$\mathcal{I}$ and $\mathcal{I}^*$ convergent function sequences

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Abstract. *In this paper, we introduce the concepts of $\mathcal{I}-$pointwise convergence, $\mathcal{I}-$uniform convergence, $\mathcal{I}^*-$pointwise convergence and $\mathcal{I}^*-$uniform convergence of function sequences and then we examine the relation between them.*

Key words: *pointwise and uniform convergence, $\mathcal{I}-$convergence, $\mathcal{I}^*-$convergence*

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1. Introduction

Steinhaus [20] introduced the idea of statistical convergence [see also Fast [10]]. If K is a subset of positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K [18] is given by $\delta(K) := \lim_n \frac{1}{n} |K_n|$, if it exists. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K := K(\varepsilon) := \{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}$ has natural density zero; in that case we write $st - \lim x = L$ $[10, 12]$. Hence x is statistically convergent to L if $\left(C_1 \chi_{K(\varepsilon)}\right)_n \to 0$ (as $n \to \infty$, for every $\varepsilon > 0$), where C_1 is the Cesáro mean of order one and χ_K is the characteristic function of the set K. Properties of statistically convergent sequences have been studied in $[2, 12, 16, 19]$.

Statistical convergence can be generalized by using a nonnegative regular summability matrix A in place of C_1.

Following Freedman and Sember [11], we say that a set $K \subseteq \mathbb{N}$ has $A-$density if $\delta_A(K) := \lim_n (A\chi_K)_n = \lim_n \sum_{k \in K} a_{nk}$ exists, where $A = (a_{nk})$ is a nonnegative regular matrix.

The number sequence $x = (x_k)$ is $A-$statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon)$ has $A-$density zero $[3, 11, 16]$.

Connor gave an extension of the notion of statistical convergence where the asymptotic density is replaced by a finitely additive set function. Let μ be a finitely additive set function taking values in $[0, 1]$ defined on a field Γ of subsets of $\mathbb{N}$ such

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that if $|A| < \infty$, then $\mu(A) = 0$; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; and $\mu(\mathbb{N}) = 1$ [4, 6].

The number of sequence $x = (x_k)$ is $\mu-$statistically convergent to L provided that $\mu\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$ for every $\varepsilon > 0$ [4, 6].

Let $X \neq \emptyset$. A class $S \subseteq 2^X$ of subsets of X is said to be an ideal in X provided that S is additive and hereditary, i.e. if S satisfies these conditions: (i) $\emptyset \in S$, (ii) $A, B \in S \Rightarrow A \cup B \in S$, (iii) $A \in S$, $B \subseteq A \Rightarrow B \in S$ [15]. An ideal is called non-trivial if $X \notin S$. A non-trivial ideal S in X is called admissible if $\{x\} \in S$ for each $x \in X$ [14].

The non-empty family of sets $\mathcal{F} \subseteq 2^X$ is a filter on X if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) for each $A \in \mathcal{F}$ and each $B \supset A$ we have $B \in \mathcal{F}$. $\mathcal{I} \subset 2^X$ is a non-trivial ideal if and only if $\mathcal{F} := \mathcal{F}(\mathcal{I}) := \{X - A : A \in \mathcal{I}\}$ is a filter on X [17].

Let $\mathcal{F}$ be a filter. $\mathcal{F}$ has property (A) if for any given countable subset $\{A_j\}$ of $\mathcal{F}$, there exists an $A \in \mathcal{F}$ such that $|A \setminus A_j| < \infty$ for each j [5].

Kostyrko, Mačaj and Šalát [14, 15] introduced two types of "ideal convergence".

Let $\mathcal{I}$ be a non-trivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ of real numbers is said to be $\mathcal{I}-$convergent to L if for every $\varepsilon > 0$ the set $A(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [14]. In this case we write $\mathcal{I}-\lim x = L$.

Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_k)$ of real numbers is said to be $\mathcal{I}^*-$convergent to L if there is a set $H \in \mathcal{I}$, such that for $M = \mathbb{N} \backslash H = \{m_1 < m_2 < ...\}$ we have $\lim_k x_{m_k} = L$. In this case we write $\mathcal{I}^*-\lim x = L$ [14].

For every admissible ideal $\mathcal{I}$ the following relation between them holds: Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. If $\mathcal{I}^*-$limit $x = L$, then $\mathcal{I}-$limit $x = L$ [14].

Note that for some ideals the converse of this result holds (see [14, Example 3.1]). Kostyrko, Mačaj and Šalát have given the necessary and sufficent condition for equivalence of $\mathcal{I}$ and $\mathcal{I}^*-$convergences. This condition is similar to the additive property for null sets in [4, 11].

An admissible ideal $\mathcal{I}$ in $\mathbb{N}$ is said to satisfy the condition (AP) if for every countable system $\{A_1, A_2, ...\}$ of mutually disjoint sets belonging to $\mathcal{I}$ there exist sets $B_j \subseteq \mathbb{N}$, $(j = 1, 2, ...)$ such that the symmetric differences $A_j \Delta B_j (j = 1, 2, ...)$ are finite and $B = \bigcup_{j=1}^{\infty} B_j$ belong to $\mathcal{I}$ [14].

It is known that $\mathcal{I}-$limit $x = L \Leftrightarrow \mathcal{I}^*-$limit $x = L$ if and only if $\mathcal{I}$ has the additive property [14]. Some results on $\mathcal{I}-$convergence may be found in [7, 8, 14, 15].

Note that if we define $\mathcal{I}_{\delta_A} = \{K \subseteq \mathbb{N} : \delta_A(K) = 0\}$, $\mathcal{I}_{\delta_{C_1}} = \{K \subseteq \mathbb{N} : \delta(K) = 0\}$ and $\mathcal{I}_\mu = \{K \subseteq \Gamma : \mu(K) = 0\}$, then we get the definition of $A-$statistical convergence, statistical convergence and $\mu-$statistical convergence, respectively.

In this paper we give the $\mathcal{I}$ analogues of results given by Duman and Orhan [9].

Throughout the paper $\mathcal{I}$ will be an admissible ideal, $D \subseteq \mathbb{R}$ and (f_n) a sequence of real functions on D.

2. $\mathcal{I}$ and $\mathcal{I}^*$ convergent function sequences

Definition 1. *(f_n) converges $\mathcal{I}^*-$pointwise to f* $\Leftrightarrow \forall\, \varepsilon > 0$ *and* $\forall\, x \in D$, $\exists\, K_x \notin \mathcal{I}$ *and* $\exists n_0 = n_{0(\varepsilon,x)} \in K_x \ni \forall n \geq n_0$ *and* $n \in K_x$, $|f_n(x) - f(x)| < \varepsilon$.

In this case we will write $f_n \to f\,(\mathcal{I}^* - \textit{convergent})$ on D.

Definition 2. *We say that* (f_n) *converges* $\mathcal{I}^*$*−uniform to* $f \iff \forall\, \varepsilon > 0$ *and* $\forall\, x \in D$, $\exists\, K \notin \mathcal{I}$ *and* $\exists n_0 = n_{0(\varepsilon)} \in K \ni \forall n \geq n_0$ *and* $n \in K$, $|f_n(x) - f(x)| < \varepsilon$.

In this case we will write $f_n \rightrightarrows f\,(\mathcal{I}^* - \textit{convergent})$ on D.

Definition 3. (f_n) *converges* $\mathcal{I}$*−pointwise to* $f \iff \forall\, \varepsilon > 0$ *and* $\forall\, x \in D$,
$\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$.

In this case we will write $f_n \to f\,(\mathcal{I} - \textit{convergent})$ on D.

Definition 4. *The sequence* (f_n) *of bounded functions on* D *converges* $\mathcal{I}$*-uniformly to* $f \iff \forall\, \varepsilon > 0$ *and* $\forall\, x \in D$, $\{n : \|f_n - f\| \geq \varepsilon\} \in \mathcal{I}$, *where the form* $\|.\|_{B(D)}$ *is the usual supremum norm on* $B(D)$, *the space of bounded functions on* D.

In this case we will write $f_n \rightrightarrows f\,(\mathcal{I} - \textit{convergent})$ on D.

As in the ordinary case the property of *Definition 1* implies that of *Definition 3*; and, of course for bounded functions, the property of *Definition 2* implies that of *Definition 4*. If $\mathcal{I}$ satisfy the condition (AP), then *Definitions 1* and *3* are equivalent, and *Definition 2* and *4* are equivalent.

The following result is a $\mathcal{I}$ analogue of the result that is well-known in analysis.

Theorem 1. *Let for all* n, f_n *be continuous on* D. *If* $f_n \rightrightarrows f\,(\mathcal{I}^* - \textit{convergent})$ *on* D, *then* f *is continuous on* D.

Proof. Assume $f_n \rightrightarrows f\,(\mathcal{I}^* - \textit{convergent})$ on D. Then for every $\varepsilon > 0$, there exists a set $K \notin \mathcal{I}$ and $\exists n_0 = n_{0(\varepsilon)} \in K$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for each $x \in D$ and for all $n \geq n_o$ and $n \in K$. Let $x_0 \in D$. Since f_{n_0} is continuous at $x_0 \in D$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}$ for each $x \in D$. Now for all $x \in D$ for which $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \varepsilon.$$

Since $x_0 \in D$ is arbitrary, f is continuous on D. □

Now from *Theorem 5* we get the following.

Corollary 1. *Let all functions* f_n *be continuous on a compact subset* D *of* $\mathbb{R}$, *and let* $\mathcal{I}$ *satisfy the condition (AP).*

If $f_n \rightrightarrows f\,(\mathcal{I} - \textit{convergent})$ *on* D, *then* f *is continuous on* D.

The next example shows that neither of the converses of *Theorem 5* and *Corollary 6* are true.

Example 1. *Let* $K \notin \mathcal{I}$ *and define* $f_n : [0,1) \to \mathbb{R}$ *by*

$$f_n(x) = \begin{cases} \frac{1}{2} & , n \notin K \\ \frac{x^n}{1+x^n} & , n \in K. \end{cases}$$

Then we have $f_n \to f = 0\,(\mathcal{I}^* - \textit{convergent})$ on $[0,1)$. Hence we get $f_n \to f = 0\,(\mathcal{I} - \textit{convergent})$ on $[0,1)$. Though all f_n and f are continuous on $[0,1)$, it follows from *Definition 4* that the $\mathcal{I} - \textit{convergent}$ of (f_n) is not uniform for

$$c_n := \sup_{x \in [0,1)} |f_n(x) - f(x)| = \frac{1}{2} \quad \text{and} \quad I - \lim c_n = \frac{1}{2} \neq 0.$$

Now we will give the following result that is an analogue of Dini's theorem.

Theorem 2. *Let $\mathcal{I}$ satisfy the condition (AP). Let D be a compact subset of $\mathbb{R}$ and (f_n) a sequence of continuous functions on D. Assume that f is continuous and $f_n \to f\,(\mathcal{I}-convergent)$ on D. Also, let (f_n) be monotonic decreasing on D; i.e. $f_n(x) \geq f_{n+1}(x)$ $(n = 1, 2, ...)$ for every $x \in D$. Then $f_n \rightrightarrows f\,(\mathcal{I}-convergent)$ on D.*

Proof. Write $g_n(x) = f_n(x) - f(x)$. By hypothesis, each g_n is continuous and $g_n \to 0\,(\mathcal{I}-convergent)$ on D, also (g_n) is a monotonic decreasing sequence on D. Now, since $g_n \to 0\,(\mathcal{I}-convergent)$ on D and $\mathcal{I}$ satisfy the condition (AP), $g_n \to 0\,(I^*-convergent)$ on D. Hence for every $\varepsilon > 0$ and each $x \in D$ there exists $K_x \notin \mathcal{I}$ and a number $n(x) = n(x, \varepsilon) \in K_x$ such that $0 \leq g_n(x) < \frac{\varepsilon}{2}$ for all $n \geq n(x)$ and $n \in K_x$. Since $g_n(x)$ is continuous at $x \in D$, for every $\varepsilon > 0$ there is an open set $J(x)$ which contains x such that $\left|g_{n(x)}(t) - g_{n(x)}(x)\right| < \frac{\varepsilon}{2}$ for all $t \in J(x)$. Hence given $\varepsilon > 0$, by monotonicity we have

$$\begin{aligned} 0 &\leq g_n(t) \leq g_{n(x)}(t) = g_{n(x)}(t) - g_{n(x)}(x) + g_{n(x)}(x) \\ &\leq \left|g_{n(x)}(t) - g_{n(x)}(x)\right| + g_{n(x)}(x) \end{aligned}$$

for every $t \in J(x)$ and for all $n \geq n(x)$ and $n \in K_x$. Since $D \subset \bigcup_{x \in D} J(x)$ and D is a compact set, by the Heine-Borel theorem D has a finite open covering such that $D \subset J(x_1) \cup J(x_2) \cup ... \cup J(x_m)$. Now, let $K := K_{x_1} \cap K_{x_2} \cap ... \cap K_{x_m}$ and $N := \max\{n(x_1), n(x_2), ..., n(x_m)\}$. Observe that $K \notin \mathcal{I}$. Then $0 \leq g_n(t) < \varepsilon$ for every $t \in D$ and for all $n \geq N$ and $n \in K$. So $g_n \rightrightarrows 0\,(\mathcal{I}^*-convergent)$ on D. Consequently $g_n \rightrightarrows 0\,(\mathcal{I}-convergent)$ on D, which completes the proof. □

Now we will give the Cauchy criterion for $\mathcal{I}-$uniform convergence but we first need a definition and a lemma:

Definition 5. (f_n) *is $\mathcal{I}-$Cauchy if for every $\varepsilon > 0$ and every $x \in D$ there is an $n(\varepsilon) \in \mathbb{N}$ such that*

$$\left\{n : \left|f_n(x) - f_{n(\varepsilon)}(x)\right| \geq \varepsilon\right\} \in \mathcal{I}$$

Lemma 1. *Let (f_n) be a sequence of a real function on D. (f_n) is $\mathcal{I}-$convergent if and only if (f_n) is $\mathcal{I}-$Cauchy.*

Proof. First we establish that a $\mathcal{I}-$convergent sequence is $\mathcal{I}-$Cauchy. Suppose that (f_n) is $\mathcal{I}-$convergent to f. Since $\left\{n : |f_n(x) - f(x)| < \frac{\varepsilon}{2}\right\} \notin \mathcal{I}$ .We can select an $n(\varepsilon) \in \mathbb{N}$ such that $\left|f_{n(\varepsilon)}(x) - f(x)\right| < \frac{\varepsilon}{2}$. The triangle inequality now yields that $\left\{n : \left|f_n(x) - f_{n(\varepsilon)}(x)\right| < \varepsilon\right\} \notin \mathcal{I}$. Since ε was arbitrary, (f_n) is $\mathcal{I}-$Cauchy.

Now suppose that (f_n) is $\mathcal{I}-$Cauchy. Select $n(1)$ such that

$$\left\{n : \left|f_n(x) - f_{n(1)}(x)\right| < 1\right\} \notin \mathcal{I}$$

and let $A_1 = \left\{n : \left|f_n(x) - f_{n(1)}(x)\right| < 1\right\}$. Suppose that

$$n(1) < n(2) < n(3) < ... < n(p)$$

have been selected in such a fashion that if $1 \leq r \leq s \leq p$ and

$$A_s = \left\{n : \left|f_n(x) - f_{n(s)}(x)\right| < 1/2^{s-1}\right\}$$

then $A_r \notin \mathcal{I}$ and $n(s) \in A_r$. Select N such that

$$\left\{n : |f_n(x) - f_N(x)| < 1/2^{P+1}\right\} \notin \mathcal{I}.$$

Since $\bigcap_1^N A_j \bigcap \left\{n : |f_n(x) - f_N(x)| < 1/2^{P+1}\right\} \notin \mathcal{I}$, there exists an

$$n(p+1) \in \bigcap_1^N A_j \bigcap \left\{n : |f_n(x) - f_N(x)| < 1/2^{P+1}\right\}$$

such that $n(p) < n(p+1)$ and

$$A_{p+1} = \left\{n : \left|f_n(x) - f_{n(p+1)}(x)\right| < 1/2^p\right\} \supseteq \left\{n : |f_n(x) - f_N(x)| < 1/2^{P+1}\right\}.$$

Observe that $A_{p+1} \notin \mathcal{I}$ and $n(p+1) \in A_s$ for all $s \leq p+1$.

Note that since $\left|f_{n(p)}(x) - f_{n(p+1)}(x)\right| < 2^{-p}$, $\left(f_{n(p)}(x)\right)$ is Cauchy, and hence there exists an $f(x)$ such that $\lim_p f_{n(p)}(x) = f(x)$. We claim that (f_n) is $\mathcal{I}-$convergent to $f(x)$. Let $\varepsilon > 0$ be given and select $p \in \mathbb{N}$ such that

$$\left|f_{n(p)}(x) - f(x)\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \varepsilon > 2^{-p}.$$

Note that if $|f_n(x) - f(x)| \geq \varepsilon$, then $\left|f_{n(p)}(x) - f_n(x)\right| \geq \frac{\varepsilon}{2} > 2^{1-p}$, and hence n is not an element of A_p. It follows that $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$ and that (f_n) is $\mathcal{I}-$convergent to $f(x)$. □

Theorem 3. *Let $\mathcal{I}$ satisfy the condition (AP) and let (f_n) be a sequence of bounded functions on D. Then (f_n) is $\mathcal{I}-$uniformly convergent on D if and only if for every $\varepsilon > 0$ there is an $n(\varepsilon) \in \mathbb{N}$ such that*

$$\left\{n : \left\|f_n - f_{n(\varepsilon)}\right\|_{B(D)} < \varepsilon\right\} \notin \mathcal{I} \tag{1}$$

Note: The sequence (f_n) satisfying property (1) is said to be $\mathcal{I}-$uniformly Cauchy on D.

Proof. Assume that (f_n) converges $\mathcal{I}-$uniformly to a function f defined on D. Let $\varepsilon > 0$. Then we have $\left\{n : \|f_n - f\|_{B(D)} < \varepsilon\right\} \notin \mathcal{I}$. We can select an $n(\varepsilon) \in \mathbb{N}$ such that $\left\{n : \left\|f_{n(\varepsilon)} - f\right\|_{B(D)} < \varepsilon\right\} \notin \mathcal{I}$. The triangle inequality yields that $\left\{n : \left\|f_n - f_{n(\varepsilon)}\right\|_{B(D)} < \varepsilon\right\} \notin \mathcal{I}$. Since ε is arbitrary, (f_n) is $\mathcal{I}-$uniformly Cauchy on D.

Conversely, assume that (f_n) is $\mathcal{I}-$uniformly Cauchy on D. Let $x \in D$ be fixed. By (1), for every $\varepsilon > 0$ there is an $n(\varepsilon) \in \mathbb{N}$ such that $\left\{n : \left|f_n(x) - f_{n(\varepsilon)}(x)\right| < \varepsilon\right\} \notin \mathcal{I}$. Hence $\{f_n(x)\}$ is $\mathcal{I}-$Cauchy, so by *Lemma 10* we have that $\{f_n(x)\}$ converges $\mathcal{I}-$convergent to $f(x)$. Then $f_n \to f\,(\mathcal{I} - \textit{convergent})$ on D. Now we shall show that this convergence must be uniform. Note that since $\mathcal{I}$ satisfy the condition (AP), by (1) there is a $K \notin \mathcal{I}$ such that $\left\|f_n - f_{n(\varepsilon)}\right\|_{B(D)} < \frac{\varepsilon}{2}$ for all $n \geq n(\varepsilon)$ and $n \in K$. So for every $\varepsilon > 0$ there is a $K \notin \mathcal{I}$ and $n(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon \tag{2}$$

for all $n, m \geq n(\varepsilon)$ and $n, m \in K$ and for each $x \in D$. Fixing n and applying the limit operator on $m \in K$ in (2), we conclude that for every $\varepsilon > 0$ there is a $K \notin \mathcal{I}$ and an $n(\varepsilon) \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_0$ and for each $x \in D$. Hence $f_n \rightrightarrows f\,(\mathcal{I}^* - convergent)$ on D, consequently $f_n \rightrightarrows f\,(\mathcal{I} - convergent)$ on D. □

3. Applications

Using $\mathcal{I}-$uniform convergence, we can also get some applications. We merely state the following theorems and omit the proofs.

Theorem 4. *Let $\mathcal{I}$ satisfy the condition (AP). If a function sequence (f_n) converges $\mathcal{I}-$uniformly on $[a,b]$ to a function f and f_n is integrable on $[a,b]$, then f is integrable on $[a,b]$. Moreover,*

$$\mathcal{I} - \lim \int_a^b f_n(x)\,dx = \int_a^b \mathcal{I} - \lim f_n(x)\,dx = \int_a^b f(x)\,dx$$

Theorem 5. *Let $\mathcal{I}$ satisfy the condition (AP). Suppose that (f_n) is a function sequence such that each (f_n) has a continuous derivative on $[a,b]$. If $f_n \to f\,(\mathcal{I} - convergent)$ on $[a,b]$ and $f_n' \rightrightarrows g\,(\mathcal{I} - convergent)$ on $[a,b]$, then $f_n \rightrightarrows f\,(\mathcal{I} - convergent)$ on $[a,b]$, where f is differentiable, and $f' = g$.*

4. Function sequences that preserve $\mathcal{I}-$ convergence

This section is motivated by a paper of Kolk [13]. Recall that function sequence (f_n) is called convergence-preserving (or conservative) on $D \subseteq \mathbb{R}$ if the transformed sequence $\{f_n(x)\}$ converges for each convergent sequence $x = (x_n)$ from D [13]. In this section, analogously, we describe the function sequences which preserve the $\mathcal{I}-$convergence of sequences. Our arguments also give a sequential characterization of the continuty of $\mathcal{I}-$limit functions of $\mathcal{I}-$uniformly convergent function sequences. This result is complementary to *Theorem 5.*

First we introduce the following definition.

Definition 6. *Let $D \subseteq \mathbb{R}$ and (f_n) be a sequence of real functions on D. Then (f_n) is called a function sequence preserving $\mathcal{I}$-convergence (or $\mathcal{I}$-convergent conservative) on D if the transformed sequence $\{f_n(x)\}$ converges $\mathcal{I}$ for each $\mathcal{I}-$convergent sequence $x = (x_n)$ from D. If (f_n) is $\mathcal{I}-$convergent conservative and preserves the limits of all $\mathcal{I}-$convergent sequences from D, then (f_n) is called $\mathcal{I}-$convergent regular on D.*

Hence, if (f_n) is conservative on D, then (f_n) is $\mathcal{I}-$convergent conservative on D. But the following example shows that the converse of this result is not true.

Example 2. *Let $K \notin \mathcal{I}$. Define $f_n : [0,1] \to \mathbb{R}$ by*

$$f_n(x) = \begin{cases} 0 \ , \ n \in K \\ 1 \ , \ n \notin K \end{cases}$$

Suppose that (x_n) from $[0,1]$ is an arbitrary sequence such that $\mathcal{I}-\lim x = L$. Then, for every $\varepsilon > 0$, $\{n : |f_n(x_n) - 0| \geq \varepsilon\} \in \mathcal{I}$. Hence $\mathcal{I}-\lim f_n(x_n) = 0$, so (f_n) is $\mathcal{I}-$convergent conservative on $[0,1]$. But observe that (f_n) is not conservative on $[0,1]$.

Now we give the first result of this section. But we need the following lemma:

Lemma 2. *Let $\mathcal{I}$ satisfy the condition (AP). If $(f_n^r(x))$ is a countable collection of sequences that are $\mathcal{I}^*-$convergent, then there exists $\lambda : \mathbb{N} \to \mathbb{N}$ such that $\lim_n f_{\lambda(k)}^r(x)$ exists for each r and $\{\lambda(k) : k \in \mathbb{N}\} \notin \mathcal{I}$.*

Proof. Let F be the filter generated by convergence in $\mathcal{I}^*-$convergence. Since each $(f_n^r(x))$ is $\mathcal{I}^*-$convergent, there is an $A^r \in F$ such that $(f_n^r(x)) \in c_{A^r}$. Since F has property (A), there is an $A \in F$ such that $|A \setminus A^r| < \infty$ for each r. Suppose $A = \{n_1, n_2, ...\}$ where $n_1 < n_2 < ...$ and $\lambda : \mathbb{N} \to \mathbb{N}$ satisfies $\lambda(k) = n_k$ for all k. Now $\lim_n f_{\lambda(k)}^r(x)$ exists for each r and $\{\lambda(k) : k \in \mathbb{N}\} = A \notin \mathcal{I}$. □

Theorem 6. *Let $\mathcal{I}$ satisfy the condition (AP) and let (f_k) be a sequence of functions defined on closed interval $[a,b] \subset \mathbb{R}$. Then (f_k) is $\mathcal{I}-$convergent conservative on $[a,b]$ if and only if (f_k) converges $\mathcal{I}-$uniformly convergent on $[a,b]$ to a continuous function.*

Proof. *Necessity.* Assume that (f_k) is $\mathcal{I}-$convergent conservative on $[a,b]$. Choose the sequence $(v_k) = (t, t, ...)$ for each $t \in [a,b]$. Since $\mathcal{I} - \lim v_k = t$, $\mathcal{I} - \lim f_k(v_k)$ exists, hence $\mathcal{I} - \lim f_k(v_k) = f(t)$ for all $t \in [a,b]$. We claim that f is continuous on $[a,b]$. To prove this we suppose that f is not continuous at a point $t_0 \in [a,b]$. Then there exists a sequence (u_k) in $[a,b]$ such that $\lim u_k = t_0$, but $\lim f(u_k)$ exists and $\lim f(u_k) \neq f(t_0)$. Since $f_k \to f\,(\mathcal{I} - convergent)$ on $[a,b]$ and $\mathcal{I}$ satisfy the condition (AP), we obtain $f_k \to f\,(\mathcal{I}^* - convergent)$ on $[a,b]$. Hence, for each j, $\{f_k(u_j) - f(u_j)\} \to 0\,(\mathcal{I}^* - convergent)$. Hence there exists $\lambda : \mathbb{N} \to \mathbb{N}$ such that $\{\lambda(k) : k \in \mathbb{N}\} \notin \mathcal{I}$ and

$$\lim \left[f_{\lambda(k)}(u_j) - f(u_j)\right] = 0$$

for each j. Now, by the "diagonal process" $[1, p.192]$ we can choose an increasing index sequence (n_k) in such a way that $\{n_k : k \in \mathbb{N}\} \notin \mathcal{I}$ and $\lim\left[f_{n_k}(u_k) - f(u_k)\right] = 0$. Now define a sequence $x = (t_i)$ by

$$t_i = \begin{cases} t_0 \ , \ i = n_k \ and \ i \ is \ odd \\ u_k \ , \ i = n_k \ and \ i \ is \ even \\ 0 \quad , \ otherwise. \end{cases}$$

Hence $t_i \to t_0\,(\mathcal{I}^* - convergent)$, which implies $\mathcal{I} - \lim t_i = t_0$. But if $i = n_k$ and i is odd, then $\lim f_{n_k}(t_0) = f(t_0)$, and if $i = n_k$ and i is even, then $\lim f_{n_k}(u_k) = \lim\left[f_{n_k}(u_k) - f(u_k)\right] + \lim f(u_k) \neq f(t_0)$. Hence $\{f_i(t_i)\}$ is not $\mathcal{I}^*-$ convergent since the sequence $\{f_i(t_i)\}$ converges to two different limit points and has two disjoint subsequences whose index set does not belong to $\mathcal{I}$. So, the sequence $\{f_i(t_i)\}$ is not $\mathcal{I}-$convergent, which contradicts the hypothesis. Thus f must be continuous on $[a,b]$. It remains to prove that (f_k) converges $\mathcal{I}-$convergent uniformly on $[a,b]$ to f. Assume that (f_k) is not $\mathcal{I}-$uniformly convergent on $[a,b]$ to f, then (f_k) is not $\mathcal{I}^*-$uniformly convergent on $[a,b]$ to f. Hence, for an arbitrary index sequence (n_k) with $\{n_k : k \in \mathbb{N}\} \notin \mathcal{I}$, there exists a number $\varepsilon_0 > 0$

and numbers $t_k \in [a,b]$ such that $|f_{n_k}(t_k) - f(t_k)| \geq 2\varepsilon_0$ $(k \in \mathbb{N})$. The bounded sequence $x = (t_k)$ contains a convergent subsequence (t_{k_i}), $\mathcal{I} - \lim t_{k_i} = \alpha$, say. By the continuity of f, $\lim f(t_{k_i}) = f(\alpha)$. So there is an index i_0 such that $|f(t_{k_i}) - f(\alpha)| < \varepsilon_0$ $(i \geq i_o)$. For the same i's, we have

$$\left|f_{n_{k_i}}(t_{k_i}) - f(\alpha)\right| \geq \left|f_{n_{k_i}}(t_{k_i}) - f(t_{k_i})\right| - |f(t_{k_i}) - f(\alpha)| \geq \varepsilon_0. \tag{3}$$

Now, defining

$$u_j = \begin{cases} \alpha & , \ j = n_{k_i} \text{ and } j \text{ is odd} \\ t_{k_i} & , \ j = n_{k_i} \text{ and } j \text{ is even} \\ 0 & , \text{ otherwise,} \end{cases}$$

we get $u_j \to \alpha\,(\mathcal{I}^* - convergent)$. Hence $\mathcal{I}-\lim u_j = \alpha$. But if $j = n_{k_i}$ and j is odd, then $\lim f(t_{k_i}) = f(\alpha)$, and if $j = n_{k_i}$ and j is even, then, by (3), $\lim f(t_{k_i}) \neq f(\alpha)$. Hence $\{f_i(t_i)\}$ is not $\mathcal{I}^*-$convergent since the sequence $\{f_i(t_i)\}$ converges to two different limit points and has two disjoint subsequences whose index set does not belong to $\mathcal{I}$ So, the sequence $\{f_i(t_i)\}$ is not $\mathcal{I}-$convergent, which contradicts the hypothesis. Thus (f_k) must be $\mathcal{I}-$uniformly convergent to f on $[a,b]$.

Sufficiency. Assume that $f_n \rightrightarrows f\,(\mathcal{I} - convergent)$ on $[a,b]$ and f is continuous. Let $x = (x_n)$ be a $\mathcal{I}-$convergent sequence in $[a,b]$ with $\mathcal{I}-\lim x_n = x_0$. Since $\mathcal{I}$ satisfy the condition (AP), $x_n \to x_0\,(\mathcal{I}^* - convergent)$, so there is an index sequence $\{n_k\}$ such that $\lim x_{n_k} = x_0$ and $\{n_k : k \in \mathbb{N}\} \notin \mathcal{I}$. By the continuity of f at x_0, $\lim f(x_{n_k}) = f(x_0)$. Hence $f(x_n) \to f(x_0)\,(\mathcal{I}^* - convergent)$. Let $\varepsilon > 0$ be given. Then there exists $K_1 \notin \mathcal{I}$ and a number $n_1 \in K_1$ such that $|f(x_n) - f(x_0)| < \frac{\varepsilon}{2}$ for all $n \geq n_1$ and $n \in K_1$. By assumption $\mathcal{I}$ satisfy the condition (AP). Hence the $\mathcal{I}-$uniform convergence is equivalent to the $\mathcal{I}^*-$uniform convergence, so there exists a $K_2 \notin \mathcal{I}$ and a number $n_2 \in K_2$ such that $|f_n(t) - f(t)| < \frac{\varepsilon}{2}$ for every $t \in [a,b]$ for all $n \geq n_2$ and $n \in K_2$. Let $N := \max\{n_1, n_2\}$ and $K := K_1 \cap K_2$. Observe that $K \notin \mathcal{I}$. Hence taking $t = x_n$ we have

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < \varepsilon$$

for all $n \geq N$ and $n \in K$. This shows that $f_n(x_n) \to f(x_0)\,(\mathcal{I}^* - convergent)$ which necessarily implies that $\mathcal{I} - \lim f_n(x_n) = f(x_0)$, whence the proof follows. □

Theorem 6 contains the following necessary and sufficient condition for the continuity of $\mathcal{I}-$convergence limit functions of function sequences that converge $\mathcal{I}-$convergent uniformly on a closed interval.

Theorem 7. *Let $\mathcal{I}$ satisfy the condition (AP) and let (f_k) be a sequence of functions that converges $\mathcal{I}-$convergent uniformly on a closed interval $[a,b]$ to a function f. The function f is continuous on $[a,b]$ if and only if (f_k) is $\mathcal{I}-$convergent conservative on $[a,b]$.*

Now, we study the $\mathcal{I}-$convergence regularity of function sequences. If (f_k) is $\mathcal{I}-$convergent regular on $[a,b]$, then obviously $\mathcal{I} - \lim f_k(t) = t$ for all $t \in [a,b]$. So, taking $f(t) = t$ in *Theorem 6*, we immediately get the following.

Theorem 8. *Let $\mathcal{I}$ satisfy the condition (AP) and let (f_k) be a sequence of functions on $[a,b]$. Then (f_k) is $\mathcal{I}-$convergent regular on $[a,b]$ if and only if $\mathcal{I}-$convergent uniformly on $[a,b]$ to the function f defined by $f(t) = t$.*

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