{U}(C) \text{ for some } X, Y \right\}$$

where $\mathcal{U}(C)$ denotes the unitary similarity orbit of $C \in M_n(\mathbf{C})$; i.e., $\mathcal{U}(C) = \{U^*CU : U \in M_n(\mathbf{C}) \text{ is unitary}\}$.

Since C_q is unitarily similar to $\underbrace{D_q \oplus \cdots \oplus D_q}_{n\text{times}}$, where $D_q = \begin{pmatrix} q & \sqrt{1-|q|^2} \\ 0 & 0 \end{pmatrix}$, it follows that

$$W_{\widetilde{C}_q}(A)=\cup\left\{W_q(A_1)+\cdots+W_q(A_n):\begin{pmatrix} A_1 & \dots & & \\ \vdots & \ddots & & \\ & & A_n & \\ & & & \ddots \end{pmatrix}\in\mathcal{U}(A)\right\} \tag{5}$$

where A_i is a 2×2 matrix and $W_q(A_i)=W_{D_q}(A_i)$, $i=1,\dots,n$.

Using this expression we can easily obtain some well-known results on the q-numerical range of an operator on a Hilbert space for the set $W_{\widetilde{C}_q}(A)$.

In what follows $\mathrm{Int}(S)$ will stand for the topological interior of $S\subseteq\mathbf{C}$.

Theorem 2. *Let $\dim H<\infty$ and let $A\in B(H)$. Let $\alpha_1,\dots,\alpha_k$ be the eigenvalues of A. Then*

$$\{q(\alpha_{j_1}+\cdots+\alpha_{j_n}): j_1,\dots,j_n\in\{1,\dots,k\}\text{ are mutually different}\}\subseteq W_{\widetilde{C}_q}(A).$$

If A is not a scalar operator and $|q|<1$, then

$$\{q(\alpha_{j_1}+\cdots+\alpha_{j_n}): j_1,\dots,j_n\in\{1,\dots,k\}\text{ are mutually different}\}\subseteq \mathrm{Int}(W_{\widetilde{C}_q}(A)).$$

Proof. For every choice $\alpha_{j_1},\dots,\alpha_{j_n}$ of eigenvalues of A there exists a unitary operator $U\in B(H)$ such that the operator U^*AU is in the lower triangular form

$$\begin{pmatrix} A_1 & & & \\ \vdots & \ddots & & \\ & & A_n & \\ & & & \ddots \end{pmatrix}$$

where A_i $(i=1,\dots,n)$ are 2×2 matrices with one diagonal entry equal to α_{j_i}. Using Theorem 2.7 of [10] and statement (5) it holds that

$$q\alpha_{j_1}+\cdots+q\alpha_{j_n}\in W_q(A_1)+\cdots+W_q(A_n)\subseteq W_{\widetilde{C}_q}(A).$$

Suppose now that A is not a scalar operator and $|q|<1$. Then at least one of A_i is not a scalar operator. Namely, if at least two eigenvalues of A are different, then some of A_i, say A_1, can be chosen to be the operator with different diagonal entries. On the other hand, if all eigenvalues of A are equal, then A is not normal. Then by Lemma 1 of [13] A_1 can be chosen to be a non-scalar operator. So, in both cases Theorem 2.7 of [10] implies

$$\begin{aligned} q\alpha_{j_1}+\cdots+q\alpha_{j_n} &\in \mathrm{Int}(W_q(A_1))+W_q(A_2)+\cdots+W_q(A_n)\\ &\subseteq \mathrm{Int}(W_q(A_1)+\cdots+W_q(A_n))\subseteq \mathrm{Int}(W_{\widetilde{C}_q}(A)). \end{aligned}$$

□

Our next result extends the classical result of an inclusion relation for q-numerical ranges for different q (see [10, Theorem 2.5]).

Theorem 3. *Suppose $q_1, q_2 \in \mathbf{C}$ satisfy $|q_2| \le |q_1| \le 1$. Then for $A \in B(H)$ we have*

$$q_2 W_{\widetilde{C}_{q_1}}(A) \subseteq q_1 W_{\widetilde{C}_{q_2}}(A).$$

Moreover, if $A = \mu I \in B(H)$ for some $\mu \in \mathbf{C}$, then

$$q_2 W_{\widetilde{C}_{q_1}}(A) = q_1 W_{\widetilde{C}_{q_2}}(A) = \{\mu n q_1 q_2\}.$$

If $A \in B(H)$ is not a scalar operator and $|q_2| < |q_1| < 1$, then

$$q_2 W_{\widetilde{C}_{q_1}}(A) \subseteq \mathrm{Int}(q_1 W_{\widetilde{C}_{q_2}}(A)).$$

Proof. First, observe that in the case of the finite dimensional space H the proof is a direct consequence of Theorem 2.5 of [10] and statement (5). The infinite dimensional case is reduced to the finite dimensional one. Namely, for $t = \mathrm{tr}(\widetilde{C}_{q_1} U^* A U) \in W_{\widetilde{C}_{q_1}}(A)$ we have $t \in W_{C_{q_1} \oplus 0_{l-2n}}(PAP|K)$ where $P : H \to K$ is the orthogonal projection onto the l-dimensional subspace K of H spanned by $e_1, \dots, e_{2n}, Ue_1, \dots, Ue_{2n}$. Then we get

$$q_2 t \in q_2 W_{C_{q_1} \oplus 0_{l-2n}}(PAP|K) \subseteq q_1 W_{C_{q_2} \oplus 0_{l-2n}}(PAP|K) \subseteq q_1 W_{\widetilde{C}_{q_2}}(A).$$

If A is not a scalar operator and $|q_2| < |q_1| < 1$, then the projection P can be chosen such that neither PAP is a scalar operator. So, we have

$$q_2 t \in q_2 W_{C_{q_1} \oplus 0_{l-2n}}(PAP|K) \subseteq \mathrm{Int}(q_1 W_{C_{q_2} \oplus 0_{l-2n}}(PAP|K)) \subseteq \mathrm{Int}(q_1 W_{\widetilde{C}_{q_2}}(A)).$$

□

Theorem 4. *Let $\dim H < \infty$ and let $A \in B(H)$. If $|q| < 1$ and A is not a scalar operator, then the boundary of $W_{\widetilde{C}_q}(A)$ is a smooth curve.*

Proof. Let t be a boundary point of $W_{\widetilde{C}_q}(A)$. Then we have $t = \mathrm{tr}(\widetilde{C}_q U^* A U)$ for some unitary $U \in B(H)$. If t is a non-differentiable boundary point of $W_{\widetilde{C}_q}(A)$, then, as in the proof of Theorem 2.1 of [8], we conclude that $\widetilde{C}_q$ and $B = U^* A U$ commute. Hence, there exists unitary $V \in B(H)$ such that both $V \widetilde{C}_q V^*$ and VBV^* are in the lower triangular form. Now,

$$t = \mathrm{tr}(V \widetilde{C}_q V^* V B V^*) = q(\alpha_{j_1} + \cdots + \alpha_{j_n})$$

for some $\alpha_{j_1}, \dots, \alpha_{j_n}$ from the spectrum of A. So, by *Theorem 2* $t \in \mathrm{Int}(W_{\widetilde{C}_q}(A))$ which contradicts the fact that t is a boundary point of $W_{\widetilde{C}_q}(A)$. □

4. The set $W_{\widetilde{C}_0}(A)$

Observe that the roles of C and A in the definition of $W_C(A)$ are symmetric, i.e., $W_C(A) = W_A(C)$. Also, note that for $q = 0$ the operator C_0 satisfies condition (e) in Theorem 2.1 of [11]. So, as a consequence of the equivalence of the conditions (e) in Theorem 2.1 of [11] and (g) in Corollary 2.2 of [11] we get the following

Corollary 7. *If* $\dim H < \infty$*, then* $W_{\widetilde{C}_0}(A)$ *is a circular disc on the complex plane centered at the origin for all* $A \in B(H)$*.*

The convexity of $W_{\widetilde{C}_0}(A)$ in the case of the infinite dimensional space H can be obtained by reducing to the finite dimensional case, as it was done in *Corollary 6*. Furthermore, if $P \in B(H)$ stands for the orthogonal projection onto the subspace K of H spanned by $\{e_1, \dots, e_{2n}\}$, then by the above corollary we have $0 \in W_{C_0}(PAP|K)$. However, $W_{C_0}(PAP|K) \subseteq W_{\widetilde{C}_0}(A)$, so $0 \in W_{\widetilde{C}_0}(A)$. Also, it is obvious that the set $W_{\widetilde{C}_0}(A)$ is circular, i.e., $\mu W_{\widetilde{C}_0}(A) = W_{\widetilde{C}_0}(A)$ for all $\mu \in \mathbf{C}$ with $|\mu| = 1$ (see *Lemma 3*). From all of this we have

Corollary 8. *If* $\dim H = \infty$*, then* $W_{\widetilde{C}_0}(A)$ *is an open or closed circular disc on the complex plane centered at the origin for all* $A \in B(H)$*.*

In what follows we will identify the radius of $W_{\widetilde{C}_0}(A)$ for hermitian A acting on the finite dimensional Hilbert space H. The proof of our theorem is based on the result of Mirsky [14, Theorem 1] (see also [17, Corollary 5] or [16]).

Theorem 5. *Let* $\dim H < \infty$*. If* $A \in B(H)$ *is hermitian with eigenvalues* $\alpha_1 \le \dots \le \alpha_k$*, then* $W_{\widetilde{C}_0}(A)$ *is a circular disc with the center at the origin and radius* $r = \frac{1}{2}(\alpha_k + \alpha_{k-1} + \dots + \alpha_{k-n+1} - \alpha_1 - \alpha_2 - \dots - \alpha_n)$*.*

Proof. By *Corollary 7* $W_{\widetilde{C}_0}(A)$ is a circular disc centered at the origin. It remains to identify its radius. Let us take any $t \in W_{\widetilde{C}_0}(A)$. By statement (5) $t = t_1 + \dots + t_n$ for some $t_i \in W_0(A_i)$ where

$$B = \begin{pmatrix} A_1 & \dots & & \\ \vdots & \ddots & & \\ & & A_n & \\ & & & \ddots \end{pmatrix} \in \mathcal{U}(A).$$

Denote by β_i and γ_i the eigenvalues of A_i and let us suppose that $\beta_i \ge \gamma_i$, $i = 1, \dots, n$. According to Theorem 1 of [14], we have $|t_i| \le \frac{1}{2}(\beta_i - \gamma_i)$ for $i = 1, \dots, n$. Since B is unitarily similar to the diagonal operator with the first $2n$ diagonal entries $\beta_1, \gamma_1, \dots, \beta_n, \gamma_n$, it follows by Corollary 2 of [4] that

$$\beta_1 + \dots + \beta_n \le \alpha_k + \alpha_{k-1} + \dots + \alpha_{k-n+1} \qquad \text{and} \qquad \gamma_1 + \dots + \gamma_n \ge \alpha_1 + \dots + \alpha_n.$$

Hence,

$$|t| \le \frac{1}{2}\sum_{i=1}^{n}(\beta_i - \gamma_i) \le \frac{1}{2}(\alpha_k + \alpha_{k-1} + \dots + \alpha_{k-n+1} - \alpha_1 - \alpha_2 - \dots - \alpha_n) = r.$$

To complete the proof it is enough to show that $r \in W_{\widetilde{C}_0}(A)$. Observe that A is unitarily similar to some diagonal operator $A_1 \oplus \dots \oplus A_n \oplus D$ where

$$A_i = \begin{pmatrix} \alpha_{k-i+1} & 0 \\ 0 & \alpha_i \end{pmatrix}.$$

Again, applying Theorem 1 of [14], we get $t_i = \frac{1}{2}(\alpha_{k-i+1} - \alpha_i) \in W_0(A_i)$. By (5) it obviously follows that $r = t_1 + \dots + t_n \in W_{\widetilde{C}_0}(A)$. □

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