Lacunary statistical convergence of double sequences

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Abstract. *In 1978 Freedman, Sember, and Raphael presented a definition for lacunary refinement as follows: $\rho = \{\bar{k}_r\}$ is called a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{\bar{k}_r\}$. They use this definition to present one side inclusion theorem with respect to the refined and non refined sequence. In 2000 Li presented the other side of the inclusion. In this paper we shall present a multidimensional analogue to the notion of refinement of lacunary sequences and use this definition to present both sides of the above inclusion theorem. In addition, we shall also present a notion of double lacunary statistically Cauchy and use this definition to establish that it is equivalent to the $S_{\theta_{r,s}}$-P-convergence.*

Key words: *double lacunary sequences, P-convergent*

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1. Introduction

We begin this section by giving some preliminaries: The concept of a statistical convergence was introduced by Fast [1] in 1951. A complex number sequence x is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\left\{k < n : |x_k - L| \geq \epsilon\right\}\right| = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, ..., n$ and the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals

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determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

The following concept is due to Fridy and Orhan [4] . Let θ be a lacunary sequence; the number sequence x is $S_\theta-$ convergent to L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\Big|\big\{k \in I_r : |x_k - L| \geq \epsilon\big\}\Big| = 0,$$

In this case we write $S_\theta - \lim x = L$ or $x_k \to L(S_\theta)$.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\{\frac{K_{m,n}}{mn}\}_{m,n=1,1}^{\infty,\infty}$ has a limit we say that K has a natural density and is defined

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, Let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero). Quite recently, Mursaleen and Edely [8], defined the statistical analogue for double sequences $x = \{x_{k,l}\}$ as follows: A real double sequence $x = \{x_{k,l}\}$ is said to be P-statistically convergent to L provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn}\big\{\text{ number of } (j,k) : j < m \text{ and } k < n, |x_{j,k} - L| \geq \epsilon\big\} = 0,$$

In this case we write $st_2 - \lim_{m,n} x_{m,n} = L$ and denote the set of all statistical convergent double sequences by st_2. In this paper, our goal is to extend a few results known in the literature from single sequences to double sequence with respect to convergence in the Pringsheim sense. The following notion of a subsequence which was presented by Patterson in [9] will help us achieve our objectives:

Definition 1. *The double sequence* $[y]$ *is a double* subsequence *of the sequence* $[x]$ *provided that there exist two increasing double index sequences* $\{n_j\}$ *and* $\{k_j\}$ *such that if* $z_j = x_{n_j,k_j}$*, then* y *is formed by*

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & -. \end{array}$$

2. Definitions and notations

Definition 2. *The double sequence* $\theta = \{(k_r, l_s)\}$ *is called* double lacunary *if there exist two increasings of integers such that*

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, The following intervals are determined by θ, $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < l \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \;\&\; l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{r,s}}$. The following definition is a multidimensional analog of definition 2.2 of Fridy and Orhan [4].

Definition 3. *Let* θ *be a double lacunary sequence; the double number sequence* x *is* S''_θ *− convergent to* L *provided that for every* $\epsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \epsilon\right\}\right| = 0.$$

In this case write $S''_\theta - \lim x = L$ *or* $x_{k,l} \to L(S''_\theta)$. Using this notion we shall also present a definition of double lacunary refinement and establish a multidimensional analog of Theorem 2.2 of Fridy and Orhan [4].

Definition 4. *The double index sequence* $\rho = \{\bar{k}_r, \bar{l}_s\}$ *is called a* double lacunary refinement *of the double lacunary sequence* $\theta = \{k_r, l_s\}$ *if* $\{k_r, l_s\} \subseteq \{\bar{k}_r, \bar{l}_s\}$.

Theorem 1. *If* ρ *is a double lacunary refinement of* θ *and* $x_{k,l} \to L(S''_\rho)$, *then* $x_{k,l} \to L(S''_\theta)$.

Proof. Suppose each $I_{r,s}$ of θ containing $\left\{\bar{k}_{r,i}, \bar{l}_{s,j}\right\}_{i=1,j=1}^{\infty,\infty}$ of θ, such that

$$k_{r-1} < \bar{k}_{r,1} < \bar{k}_{r,2} < \cdots < \bar{k}_{r,\alpha_r} = k_r,$$

$$\bar{I}_{r,i} = (\bar{k}_{r,i-1}, \bar{k}_{r,i}],$$

$$l_{s-1} < \bar{l}_{s,1} < \bar{l}_{s,2} < \cdots < \bar{l}_{s,\beta_s} = l_s,$$

$$\bar{I}_{s,j} = (\bar{l}_{s,j-1}, \bar{l}_{s,j}],$$

and

$$\bar{I}_{r,s,i,j} = \{(k,l) : \bar{k}_{r,i-1} < k \leq \bar{k}_{r,i} \& \bar{l}_{s,j-1} < l \leq \bar{l}_{s,j}\}$$

for all r and s, let α_r and β_s be greater than or equal to one; this implies that $\{k_r, l_s\} \subseteq \{\bar{k}_r, \bar{l}_s\}$. Let $\{I_{i,j}\}_{i,j=1,1}^{\infty,\infty}$ be a sequence of abutting blocks $\{\bar{I}_{r,s,i,j}\}$ ordered by increasing a lower right index. Also since $x_{k,l} \to L(S''_\rho)$ we have the following for each $\epsilon > 0$:

$$P - \lim_{i,j} \sum_{\bar{I}_{i,j} \subset I_{r,s}} \frac{1}{\bar{h}_{r,s}} \left|\left\{(k,l) \in \bar{I}_{i,j} : |x_{k,l} - L| \geq \epsilon\right\}\right| = 0.$$

Where $h_{r,s} = h_r \bar{h}_s$, $h_{r,i} = \bar{k}_{r,i} - \bar{k}_{r,i-1}$, and $\bar{h}_{s,j} = \bar{l}_{s,j} - \bar{l}_{s,j-1}$. Also for each $\epsilon > 0$, we obtain the following:

$$\frac{1}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:|x_{k,l}-L|\geq\epsilon\}\right|$$
$$=\frac{1}{h_{r,s}}\sum_{\bar{I}_{i,j}\subseteq I_{r,s}}\bar{h}_{i,j}\frac{1}{\bar{h}_{i,j}}\left|\left\{(k,l)\in\bar{I}_{i,j}:|x_{k,l}-L|\geq\epsilon\right\}\right|.$$

First observe that

$$t_{i,j}=\frac{1}{\bar{h}_{i,j}}\left|\left\{(k,l)\in\bar{I}_{i,j}:|x_{k,l}-L|\geq\epsilon\right\}\right|$$

is a Pringsheim null sequence. The transformation

$$(At)_{r,s}=\frac{1}{h_{r,s}}\sum_{\bar{I}_{i,j}\subseteq I_{r,s}}\bar{h}_{i,j}\frac{1}{\bar{h}_{i,j}}\left|\left\{(k,l)\in\bar{I}_{i,j}:|x_{k,l}-L|\geq\epsilon\right\}\right|$$

satisfies all conditions for a matrix transformation to map a Pringsheim null sequence into a Pringsheim null sequence [6]. Therefore $S_{\rho}^{''}\subseteq S_{\theta}^{''}$. This completes the proof of this theorem. □

The following theorems are multidimensional analogues of some results of Li [7].

Theorem 2. *Suppose $\rho=\{\bar{k}_r,\bar{l}_s\}$ is a lacunary refinement of the double lacunary sequence $\theta=\{k_r,l_s\}$. Let $I_{r,s}$ and $\bar{I}_{r,s}$; $r,s=1,2,3,\ldots$ be defined as above. If there exists $\delta>0$ such that*

$$\frac{|\bar{I}_{\alpha,\beta}|}{|I_{r,s}|}\geq\delta\quad\text{for every }\bar{I}_{\alpha,\beta}\subseteq I_{r,s}.$$

Then $x_{k,l}\to L(S_{\theta}^{''})$ implies $x_{k,l}\to L(S_{\rho}^{''})$ (i.e. $S_{\theta}^{''}\subseteq S_{\rho}^{''}$).

Proof. Given any $\epsilon>0$ and every $\bar{I}_{\alpha,\beta}$ we can find $I_{r,s}$ such that $\bar{I}_{\alpha,\beta}\subseteq I_{r,s}$, and we obtain the following:

$$\left(\frac{1}{|I_{r,s}|}\right)\left|\{(k,l)\in I_{r,s}:|x_{k,l}-L|\geq\epsilon\}\right|$$
$$=\left(\frac{|\bar{I}_{\alpha,\beta}|}{|I_{r,s}|}\right)\left(\frac{1}{|\bar{I}_{\alpha,\beta}|}\right)\left|\{(k,l)\in I_{r,s}:|x_{k,l}-L|\geq\epsilon\}\right|$$
$$\leq\left(\frac{|\bar{I}_{\alpha,\beta}|}{|I_{r,s}|}\right)\left(\frac{1}{|\bar{I}_{\alpha,\beta}|}\right)\left|\left\{(k,l)\in\bar{I}_{\alpha,\beta}:|x_{k,l}-L|\geq\epsilon\right\}\right|$$
$$\leq\left(\frac{1}{\delta}\right)\left(\frac{1}{|\bar{I}_{\alpha,\beta}|}\right)\left|\left\{(k,l)\in\bar{I}_{\alpha,\beta}:|x_{k,l}-L|\geq\epsilon\right\}\right|.$$

This grants us the theorem. □

Theorem 3. *Suppose $\rho=\{\bar{k}_r,\bar{l}_s\}$ and $\theta=\{k_r,l_s\}$ are two lacunary sequences. Let $I_{r,s}$ and $\bar{I}_{r,s}$; $r,s=1,2,3,\ldots$ be defined as above and $I_{r,s,\alpha,\beta}=I_{r,s}\cap\bar{I}_{\alpha,\beta}$; $r,s,\alpha,\beta=1,2,3,\ldots$. If there exists $\delta>0$, such that*

$$\frac{|I_{r,s,\alpha,\beta}|}{|I_{r,s}|}\geq\delta\quad\text{for every }\alpha,\beta,r,s=1,2,\ldots\text{ provided that }I_{r,s,\alpha,\beta}\neq\emptyset.$$

Then $x_{k,l} \to L(S''_\theta)$ *implies* $x_{k,l} \to L(S''_\rho)$ *(i.e.* $S''_\theta \subseteq S''_\rho$*).*

Proof. Let $\alpha = \rho \cup \theta$. Then α is a lacunary refinement of the lacunary double sequences ρ and θ. The blocks of α are $\{I_{r,s,\alpha,\beta} = I_{r,s} \cap \bar{I}_{\alpha,\beta} : I_{r,s,\alpha,\beta} \neq \emptyset\}$. *Theorem 2* and $\frac{|I_{r,s,\alpha,\beta}|}{|I_{r,s}|} \geq \delta$ for every $\alpha, \beta, r, s = 1, 2, \ldots$ provided that $I_{r,s,\alpha,\beta} \neq \emptyset$ implies $x_{k,l} \to L(S''_\theta)$ implies $x_{k,l} \to L(S''_\alpha)$. Since α is a lacunary refinement of ρ, *Theorem 1* grants us the following: $x_{k,l} \to L(S''_\alpha)$ implies $x_{k,l} \to L(S''_\rho)$. Thus $S''_\theta \subseteq S''_\rho$. □

Theorem 4. *Suppose* $\rho = \{\bar{k}_r, \bar{l}_s\}$ *and* $\theta = \{k_r, l_s\}$ *are two lacunary sequences. Let* $I_{r,s}$ *and* $\bar{I}_{r,s}$; $r, s = 1, 2, 3, \ldots$ *be defined as above and* $I_{r,s,\alpha,\beta} = I_{r,s} \cap \bar{I}_{\alpha,\beta}$; $r, s, \alpha, \beta = 1, 2, 3, \ldots$. *If there exists* $\delta > 0$*, such that*

$$\frac{|I_{r,s,\alpha,\beta}|}{|I_{r,s}| + |I_{\alpha,\beta}|} \geq \delta \quad \text{for every } \alpha, \beta, r, s = 1, 2, \ldots \text{ provided that } I_{r,s,\alpha,\beta} \neq \emptyset.$$

Then $x_{k,l} \to L(S''_\theta)$ *if and only if* $x_{k,l} \to L(S''_\rho)$ *(i.e.* $S''_\theta \subseteq S''_\rho$*).*

Proof. *Theorem 1* grants us the necessary conditions of this theorem and *Theorem 3* grants us the sufficient conditions. □

Remark 1. *In [[12], Theorem 2.1 part A] the authors established the following: Let* θ *be a double lacunary sequence and if* $x \to L(N''_\theta)$ *then* $x \to L(S'')$*, thus similar to the example presented on page 511 of Freeman et. al. [3]. We can easily construct two double lacunary sequences* θ_1 *and* θ_2 *such that* $x \to 0(N''_{\theta_1})$ *and* $x \to 1(N''_{\theta_2})$ *for a prescribed bounded double sequence. Thus [[12], Theorem 2.1 part A] assures that* $x \to 0(S''_{\theta_1})$ *and* $x \to 1(S''_{\theta_2})$*. Therefore limits are not unique for different lacunary sequences.* In this final section of our paper we present S''_θ- analog of the Cauchy criterion for double sequences and also we prove that it is equivalent to S''_θ-P- convergence as in [5]. In addition, two implications of this Cauchy criterion shall also be presented.

Definition 5. *Let* θ *be a double lacunary sequence; the double number sequence* x *is said to be an* S''_θ*-Cauchy double sequence if there exists a double subsequence* $\{x_{\bar{k}_r,\bar{l}_s}\}$ *of* x *such that* $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ *for each* (r, s) P-$\lim_{r,s} x_{k_r,l_s} = L$ *and for every* $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k, l) \in I_{r,s} : |x_{r,s} - x_{\bar{k}_r,\bar{l}_s}| \geq \epsilon\right\}\right| = 0.$$

Theorem 5. *The double sequence* x *is* S''_θ *- P- convergent if and only if* x *is an* S''_θ*- Cauchy double sequence.*

Proof. Let $x_{k,l} \to L(S''_\theta)$ and $K^{i,j} = \{(k, l) \in \mathcal{N} \times \mathcal{N} : |x_{k,l} - L| < \frac{1}{ij}\}$ for each $(i, j) \in \mathcal{N} \times \mathcal{N}$ we obtain the following $K^{i+1,j+1} \subseteq K^{i,j}$ and

$$\frac{|K^{i,j} \cap I_{r,s}|}{h_{rs}} \to 1 \text{ as } r, s \to \infty.$$

This implies that there exist m_1 and n_1 such that $r \geq m_1$ and $s \geq n_1$ and

$$\frac{|K^{1,1} \cap I_{r,s}|}{h_{rs}} > 0$$

that is $K^{1,1} \cap I_{r,s} \neq \emptyset$. We next choose $m_2 > m_1$ and $n_2 > n_1$ such that $r > m_2$ and $s > n_2$ implies that $K^{2,2} \cap I_{r,s} \neq \emptyset$. Thus for each pair (r,s) such that $m_1 \leq r < m_2$ and $n_1 \leq s < n_2$ we choose $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ such that $(\bar{k}_r, \bar{l}_s) \in K^{r,s} \cap I_{r,s}$ that is $|x_{\bar{k}_r,\bar{l}_s} - L| < 1$. In general we choose $m_{i+1} > m_2$ and $n_{j+1} > n_j$ such that $r > m_{i+1}$ and $s > n_{j+1}$, this implies $I_{r,s} \cap K^{i+1,j+1} \neq \emptyset$. Thus for all (r,s) such that for $m_i \leq r < m_{i+1}$ and $n_j \leq s < n_{j+1}$ choose $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ i.e. $|x_{\bar{k}_r,\bar{l}_s} - L| < \frac{1}{ij}$. Thus $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ for each pair (r,s) and $|x_{\bar{k}_r,\bar{l}_s} - L| < \frac{1}{ij}$ implies $P - \lim_{r,s} x_{\bar{k}_r,\bar{l}_s} = L$. Also, for each $\epsilon > 0$

$$\frac{1}{h_{rs}} \left|\{(k,l) \in I_{r,s} : |x_{k,l} - x_{\bar{k}_r,\bar{l}_s}| \geq \epsilon\}\right| \leq \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \frac{\epsilon}{2}\right\}\right| + \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : |x_{\bar{k}_r,\bar{l}_s} - L| \geq \frac{\epsilon}{2}\right\}\right|$$

Since $x_{k,l} \to L(S_\theta^{''})$ and $P - \lim_{r,s} x_{\bar{k}_r,\bar{l}_s} = L$ it follows that x is an $S_\theta^{''}$-Cauchy double sequence. Now suppose that x is an $S_\theta^{''}$-Cauchy double sequence, then

$$|\{(k,l) \in I_{r,s} : |x_{k,l} - L| \geq \epsilon\}| \leq \left|\left\{(k,l) \in I_{r,s} : |x_{k,l} - x_{\bar{k}_r,\bar{l}_s}| \geq \frac{\epsilon}{2}\right\}\right| + \left|\left\{(k,l) \in I_{r,s} : |x_{\bar{k}_r,\bar{l}_s} - L| \geq \frac{\epsilon}{2}\right\}\right|.$$

Therefore $x_{k,l} \to L(S_\theta^{''})$. Thus the theorem is proven.

□

Corollary 1. *If x is an $S_\theta^{''}$ - P- convergent double sequence, then x has a P - convergent subsequence.* The proof of this corollary is an immediate consequence of *Theorem 5*. Let us consider the following double difference diagonal operator: $\triangle^{''} x_{p,q} = (x_{p,q} - x_{p+1,q+1})$ and establish the following Tauberian theorem.

Theorem 6. *If $x \to L(S_\theta^{''})$ and*

$$\max_{\{(p,q):k \leq p < k_r \& l \leq q < l_s\}} \triangle^{''} x_{p,q} = o\left(\frac{1}{h_{r,s}^2}\right) \text{ as } (r,s) \to \infty \text{ in the Pringsheim sense}$$

then P-$\lim_{k,l} x_{k,l} = L$.

Proof. Since $x \to L(S_\theta^{''})$, *Corollary 1* assures that there exists a P-convergent subsequence $\{x_{k_r,l_s}\}$ of x. Since $\{k_r, l_s\} \in I_{r,s}$ we obtain the following:

$$\begin{aligned}
|x_{k,l} - x_{k_r,l_s}| &\leq \sum_{\{(p,q):k \leq p < k_r \& l \leq q < l_s\}} |\triangle^{''} x_{p,q}| \\
&\leq (k_r - k)(l_s - l) \max_{\{(p,q):k \leq p < k_r \& l \leq q < l_s\}} \triangle^{''} x_{p,q} \\
&\leq (k_r - k)(l_s - l) o\left(\frac{1}{h_{r,s}^2}\right) \\
&= o\left(\frac{(k_r - k)(l_s - l)}{h_{r,s}^2}\right) \\
&= o(1)
\end{aligned}$$

where $k_{r-1} \leq k \leq p < k_r$ and $l_{s-1} \leq l \leq q < l_s$. Since $\{x_{k_r,l_s}\}$ is P-convergent to L, x is also P-convergent to L. $\square$

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