The equivalence of Picard, Mann and Ishikawa iterations dealing with quasi-contractive operators

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Abstract. *We show that the Ishikawa iteration, the corresponding Mann iteration and the Picard iteration are equivalent when applied to quasi-contractive operators.*

Key words: *Picard iteration, Mann iteration, Ishikawa iteration, quasi-contractive operators*

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1. Introduction

Let X be a real Banach space, D a nonempty, convex subset of X, and T a selfmap of D, let $x_0 = u_0 = p_0 \in D$. The Mann iteration (see [3]) is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n. \tag{1}$$

The Ishikawa iteration is defined (see [2]) by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2}$$

where $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset [0, 1)$.

The Picard iteration is given by

$$p_{n+1} = T p_n. \tag{3}$$

Definition 1. [4] *The operator* $T : X \to X$ *satisfies condition* Z *if and only if there exist real numbers* a, b, c *satisfying* $0 < a < 1,\ 0 < b, c < 1/2$ *such that for each pair* x, y *in* X, *at least one condition is true*

- (z_1) $\|Tx - Ty\| \le a\ \|x - y\|$,
- (z_2) $\|Tx - Ty\| \le b\ (\|x - Tx\| + \|y - Ty\|)$,

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- (z_3) $\|Tx - Ty\| \leq c\ (\|x - Ty\| + \|y - Tx\|)$.

It is known, see Rhoades [5], that (z_1), (z_2) and (z_3) are independent conditions.

In [6] the following conjecture was given: "if the Mann iteration converges, then so does the Ishikawa iteration". In a series of papers [6], [7], [8], [9], [10], Professor B. E. Rhoades and I have given a positive answer to this Conjecture, showing the equivalence between Mann and Ishikawa iterations for strongly and uniformly pseudocontractive maps. In this paper we show that the convergence of the Mann iteration is equivalent to the convergence of the Ishikawa iteration and both are equivalent to the Picard iteration, when applied to a map which satisfies condition Z. A map satisfying condition Z is independent, see Rhoades [4], of the class of strongly pseudocontractive maps.

Lemma 1. [12] *Let $(a_n)_n$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \sigma_n, \tag{4}$$

where $\lambda_n \in (0,1)$, $\forall n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main result

Let $F(T)$ denote the fixed point set with respect to D for the map T. Suppose that $x^* \in F(T)$.

Theorem 1. *Let X be a normed space, D a nonempty, convex, closed subset of X and $T : D \to D$ an operator satisfying condition Z. If $u_0 = x_0 \in D$, then the following are equivalent:*

(i) the Mann iteration (1) converges to x^*,

(ii) the Ishikawa iteration (2) converges to x^*.

Proof. Consider $x, y \in D$. Since T satisfies condition Z, at least one of the conditions from (z_1), (z_2) and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned}\|Tx - Ty\| &\leq b\ (\|x - Tx\| + \|y - Ty\|)\\ &\leq b\,(\|x - Tx\| + (\|y - x\| + \|x - Tx\| + \|Tx - Ty\|)),\end{aligned}$$

thus

$$(1 - b)\,\|Tx - Ty\| \leq b\,\|x - y\| + 2b\,\|x - Tx\|.$$

From $0 \leq b < 1$ we get

$$\|Tx - Ty\| \leq \frac{b}{1-b}\,\|x - y\| + \frac{2b}{1-b}\,\|x - Tx\|.$$

If (z_3) holds, then we obtain

$$\begin{aligned}\|Tx - Ty\| &\leq c\ (\|x - Ty\| + \|y - Tx\|)\\ &\leq c\,(\|x - Tx\| + \|Tx - Ty\| + \|x - y\| + \|x - Tx\|),\end{aligned}$$

hence

$$(1-c)\|Tx-Ty\| \leq c\|x-y\|+2c\|x-Tx\| \quad i.e.$$
$$\|Tx-Ty\| \leq \frac{c}{1-c}\|x-y\|+\frac{2c}{1-c}\|x-Tx\|.$$

Denote

$$\delta := \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\},$$

to obtain

$$0 \leq \delta < 1.$$

Finally, we get

$$\|Tx-Ty\| \leq \delta\|x-y\|+2\delta\|x-Tx\|, \forall x, y \in D. \tag{5}$$

Formula (5) was obtained as in [1].

We will prove the implication $(i) \Rightarrow (ii)$. Suppose that $\lim_{n\to\infty} u_n = x^*$. Using

$$\lim_{n\to\infty}\|x_n-u_n\| = 0, \tag{6}$$

and

$$0 \leq \|x^*-x_n\| \leq \|u_n-x^*\|+\|x_n-u_n\|$$

we get

$$\lim_{n\to\infty} x_n = x^*.$$

The proof is complete if we prove relation (6).

Using now (1), (2) and (5) with

$$x := u_n, \quad y := y_n,$$

we have

$$\begin{aligned}\|u_{n+1}-x_{n+1}\| &\leq \|(1-\alpha_n)(u_n-x_n)+\alpha_n(Tu_n-Ty_n)\| \\ &\leq (1-\alpha_n)\|u_n-x_n\|+\alpha_n\|Tu_n-Ty_n\| \\ &\leq (1-\alpha_n)\|u_n-x_n\|+\alpha_n\delta\|u_n-y_n\| \\ &\quad +2\alpha_n\delta\|u_n-Tu_n\|.\end{aligned} \tag{7}$$

Using (5) with $x := u_n,\ y := y_n$, we have

$$\begin{aligned}\|u_n-y_n\| &\leq \|(1-\beta_n)(u_n-x_n)+\beta_n(u_n-Tx_n)\| \\ &\leq (1-\beta_n)\|u_n-x_n\|+\beta_n\|u_n-Tx_n\| \\ &\leq (1-\beta_n)\|u_n-x_n\|+\beta_n\|u_n-Tu_n\| \\ &\quad +\beta_n\|Tu_n-Tx_n\| \\ &\leq (1-\beta_n)\|u_n-x_n\|+\beta_n\|u_n-Tu_n\| \\ &\quad +\beta_n\delta\|u_n-x_n\|+2\delta\beta_n\|u_n-Tu_n\| \\ &= (1-\beta_n(1-\delta))\|u_n-x_n\| \\ &\quad +\beta_n\|u_n-Tu_n\|(1+2\delta).\end{aligned} \tag{8}$$

Relations (7) and (8) lead to

$$\begin{aligned}
\|u_{n+1}-x_{n+1}\| &\leq (1-\alpha_n)\|u_n-x_n\| \\
&\quad +\alpha_n\delta(1-\beta_n(1-\delta))\|u_n-x_n\| \\
&\quad +\alpha_n\beta_n\delta\|u_n-Tu_n\|(1+2\delta) \\
&\quad +\alpha_n\delta\|u_n-y_n\| \\
&= (1-\alpha_n(1-\delta(1-\beta_n(1-\delta))))\|u_n-x_n\| \\
&\quad +\alpha_n\delta\|u_n-Tu_n\|(\beta_n(1+2\delta)+2\delta).
\end{aligned} \tag{9}$$

Denote by

$$\begin{aligned}
a_n &:= \|u_n-x_n\|, \\
\lambda_n &:= \alpha_n(1-\delta(1-\beta_n(1-\delta))) \subset (0,1), \\
\sigma_n &:= \alpha_n\delta\|u_n-Tu_n\|(\beta_n(1+2\delta)+2\delta).
\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n-x^*\|=0,$ T satisfies condition Z, and $x^*\in F(T)$, from (5) we obtain

$$\begin{aligned}
0 \leq \|u_n-Tu_n\| &\leq \|u_n-x^*\|+\|x^*-Tu_n\| \\
&\leq (\delta+1)\|u_n-x^*\| \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n-Tu_n\|=0$; that is, $\sigma_n=o(\lambda_n)$. *Lemma 1* leads to

$$\lim_{n\to\infty}\|u_n-x_n\|=0.$$

We will prove now that if the Ishikawa iteration converges, then the Mann iteration does too. Using (5) with

$$x:=y_n, \quad y:=u_n,$$

we obtain

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| &\leq \|(1-\alpha_n)(x_n-u_n)+\alpha_n(Ty_n-Tu_n)\| \\
&\leq (1-\alpha_n)\|x_n-u_n\|+\alpha_n\|Ty_n-Tu_n\| \\
&\leq (1-\alpha_n)\|x_n-u_n\|+\alpha_n\delta\|y_n-u_n\| \\
&\quad +2\alpha_n\delta\|y_n-Ty_n\|.
\end{aligned} \tag{10}$$

The following relation holds

$$\begin{aligned}
\|y_n-u_n\| &\leq \|(1-\beta_n)(x_n-u_n)+\beta_n(Tx_n-u_n)\| \\
&\leq (1-\beta_n)\|x_n-u_n\|+\beta_n\|Tx_n-u_n\| \\
&\leq (1-\beta_n)\|x_n-u_n\|+\beta_n\|Tx_n-x_n\| \\
&\quad +\beta_n\|x_n-u_n\| \\
&\leq \|x_n-u_n\|+\beta_n\|Tx_n-x_n\|.
\end{aligned} \tag{11}$$

Substituting (11) in (10), we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq (1-\alpha_n)\|x_n-u_n\|+\alpha_n\delta\left(\|x_n-u_n\|+\beta_n\|Tx_n-x_n\|\right) \\ &\quad +2\alpha_n\delta\|y_n-Ty_n\| \\ &\leq (1-(1-\delta)\alpha_n)\|x_n-u_n\|+\alpha_n\beta_n\delta\|Tx_n-x_n\| \\ &\quad +2\alpha_n\delta\|y_n-Ty_n\|.\end{aligned} \tag{12}$$

Denote by

$$\begin{aligned}a_n &:= \|x_n-u_n\|, \\ \lambda_n &:= \alpha_n(1-\delta)\subset(0,1), \\ \sigma_n &:= \alpha_n\beta_n\delta\|Tx_n-x_n\|+2\alpha_n\delta\|y_n-Ty_n\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-x^*\|=0,$ T satisfies condition Z, and $x^*\in F(T)$, from (5) we obtain

$$0\leq\|x_n-Tx_n\|\leq\|x_n-x^*\|+\|x^*-Tx_n\|\leq(\delta+1)\|x_n-x^*\|\to 0 \text{ as } n\to\infty,$$

and

$$\begin{aligned}0 &\leq \|y_n-Ty_n\| \\ &\leq \|y_n-x^*\|+\|x^*-Ty_n\| \\ &\leq (\delta+1)\|y_n-x^*\| \\ &\leq (\delta+1)\ \left[(1-\beta_n)\|x_n-x^*\|+\beta_n\|Tx_n-x^*\|\right] \\ &\leq (\delta+1)\left[(1-\beta_n)\|x_n-x^*\|+\beta_n\delta\|x_n-x^*\|\right] \\ &\leq (\delta+1)(1-\beta_n(1-\delta))\|x_n-x^*\|\to 0 \text{ as } n\to\infty,\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ and $\lim_{n\to\infty}\|y_n-Ty_n\|=0$, that is, $\sigma_n=o(\lambda_n)$. *Lemma 1* and (12) lead to $\lim_{n\to\infty}\|x_n-u_n\|=0$. Thus, we get $\|x^*-u_n\|\leq\|x_n-u_n\|+\|x_n-x^*\|\to 0$. □

Theorem 2. *Let X be a normed space, D a nonempty, convex, closed subset of X and $T:D\to D$ an operator satisfying condition Z. Let $u_0=p_0\in D$, then:*

(i) If the Mann iteration (1) *converges to x^* and*

$$\lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\alpha_n}=0,$$

then the Picard iteration (3) *converges to x^*.*

(ii) If the Picard iteration (3) *converges to x^* and*

$$\lim_{n\to\infty}\frac{\|p_{n+1}-p_n\|}{\alpha_n}=0,$$

then the Mann iteration (1) *converges to x^*.*

Proof. Suppose that the Mann iteration converges. We will prove that the Picard iteration converges, too. Relations (1) and (5) with

$$x:=u_n,\quad y:=p_n,$$

lead to

$$\begin{aligned}\|u_{n+1}-p_{n+1}\| &\leq (1-\alpha_n)\|u_n-Tp_n\|+\alpha_n\|Tu_n-Tp_n\| \\ &\leq (1-\alpha_n)\|u_n-Tp_n\|+\alpha_n\delta\|u_n-p_n\| \\ &\quad +2\alpha_n\delta\|Tu_n-u_n\| \\ &= (1-\alpha_n)\|u_n-p_{n+1}\|+\alpha_n\delta\|u_n-p_n\| \\ &\quad +2\alpha_n\delta\|Tu_n-u_n\| \\ &\leq (1-\alpha_n)\|u_{n+1}-p_{n+1}\|+(1-\alpha_n)\|u_{n+1}-u_n\| \\ &\quad +\alpha_n\delta\|u_n-p_n\|+2\alpha_n\delta\|Tu_n-u_n\|.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\alpha_n\|u_{n+1}-p_{n+1}\| &\leq (1-\alpha_n)\|u_{n+1}-u_n\| \\ &\quad +\alpha_n\delta\|u_n-p_n\|+2\alpha_n\delta\|Tu_n-u_n\| \text{ i.e.} \\ \|u_{n+1}-p_{n+1}\| &\leq \delta\|u_n-p_n\|+2\delta\|Tu_n-u_n\|+\frac{(1-\alpha_n)}{\alpha_n}\|u_{n+1}-u_n\|\end{aligned}$$

Set in *Lemma 1*:

$$\begin{aligned}a_n &:= \|u_n-p_n\|, \\ 1-\lambda &:= \delta\in(0,1), \\ \sigma_n &:= 2\delta\|Tu_n-u_n\|+\frac{(1-\alpha_n)}{\alpha_n}\|u_{n+1}-u_n\|,\end{aligned}$$

to obtain $\lim_{n\to\infty}\|u_n-p_n\|=0$. Hence, one get $\|x^*-p_n\|\leq\|u_n-p_n\|+\|u_n-x^*\|\to 0$.

Suppose now that the Picard iteration converges. We prove that the Mann iteration converges as well. Using (5) with

$$x:=p_n,\quad y:=u_n,$$

and the following

$$\|u_n-Tp_n\|\leq\|u_n-p_n\|+\|p_n-Tp_n\|,$$

we get

$$\begin{aligned}\|u_{n+1}-p_{n+1}\| &\leq (1-\alpha_n)\|u_n-Tp_n\|+\alpha_n\|Tp_n-Tu_n\| \\ &\leq (1-\alpha_n)\|u_n-p_n\|+(1-\alpha_n)\|p_n-Tp_n\| \\ &\quad +\alpha_n\delta\|p_n-u_n\|+2\alpha_n\delta\|Tp_n-p_n\| \\ &= (1-(1-\delta)\alpha_n)\|u_n-p_n\|+(1-\alpha_n)\|p_n-Tp_n\| \\ &\quad +2\alpha_n\delta\|Tp_n-p_n\| \\ &\leq (1-(1-\delta)\alpha_n)\|u_n-p_n\|+(1+2\alpha_n)\|p_n-p_{n+1}\| \\ &= (1-(1-\delta)\alpha_n)\|u_n-p_n\|+o(\alpha_n).\end{aligned}$$

Set in *Lemma 1*:

$$a_n := \|u_n - p_n\|,$$
$$\alpha_n := (1-\delta)\,\alpha_n \in (0,1),\ \forall n \in N,$$
$$\sigma_n := (1+2\alpha_n)\,\|p_n - p_{n+1}\|,$$

to obtain $\lim_{n\to\infty}\|u_n - p_n\| = 0$. Hence, one obtains $\|x^* - u_n\| \le \|u_n - p_n\| + \|p_n - x^*\| \to 0$. □

Theorem 1 and *Theorem 2* lead to the following Corollary:

Corollary 1. *Let X be a normed space, D a nonempty, convex, closed subset of X and $T : D \to D$ an operator satisfying condition Z. If $u_0 = x_0 \in D$, then the following are equivalent:*

(i) the Mann iteration (1) *converges to x^*,*

(ii) the Ishikawa iteration (2) *converges to x^*,*

(iii) the Picard iteration (3) *converges to x^*.*

Theorem 1 generalizes Theorem 2 from [11]. In Theorem 2, the map T satisfies only condition (z_1) :

Theorem 3. [11] *Let X be a normed space, and B a nonempty convex subset of X. Let $T : B \to B$ be a contraction with constant $L \in (0,1)$. Let $x_0 = u_0 \in B$. The following two assertions are equivalent:*

(i) the Mann iteration $(u_n)_n$ converges to x^*,

(ii) the Ishikawa iteration $(x_n)_n$ converges to x^*.

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