A class of random complementarity problems in Hilbert spaces*

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Abstract. *The purpose of this paper is to study the existence and approximation problem of random solutions for a class of random complementarity problems in the setting of Hilbert spaces.*

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1. Introduction and preliminaries

Complementarity problem introduced and studied by Cottle and Dantzig [4], Lemke [7] in the early 1960's has enjoyed a vigorous growth for the last thirty years. Complementarity problem is closely related to the theory of variational inequality, which plays an important and fundamental role in control and optimization, economics and transportation equilibrium, contact problems in elasticity and fluid flow through porous media, management sciences and operational research. Recently with the development of the theory of variational inequalities, many great developments have been made in the theory and applications of complementarity problems (see, for example, [3, 5, 6, 8, 9] and the references therein).

The purpose of this paper is to introduce and study the existence and approximation problem of random solutions for a class of random complementarity problems in the setting of Hilbert spaces. For the sake of convenience, we first recall some definitions, notations and conclusions.

Throughout this paper, we assume that X is a real separable Hilbert space, (Ω, μ) is a measurable space, $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ are the inner product and the norm in X, respectively, $\beta(X)$ is the $\sigma-$algebra of all Borel subsets of X.

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A mapping $f : \Omega \to X$ is said to be *measurable*, if for each $B \in \beta(X)$ the set $\{\omega \in \Omega : f(\omega) \in B\} \in \mu$. A mapping $T : \Omega \times X \to X$ is said to be random, if for each given $x \in X$ the mapping $\omega \mapsto T(\omega, x)$ is measurable. A measurable function $f : \Omega \to X$ is called a random fixed point of the random operator $T : \Omega \times X$, if $T(\omega, f(\omega)) = f(\omega)$, for all $\omega \in \Omega$.

A random operator $T : \Omega \times X \to X$ is said to be continuous, if for any given $\omega \in \Omega, T(\omega, \cdot) : X \to X$ is continuous.

Lemma 1 [11]. *Let E be a separable metric space and Y a metric space. Let $T : \Omega \times E \to Y$ be measurable in $\omega \in \Omega$ and continuous in $x \in E$. If $g : \Omega \to E$ is a measurable function, then $T(\cdot, g(\cdot)) : \Omega \to Y$ is measurable.*

Definition 1. *Let $T : \Omega \times X \to X$ be a random mapping.*

(1) T is said to be monotone, if for any $x, y \in X$ we have

$$\langle T(\omega, x) - T(\omega, y), x - y\rangle \geq 0, \quad \forall\, \omega \in \Omega;$$

(2) T is said to be strongly monotone, if there exists a measurable function $\alpha : \Omega \to (0, \infty)$ such that for any $x, y \in X$, we have

$$\langle T(\omega, x) - T(\omega, y), x - y\rangle \geq \alpha(\omega)||x - y||^2, \quad \forall\, \omega \in \Omega;$$

(3) T is said to be Lipschitzian continuous, if there exists a measurable function $\gamma : \Omega \to (0, \infty)$ such that for any $x, y \in X$, we have

$$||T(\omega, x) - T(\omega, y)|| \leq \gamma(\omega)||x - y||, \quad \forall\, \omega \in \Omega;$$

(4) T is said to be demi-continuous, if the mapping

$$\lambda \mapsto T(\omega, \lambda x + (1 - \lambda)y) : [0, 1] \to X$$

satisfies the following conditions: for any given sequence $\{\lambda_n\} \subset [0, 1]$ with $\lambda_n \to \lambda_0$ we have

$$T(\omega, \lambda_n x + (1 - \lambda_n)y) \to T(\omega, \lambda_0 x + (1 - \lambda_0)y) \text{ (weakly) } \forall\, \omega \in \Omega,\ x, y \in X.$$

Definition 2. *Let K be a closed convex set, $T : \Omega \times X \to X$ a random mapping. The so-called random complementarity problem with respect to T is to find a measurable mapping $x_* : \Omega \to K$ such that*

$$T(\omega, x_*(\omega)) \in K^*, \quad \langle T(\omega, x_*(\omega)), x_*(\omega))\rangle = 0 \ \ \forall \omega \in \Omega, \tag{1.1}$$

where

$$K^* = \{f \in X : \langle f, x\rangle \geq 0, \ \ \forall x \in K\}.$$

Lemma 2. *Let X be a separable Hilbert space, $K \subset X$ a nonempty closed convex cone and $T : \Omega \times X \to X$ a demi-continuous monotone random mapping, then the following conclusions are equivalent:*

(1) $x : \Omega \to K$ is a random solution of random complementarity problem (1.1);

(2) $x : \Omega \to K$ is a random solution of the following random variational inequality:

$$\langle T(\omega, x(\omega)), y - x(\omega)\rangle \geq 0, \ \ \forall\, y \in K, \ \omega \in \Omega; \tag{1.2}$$

(3) $x : \Omega \to K$ is a random solution of the following random variational inequality:

$$\langle T(\omega, y), y - x(\omega)\rangle \geq 0, \ \ \forall\, y \in K, \ \omega \in \Omega; \tag{1.3}$$

Proof. (1) $\Rightarrow$ (2). Let $x : \Omega \to K$ be a random solution of random complementarity problem (1.1), hence we have

$$T(\omega, x(\omega)) \in K^*, \ \ \langle T(\omega, x(\omega)), x(\omega)\rangle = 0, \ \ \omega \in \Omega.$$

therefore we have

$$\begin{aligned}\langle T(\omega, x(\omega)), y - x(\omega)\rangle &= \langle T(\omega, x(\omega)), y\rangle - \langle T(\omega, x(\omega)), x(\omega)\rangle \\ &= \langle T(\omega, x(\omega)), y\rangle. \ \ \forall y \in K, \ \omega \in \Omega.\end{aligned}$$

Since $T(\omega, x(\omega)) \in K^*$, we have

$$\langle T(\omega, x(\omega)), y\rangle \geq 0, \ \ \forall y \in K, \ \omega \in \Omega.$$

Hence we have

$$\langle T(\omega, x(\omega)), y - x(\omega)\rangle \geq 0, \ \ \forall\, y \in K, \ \omega \in \Omega.$$

(2) $\Rightarrow$ (3). Let $x : \Omega \to K$ be a random solution of (1.2), we have

$$\langle T(\omega, x(\omega)), y - x(\omega)\rangle \geq 0, \ \ \forall\, y \in K, \ \omega \in \Omega.$$

Since $T : \Omega \times X \to X$ is monotone, we have

$$\begin{aligned}0 &\leq \langle T(\omega, y) - T(\omega, x(\omega)), y - x(\omega)\rangle \\ &= \langle T(\omega, y), y - x(\omega)\rangle - \langle T(\omega, x(\omega)), y - x(\omega)\rangle.\end{aligned}$$

And so we have

$$\langle T(\omega, y), y - x(\omega)\rangle \geq \langle T(\omega, x(\omega)), y - x(\omega)\rangle \geq 0, \ \ \forall y \in K, \ \omega \in \Omega.$$

(3) $\Rightarrow$(1). Let $x : \Omega \to K$ be a random solution of (1.3), we have

$$\langle T(\omega, y), y - x(\omega)\rangle \geq 0, \ \ \forall\, y \in K, \ \omega \in \Omega. \tag{1.4}$$

For any given $u \in K$ and $\lambda \in (0, 1]$, letting $y = x(\omega) + \lambda(u - x(\omega)) \in K$ and substituting it into (1.4), we have

$$\langle T(\omega, x(\omega) + \lambda(u - x(\omega))), \lambda(u - x(\omega))\rangle \geq 0, \ \ \omega \in \Omega. \tag{1.5}$$

First divide two sides of (1.5) by λ and then letting $\lambda \to 0$, by virtue of the demicontinuity of T, we have

$$\langle T(\omega, x(\omega)), u - x(\omega))\rangle \geq 0 \ \ \forall u \in K,\ \omega \in \Omega. \tag{1.6}$$

Taking $u = 2x(\omega)$ in (1.6), we have

$$\langle T(\omega, x(\omega)), x(\omega))\rangle \geq 0 \ \ \forall \omega \in \Omega. \tag{1.7}$$

Again taking $u = 0$ in (1.6), we have

$$\langle T(\omega, x(\omega)), -x(\omega))\rangle \geq 0 \ \ \forall \omega \in \Omega. \tag{1.8}$$

Therefore we have

$$\langle T(\omega, x(\omega)), x(\omega))\rangle = 0 \ \ \forall \omega \in \Omega.$$

Next we prove that $T(\omega, x(\omega)) \in K^*$, $\forall \omega \in \Omega$. Suppose the contrary, there exists some $\omega_0 \in \Omega$ such that $T(\omega_0, x(\omega_0)) \notin K^*$. Therefore there exists some $y_0 \in K$ such that $\langle T(\omega_0, x(\omega_0)), y_0\rangle < 0$. Since $y_0 \in K$, from (1.6) we have

$$\begin{aligned} 0 &\leq \langle T(\omega_0, x(\omega_0)), y_0 - x(\omega_0)\rangle \\ &= \langle T(\omega_0, x(\omega_0)), y_0)\rangle - \langle T(\omega_0, x(\omega_0)), x(\omega_0)\rangle \\ &= \langle T(\omega_0, x(\omega_0)), y_0)\rangle < 0, \end{aligned}$$

a contradiction. This implies that $T(\omega, x(\omega)) \in K^* \ \ \forall \omega \in \Omega$.

This completes the proof of *Lemma 2*. □

2. Main results

We are now inthe position to give the main result of this paper.

Theorem 1. *Let X be a real separable Hilbert space, K a nonempty closed convex cone in X, $T : \Omega \times X \to X$ a strongly monotone and Lipschitzian continuous random mapping and the corresponding strongly monotone measurable function and let the corresponding Lipschitzian measurable function of T be $\alpha : \Omega \to (0, \infty)$ and $\gamma : \Omega \to (0, \infty)$, respectively. If the following condition is satisfied:*

$$0 < \gamma^2(\omega) < 2\alpha(\omega) \leq \gamma^2(\omega) + 1, \ \ \forall \omega \in \Omega, \tag{2.1}$$

then

(1) the random complementarity problem (1.1) has a unique random solution $x_* : \Omega \to K$;

(2) for any given $x_0 \in K$ the following random iterative sequence:

$$x_{n+1}(\omega) = S(\omega, x_n(\omega)), \ \ \forall n \geq 0,\ \omega \in \Omega \tag{2.2}$$

converges strongly to the unique random fixed point $x_*(\omega)$ and has the following error estimation:

$$||x_n(\omega) - x_*(\omega)|| \leq \frac{\theta^n(\omega)}{1 - \theta(\omega)} ||x_1(\omega) - x_*(\omega)||$$

where the mapping $S : \Omega \times K \to K$ is defined by the following (2.6) and

$$\theta(\omega) = \sqrt{1 + \gamma^2(\omega) - 2\alpha(\omega)} < 1, \ \ \forall \omega \in \Omega. \tag{2.3}$$

Proof. Since $K \subset X$ is a closed convex cone, by the well-known minimizing vector theorem in Hilbert space (see, for example, Rudin [10]), for each $y \in K$ and each $\omega \in \Omega$, there exists a unique $x(\omega) \in K$ such that

$$||x(\omega) - y + T(\omega, y)|| \le ||v - y + T(\omega, y)||, \ \ \forall v \in K.$$

i.e.,

$$x(\omega) = P_K(y - T(\omega, y)), \tag{2.4}$$

where P_K is the projection operator from X onto K and so it is a nonexpansive mapping. By the well-known projection theorem in Hilbert space (see Chang [2, p. 9, Proposition 1.3.2]), we have

$$\langle y - T(\omega, y) - x(\omega), x(\omega) - v \rangle \ge 0, \ \ \forall v \in K. \tag{2.5}$$

Define a mapping $S : \Omega \times K \to K$ by:

$$S(\omega, y) = P_K(y - T(\omega, y)). \tag{2.6}$$

Next we prove that $S : \Omega \times K \to K$ is a random Banach contraction mapping. Indeed, for any y_1, $y_2 \in K$, we have

$$\begin{aligned} ||S(\omega, y_1) - S(\omega, y_2)||^2 &= ||P_K(y_1 - T(\omega, y_1)) - P_K(y_2 - T(\omega, y_2))||^2 \\ &\le ||y_1 - T(\omega, y_1) - (y_2 - T(\omega, y_2))||^2 \\ &= ||y_1 - y_2||^2 + ||T(\omega, y_1) - T(\omega, y_2)||^2 \\ &\quad - 2\langle y_1 - y_2, T(\omega, y_1) - T(\omega, y_2)\rangle, \ \ \omega \in \Omega. \end{aligned} \tag{2.7}$$

By the assumption, T is $\alpha-$strongly monotone and $\gamma-$Lipschitzian, where α and $\gamma : \Omega \to (0, \infty)$ are measurable functions satisfying condition (2.1). Hence from (2.7) we have

$$||S(\omega, y_1) - S(\omega, y_2)||^2 \le \theta^2(\omega)||y_1 - y_2||^2, \ \ \forall \omega \in \Omega,$$

i.e.,

$$||S(\omega, y_1) - S(\omega, y_2)|| \le \theta(\omega)||y_1 - y_2||, \ \ \forall \omega \in \Omega, \tag{2.8}$$

where $\theta(\omega) = \sqrt{1 + \gamma^2(\omega) - 2\alpha(\omega)} : \Omega \to (0, 1)$ is a measurable function. This implies that $S : \Omega \times K \to K$ is a random Banach contractive mapping. Therefore by the well-known random Banach contractive mapping theorem (see, for example, A. T. Bharucha-Reid [1] or Chang [3]), S has a random fixed point $x_*(\omega) : \Omega \to K$. Therefore we have

$$x_*(\omega) = S(\omega, x_*(\omega)) = P_K(x_*(\omega) - T(\omega, x_*(\omega))) \ \ \forall \omega \in \Omega. \tag{2.9}$$

Substituting (2.9) into (2.5) we have

$$\langle x_*(\omega) - T(\omega, x_*(\omega)) - x_*(\omega), x_*(\omega) - v \rangle \ge 0, \ \ \forall v \in K, \ \omega \in \Omega.$$

i.e.,

$$\langle T(\omega, x_*(\omega)), v - x_*(\omega)\rangle \geq 0, \ \ \forall v \in K, \ \omega \in \Omega.$$

This implies that $x_* : \Omega \to K$ is a random solution of random variational inequality (1.2). By *Lemma 2*, we know that $x^*(\omega)$ is a random solution of random complementarity problem (1.1).

On the other hand, for any given $x_0 \in K$ let $\{x_n(\omega)\}$ be the iterative sequence defined by (2.2). From *Lemma 1*, it is easy to see that $\{x_n(\omega)\}$ is a random sequence from Ω to K. By the well-known method, we can prove that $\{x_n(\omega)\}$ converges strongly to $x_*(\omega)$ and has the following error estimation:

$$||x_n(\omega) - x_*(\omega)|| \leq \frac{\theta^n(\omega)}{1-\theta(\omega)}||x_1(\omega) - x_*(\omega)|| \ \ \forall \omega \in \Omega.$$

This completes the proof of the theorem. □

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