On the strong laws of large numbers for ρ-mixing sequences*

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Abstract. *The connection between general moment conditions and the applicability of strong laws of large numbers for a sequence of identically distributed ρ-mixing random variables are obtained. The results obtained generalize the result of Petrov (1996, Statist. Probab. Lett. 26, 377-380) to ρ-mixing sequences and NA sequences.*

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. There are two famous theorems on the strong laws of large numbers for such a sequence: Kolmogorov's theorem and the Marcinkiewics-Zygmund theorem (see Loève, 1963; Stout, 1974 or Petrov, 1995). Let $S_n = \sum_{i=1}^n X_i$. According to Kolmogorov's theorem, there exists a constant b such that

$$\lim_{n\to\infty} \frac{|S_n|}{n} = b \quad a.s.$$

if and only if $E|X_1| < \infty$ and $b = EX_1$.

By the Marcinkiewics-Zygmund theorem, if $0 < p < 2$, then the relations $\lim\limits_{n\to\infty} \frac{|S_n - nb|}{n^{1/p}} = 0 \quad a.s.$ and $E|X_1|^p < \infty$ are equivalent. Here $b = 0$ if $0 < p < 1$, and $b = EX_1$ if $1 \leq p < 2$.

Etemadi (1981) proved that Kolmogorov's theorem remains true if we replace the independence condition by the weaker condition of pairwise independence of random variables. Martikainen (1992) proved some other strong laws of large numbers for pairwise independence of random variables. Petrov (1996) proved the connection

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between general moment conditions and the applicability of strong laws of large numbers for a sequence of identically distributed pairwise independence random variables sequences.

Petrov (1996) proved the following two theorems.

Theorem 1. *Let $\{X, X_i, i \geq 1\}$ be a sequence of pairwise independence identically distributed random variables.*
If

$$\frac{S_n}{a_n} \to 0 \quad a.s.$$

then $Ef(X_1) < \infty$.

Theorem 2. *Let $\{X, X_i, i \geq 1\}$ be a sequence of pairwise independence identically distributed random variables.*
If

$$Ef(X_1) < \infty \qquad and \qquad \sum_{k=n}^{\infty} \frac{1}{a_k} = O(\frac{n}{a_n}),$$

then

$$\frac{S_n}{a_n} \to 0 \quad a.s.$$

A sequence $\{X_n, n \geq 1\}$ is said to be a ρ -mixing sequence, if $n \to \infty$, we have

$$\rho(n) = \sup_{k \geq 1, X \in L^2(F_1^k), Y \in L^2(F_{k+n}^{\infty})} |cov(X,Y)|/\|X\|_2\|Y\|_2 \to 0,$$

where F_n^m is a σ -field generated by the random variables $X_n, X_{n+1}, \cdots, X_m$. Here $\|X\|_p = (E|X|^p)^{\frac{1}{p}}$. As for the ρ -mixing sequence, there are many results, such as Shao (1995) for moment inequalities and strong laws of large numbers.

As for negatively associated (NA) random variables, Joag (1983) gave the following definition.

Definition 1 [Joag, 1983]. *A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be* negatively associated *(NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, ..., n\}$, we have*

$$Cov(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su et al. (1996) derived some moment inequalities of partial sums and a weak convergence for a strong stationary NA sequence. Lin (1997) set up an invariance principal for NA sequences. Su and Qin (1997) also studied some limiting results for NA sequences. More recently, Liang (1999, 2000) considered some complete convergence for weighted sums of NA sequences. Those results, especially some moment inequality by Wang and Xu (2002), Shao (2000) and Yang

(2000), undoubtedly propose an important theory guide for further application of the NA sequence.

In this paper we will examine the connection between general moment conditions and the applicability of strong laws of large numbers for a sequence of identically distributed ρ-mixing random variables. The results obtained generalize the result of Petrov (1996, Statist. Probab. Lett. 26, 377–380) to ρ-mixing sequences and NA sequences.

Throughout this paper C will represent a positive constant, though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \le Cb_n$.

2. General moment conditions and the strong laws of large numbers

Let $f(x)$ be an even continuous function that is positive and strictly increasing in the region $x > 0$ and satisfying the condition $f(x) \to \infty$ as $x \to \infty$. We put

$$a_n = f^{-1}(n), \tag{1}$$

where f^{-1} is the inverse of f. By the properties of f, we have $a_n \uparrow \infty$.

Theorem 3. *Let $\{X, X_i, i \ge 1\}$ be a sequence of ρ-mixing identically distributed random variables. If*

$$\sum_{i=0}^{\infty} \rho(2^i) < \infty, \qquad \text{and} \qquad \frac{S_n}{a_n} \to 0 \quad a.s. \tag{2}$$

then

$$Ef(X_1) < \infty. \tag{3}$$

In order to prove our results, we need the following lemmas.

Lemma 1 [Shao, 1995]. *Let $\{X_i, i \ge 1\}$ be a sequence of ρ-mixing random variables, let $S_n = \sum_{i=1}^n X_i$, $EX_i = 0, E|X_i|^p < \infty$ for some $p \ge 2$ and for every $i \ge 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1\le i\le n} |S_i|^p \le C\Big\{ \exp\big(C \sum_{i=0}^{[\log n]} \rho(2^i)\big)(n \max_{1\le i\le n} E|X_i|^2)^{p/2} + n \exp\big(C \sum_{i=0}^{[\log n]} \rho^{2/p}(2^i)\big) \max_{1\le i\le n} E|X_i|^p \Big\}$$

Lemma 2. *If events $A_1, A_2, ..., A_n$ satisfy $Var(\sum_{k=1}^n I_{A_k}) \le K \sum_{k=1}^n P(A_k)$, then*

$$\big(1 - P(\bigcup_{j=1}^n A_j)\big)^2 \sum_{k=1}^n P(A_k) \le KP(\bigcup_{j=1}^n A_j). \tag{4}$$

Proof. Let $\beta = 1 - P(\bigcup_{k=1}^n A_k)$. If $\beta = 0$, then (4) is obvious. If $\beta > 0$, by $Var(\sum_{k=1}^n I_{A_k}) \le K \sum_{k=1}^n P(A_k)$, then

$$\begin{aligned}
\sum_{j=1}^n P(A_j) &= \sum_{j=1}^n P\Big\{A_j \bigcap (\bigcup_{i=1}^n A_i)\Big\} \\
&= E\Big\{\sum_{j=1}^n (I_{A_j} - P(A_j)) I_{\{\bigcup_{i=1}^n A_i\}}\Big\} + \sum_{j=1}^n P(A_j)(1-\beta) \\
&\le \Big(Var(\sum_{j=1}^n I_{A_j}) P(\bigcup_{k=1}^n A_k)\Big)^{1/2} + \sum_{j=1}^n P(A_j)(1-\beta) \\
&\le \Big\{K \sum_{j=1}^n P(A_j) P(\bigcup_{k=1}^n A_k)\Big\}^{1/2} + \sum_{j=1}^n P(A_j)(1-\beta) \\
&\le \frac{K}{2\beta} P\Big(\bigcup_{k=1}^n A_k\Big) + \Big(\frac{\beta}{2} + 1 - \beta\Big) \sum_{j=1}^n P(A_j).
\end{aligned}$$

Thus

$$\sum_{j=1}^n P(A_j) \le \frac{K}{2\beta} P\Big(\bigcup_{k=1}^n A_k\Big) + \Big(1 - \frac{\beta}{2}\Big) \sum_{j=1}^n P(A_j).$$

So we have

$$\sum_{j=1}^n P(A_j) \le \frac{K}{\beta^2} P\Big(\bigcup_{k=1}^n A_k\Big).$$

□

Lemma 3. *Let $\{X_i, i \ge 1\}$ be a sequence of ρ-mixing random variables. Let $\{a_n\}$ be a sequence of positive numbers such that $a_{n-1}/a_n = O(1)$ and (2) are satistied. Then*

$$\sum_{n=1}^\infty P(|X_n| > a_n) < \infty. \tag{5}$$

Proof. We observe that the equality

$$\frac{X_n}{a_n} = \frac{S_n}{a_n} - \frac{a_{n-1}}{a_n} \frac{S_{n-1}}{a_{n-1}},$$

then

$$\frac{|X_n|}{a_n} = \Big|\frac{S_n}{a_n} - \frac{a_{n-1}}{a_n} \frac{S_{n-1}}{a_{n-1}}\Big| \le \Big|\frac{S_n}{a_n}\Big| + \Big|\frac{a_{n-1}}{a_n} \frac{S_{n-1}}{a_{n-1}}\Big|.$$

Using $a_{n-1}/a_n = O(1)$ and (2),

$$\frac{|X_n|}{a_n} \to 0 \quad a.s. \tag{6}$$

So we have

$$P\Big(\frac{|X_n|}{a_n} > 1, \quad i.o.\Big) = 0.$$

Then we have

$$P\Big(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\Big\{\frac{|X_n|}{a_n}>1\Big\}\Big)=0.$$

Let $A_n=\{|X_n|>a_n\}$.
So we have

$$P\Big(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}A_n\Big)=0.$$

There exists an integer m, then

$$P\Big(\bigcup_{n=m}^{\infty}A_n\Big)<\frac{1}{2}.$$

By *Lemma 1*, for all t, we have

$$Var\Big(\sum_{n=m}^{m+t}I(|X_n|>a_n)\Big)\leq C\sum_{n=m}^{m+t}P(|X|>a_n).$$

By *Lemma 2*, we have

$$\sum_{n=m}^{\infty}P(A_n)\leq CP(\bigcup_{n=m}^{\infty}A_n),$$

So we have

$$\sum_{n=1}^{\infty}P(|X_n|>a_n)=\sum_{n=1}^{m-1}P(A_n)+\sum_{n=m}^{\infty}P(A_n)\leq m-1+\frac{1}{2}C<\infty.$$

□

Proof of Theorem 3. By *Lemma 3*, $a_n=f^{-1}(n)$ and $f(x)$ is an even continuous function that is positive and strictly increasing in the region $x>0$ and satisfies the condition $f(x)\to\infty$ as $x\to\infty$. Then

$$\sum_{n=1}^{\infty}P(|X_n|>a_n)<\infty.$$

So

$$\sum_{n=1}^{\infty}P(f(X_n)>n)<\infty.$$

Thus $Ef(X_n)<\infty$. □

Corollary 1. *Let $\{X,X_i,i\geq 1\}$ be a sequence of ρ-mixing identically distributed random variables. If $\sum_{i=0}^{\infty}\rho(2^i)<\infty$ and $S_n/n^{1/p}\to 0\quad a.s.$, then $E|X|^p<\infty$.*

Proof. Let $f(x)=|x|^p, p>0$, then $a_n=f^{-1}(n)=n^{1/p}$. Thus by *Theorem 3* we have the corollary. □

Remark 1. Corollary 1 *is the Marcinkiewics-Zygmund theorem for the ρ-mixing sequence.*

Lemma 4 [Shao, 2000]. *Let* $\{X_i, i \geq 1\}$ *be a sequence of NA random variables,* $EX_i = 0$, $E|X_i|^p < \infty$ *for some* $p \geq 2$ *and for every* $i \geq 1$*. Then there exists* $C = C(p)$*, such that*

$$E \max_{1\leq k\leq n} \Big|\sum_{i=1}^{k} X_i\Big|^p \leq C\Big\{\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\Big\}.$$

Theorem 4. *Let* $\{X, X_i, i \geq 1\}$ *be a sequence of NA and identically distributed random variables. If*

$$\frac{S_n}{a_n} \to 0 \quad a.s. \tag{7}$$

then

$$Ef(X_1) < \infty. \tag{8}$$

Proof. Using *Lemma 4*, the proof of *Theorem 4* is similar to the proof of *Theorem 3.* □

Corollary 2. *Let* $\{X, X_i, i \geq 1\}$ *be a sequence of NA and identically distributed random variables. If* $\sum_{i=0}^{\infty} \rho(2^i) < \infty$ *and* $S_n/n^{1/p} \to 0 \quad a.s.$*, then* $E|X|^p < \infty$*.*

Proof. Let $f(x) = |x|^p, p > 0$, then $a_n = f^{-1}(n) = n^{1/p}$. Thus by *Theorem 4*, we have *Corollary 2.* □

Remark 2. Corollary 2 *is the Marcinkiewics-Zygmund theorem for NA sequence.*

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