Diophantine quadruples in the ring $\mathbb{Z}[\sqrt{2}]$

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Abstract. *The set of integers of the quadratic field $\mathbb{Q}(\sqrt{d})$ has the property $D(z)$ if the product of its any two distinct elements increased by z is a perfect square in $\mathbb{Q}(\sqrt{d})$. In case $d = 2$, we prove that there exist infinitely many integer quadruples with the property $D(z)$ if and only if z can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{2})$.*

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1. Introduction

Let z be an integer in the quadratic field $\mathbb{Q}(\sqrt{d})$. A *Diophantine quadruple with the property $D(z)$*, or shortly a *$D(z)$-quadruple*, is a set of four non-zero integers $D = \{z_1, z_2, z_3, z_4\} \subset \mathbb{Q}(\sqrt{2})$ with the property that the product of any two distinct elements of this set increased by z is a perfect square in $\mathbb{Q}(\sqrt{2})$. If a set D satisfies the above condition, then D is called a *set with the property $D(z)$*, even if not all elements of D are non-zero integers.

Several authors considered the problem of existence of Diophantine quadruples. The first one who considered this problem in rationals was the Greek mathematician Diophantus of Alexandria in the third century. He noted that the set $\{1, 33, 68, 105\}$ has the property $D(256)$. Many centuries later, Fermat found out that $\{1, 3, 8, 120\}$ is a $D(1)$-quadruple. This problem is almost completely solved in the ring of integers. In [2], [8] and [9], it was proved that if $n \in \mathbb{Z}$ and $n \equiv 2 (\mathrm{mod}\ 4)$, then there does not exist a Diophantine quadruple with the property $D(n)$. The converse of this statement was proved by Dujella in [3]. To be more precise, Dujella showed that if $n \not\equiv 2 (\mathrm{mod}\ 4)$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists a $D(n)$-quadruple. If $n \in S$ we still do not know the answer to the question of the existence of $D(n)$-quadruples. A recent result of Dujella and Fuchs shows that there does not exist a $D(-1)$-quintuple (see [7]).

It is interesting that the existence of a $D(n)$-quadruple in $\mathbb{Z}$ can be described in terms of representability of number n as a difference of two squares. Indeed,

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it can be easily seen that $n \in \mathbb{Z}$ can be represented as a difference of squares of two integers if and only if $n \not\equiv 2 (\mathrm{mod}\ 4)$. So, we can conclude that there exists a $D(n)$-quadruple in $\mathbb{Z}$ if and only if n can be represented as a difference of squares of two integers, up to finitely many possible exceptions listed in the set S.

In the ring of Gaussian integers $\mathbb{Z}[i]$, a similar result can be obtained. Namely, it was proved by Dujella in [5] that if b is odd or $a \equiv b \equiv 2 (\mathrm{mod}\ 4)$, then there does not exist a $D(a+bi)$-quadruple and if $a+bi$ is not of such form and $a+bi \notin \{\pm 2, \pm 1, \pm 2i, \pm 4i\}$, then there exists a $D(a+bi)$-quadruple. Further, a Gaussian integer z can be represented as a difference of squares of two Gaussian integers if and only if z is of the following forms $2m+1+2ni$, $4m+4ni$, $4m+(4n+2)i$, $4m+2+4ni$ (see [11], p. 449). Therefore, $z = a+bi$ is not a difference of two squares if and only if b is odd or $a \equiv b \equiv 2 (\mathrm{mod}\ 4)$.

Thus one can state the Conjecture: *Let z be an integer in the quadratic field $\mathbb{Q}(\sqrt{d})$, where d is not a perfect square. There exists a $D(z)$-quadruple if and only if z can be represented as a difference of squares of two integers, up to finitely many possible exceptions.* The first step in the verification of this Conjecture is to investigate differences of two squares in quadratic fields. The partial solution of that problem was given in [6]. It was shown that the representability of an integer as a difference of squares can be described in terms of solvability of certain Pellian equations. In our paper we will prove the above conjecture in case $d = 2$. In this case the ring of integers is $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$.

Here is our main result:

Theorem 1. *Let $z \in \mathbb{Z}[\sqrt{2}]$. There exist infinitely many $D(z)$-quadruples if and only if z can be represented as a difference of squares of two elements in $\mathbb{Z}[\sqrt{2}]$.*

Let us mention that recently Abu Muriefah and Al-Rashed in [1] gave some partial results about the existence of Diophantine quadruples in the ring $\mathbb{Z}[\sqrt{-2}]$.

2. Preliminaries

Our first aim is to determine which forms of numbers in $\mathbb{Z}[\sqrt{2}]$ can be represented as a difference of squares of two integers. This problem was considered more generally in [6]. Theorem 1 in [6] implies that $z \in \mathbb{Z}[\sqrt{2}]$ can be represented as a difference of two squares if and only if z has one of the following forms:

$$z = 2m+1+2n\sqrt{2},\ z = 4m+4n\sqrt{2},\ z = 4m+2+4n\sqrt{2},\ z = 4m+2+(4n+2)\sqrt{2},$$

where $m, n \in \mathbb{Z}$.

Following lemmas will be useful for proving that there exists a $D(z)$-quadruple for each number z of the above form.

Lemma 1 [Theorem 1, [4]]. *The sets*

$$\{m, (3k+1)^2 m + 2k, (3k+2)^2 m + 2k + 2, 9(2k+1)^2 m + 8k + 4\},$$

$$\{m, mk^2 - 2k - 2, m(k+1)^2 - 2k, m(2k+1)^2 - 8k - 4\}$$

have the property $D(2m(2k+1)+1)$.

Lemma 2. *Let $\{z_1, z_2, z_3, z_4\} \subset \mathbb{Z}[\sqrt{2}]$ be a set with the property $D(z)$ and $w \in \mathbb{Z}[\sqrt{2}]$. Then $\{z_1 w, z_2 w, z_3 w, z_4 w\}$ has the property $D(zw^2)$.*

The number w from *Lemma 2* is often chosen as a solution of Pellian equations $x^2 - 2y^2 = -1$, $x^2 - 2y^2 = 1$, $x^2 - 2y^2 = 2$ or $x^2 - 2y^2 = -2$. We have to point out that the quadratic field $\mathbb{Q}(\sqrt{2})$ is a very special one in this respect, because all three Pellian equations $x^2 - 2y^2 = -1$, $x^2 - 2y^2 = 2$ and $x^2 - 2y^2 = -2$ are solvable (in integers). Namely, for any positive square-free integer $d \neq 2$, at most one of the following three equations is solvable: $x^2 - dy^2 = -1$, $x^2 - dy^2 = 2$ and $x^2 - dy^2 = -2$, see [10, §28]. Hence, our proof cannot be immediately generalized to the arbitrary quadratic field $\mathbb{Q}(\sqrt{d})$.

3. The existence of $D(z)$-quadruples in $\mathbb{Z}[\sqrt{2}]$

Proposition 1. *Let $z \in \mathbb{Z}[\sqrt{2}]$ be of the form $2m + 1 + 2n\sqrt{2}$, $m, n \in \mathbb{Z}$. Then there exist infinitely many $D(z)$-quadruples.*

Proof. The proof splits into four parts.

1) Let $z = 4m + 3 + 4n\sqrt{2}$, where $m, n \in \mathbb{Z}$. We will show that there exist $k, l \in \mathbb{Z}[\sqrt{2}]$ such that

$$4m + 3 + 4n\sqrt{2} = 2(2k + 1)l + 1 \tag{1}$$

If $l = \alpha + \beta\sqrt{2}$ and $k = \gamma + \delta\sqrt{2}$, then equation (1) can be written as

$$\begin{aligned} 4\alpha\gamma + 8\beta\delta &= 4m - 2\alpha + 2, \\ 4\beta\gamma + 4\alpha\delta &= 4n - 2\beta. \end{aligned} \tag{2}$$

Equations (2) can be understood as a linear system in two unknowns γ and δ. Obviously, the solutions of (2) are given by

$$\gamma = ((2m - \alpha + 1)\alpha - 2(2n - \beta)\beta)/(2(\alpha^2 - 2\beta^2)), \tag{3}$$

$$\delta = ((2n - \beta)\alpha - (2m - \alpha + 1)\beta)/(2(\alpha^2 - 2\beta^2)) \tag{4}$$

Now, let $\alpha + \beta\sqrt{2}$ be a solution of Pell equation $x^2 - 2y^2 = 1$. This assumption implies that α must be odd, β must be even and the denominator in (3) and (4) is 2. Hence, it is easy to verify that numbers γ and δ, given by (3) and (4), are integers.

According to *Lemma 1* and (1), we obtain that the set

$$\{l, (3k + 1)^2 l + 2k, (3k + 2)^2 l + 2k + 2, 9(2k + 1)^2 l + 8k + 4\} \tag{5}$$

represents a Diophantine quadruple with property $D(4m + 3 + 4n\sqrt{2})$. Obviously, there are infinitely many such quadruples, because there are infinitely many solutions of the Pell equation.

One may ask what if there exists a $z = 4m+3+4n\sqrt{2}$ such that set (5) does not represent a $D(z)$-quadruple, i.e. when at least two elements of (5) are equal or some of the elements are zero. This problem can be solved by using the following idea. For fixed l, there are only finitely many parameters k such that some elements of (5) are equal or zero. So, let us denote by S the set of all integers $z = 2(2k+1)l+1$, where l is fixed, such that the set (5) does not represent a Diophantine quadruple, i.e. such that some elements of (5) are equal or zero. Let $z_0 \in S$ and let $w = s+t\sqrt{2}$

be some solution of the Pell equation $x^2 - 2y^2 = 1$. It can be easily verified that number $z_0 w^2$ is of the same form as number z_0, i.e. of the form $4m + 3 + 4n\sqrt{2}$. Indeed,

$$z_0 w^2 = (4m_0 + 3 + 4n_0\sqrt{2})(s + t\sqrt{2})^2 = 4a + 3(s^2 + 2t^2) + (4b + 2st(4n_0 + 3))\sqrt{2},$$

where $a = m_0(s^2 + 2t^2) + 4n_0 st$ and $b = n_0(s^2 + 2t^2)$. Because s is necessarily odd and t is even, it follows that $s^2 + 2t^2 \equiv 1 \pmod 4$ and that number $z_0 w^2$ is of the form $4m + 3 + 4n\sqrt{2}$. Since there are infinitely many solutions of the Pell equation, we have $z_0 w^2 \notin S$ for infinitely many w's. The $D(z_0 w^2)$-quadruple, for corresponding k, is given by (5). The $D(z_0)$-quadruple can be obtained by multiplying the elements of (1) by $s - t\sqrt{d}$, according to *Lemma 2*.

2) In the same manner as in case 1, we will try to represent the number $z = 4m + 3 + (4n + 2)\sqrt{2}$ in the form $2(2k + 1)l + 1$. Precisely, for given $m, n \in \mathbb{Z}$ we will prove that there exist $k, l \in \mathbb{Z}[\sqrt{2}]$ such that

$$4m + 3 + (4n + 2)\sqrt{2} = 2(2k + 1)l + 1. \tag{6}$$

In this case, let $l = \alpha + \beta\sqrt{2}$ be a solution of $x^2 - 2y^2 = -1$. That means that α and β are odd numbers. Under the notation $k = \gamma + \delta\sqrt{2}$, equation (6) becomes a linear system in two unknowns γ and δ

$$\begin{aligned} 4\alpha\gamma + 8\beta\delta &= 4m - 2\alpha + 2, \\ 4\beta\gamma + 4\alpha\delta &= 4n + 2 - 2\beta. \end{aligned} \tag{7}$$

Evidently, the solutions of (7)

$$\begin{aligned} \gamma &= -((2m - \alpha + 1)\alpha - 2(2n - \beta + 1)\beta)/2, \\ \delta &= -((2n - \beta + 1)\alpha - (2m - \alpha + 1)\beta)/2. \end{aligned}$$

are integers. So, set (5) is a Diophantine quadruple with the property $D(4m + 3 + (4n + 2)\sqrt{2})$, according to *Lemma 1* and (6). Obviously, for different solutions of the equation $x^2 - 2y^2 = -1$ we obtain different quadruples, so, there are infinitely many such quadruples. Eventual problems, when the set (5) does not represent a Diophantine quadruple, can be solved as in case 1.

3) As in previous cases 1 and 2, for given $m, n \in \mathbb{Z}$ we can determine $k, l \in \mathbb{Z}[\sqrt{2}]$ such that

$$4m + 1 + (4n + 2)\sqrt{2} = 2(2k + 1)l + 1. \tag{8}$$

Under the notation of cases 1 and 2, equation (8) can be understood as a linear system in unknowns γ and δ, whose solutions are given by

$$\gamma = ((2m - \alpha)\alpha - 2(2n - \beta + 1)\beta)/(2(\alpha^2 - 2\beta^2)), \tag{9}$$

$$\delta = ((2n - \beta + 1)\alpha - (2m - \alpha)\beta)/(2(\alpha^2 - 2\beta^2)). \tag{10}$$

For l we take solutions of equations $x^2 - 2y^2 = \pm 2$ and we have to show that γ and δ given by (9) and (10) are integers. In fact, we have to verify that the numerators are divisible by 4. Indeed, if m is odd, then for $l = \alpha + \beta\sqrt{2}$ we take a solution of the equation $x^2-2y^2=2$. This implies that $\alpha\equiv 2(\text{mod } 4)$ and that β is odd. So, $2m-\alpha\equiv 0(\text{mod } 4)$ and $(2n-\beta+1)\alpha\equiv 0(\text{mod } 4)$. Similarly, if m is even, then for $l=\alpha+\beta\sqrt{2}$ we chose a solution of another equation $x^2-2y^2=-2$, what means that $\alpha\equiv 0(\text{mod } 4)$ and b is odd. Again, this implies that $\gamma,\delta\in\mathbb{Z}$.

Further conclusions can be drawn in the same way as it was done in cases 1 and 2.

4) In this last case, we show the existence of a Diophantine quadruple with property $D(4m+1+4n\sqrt{2})$. First, we express a given integer of the form $4m+1+4n\sqrt{2}$ as $(4a+3+(4b+2)\sqrt{2})w^2$, where $a,b\in\mathbb{Z}$ and $w\in\mathbb{Z}[\sqrt{2}]$. Indeed, it can be easily seen that the equation

$$4m+1+4n\sqrt{2}=(4a+3+(4b+2)\sqrt{2})w^2$$

holds for $w=1+\sqrt{2}$ (the fundamental solution of the equation $x^2-2y^2=-1$) and for integers $a=3m-4n$ and $b=3n-2m-1$. Now, we are finished with the proof, because in the case 2 we proved that there exists infinitely many $D(4a+3+(4b+2)\sqrt{2})$-quadruples and these quadruples multiplied by w are exactly $D(4m+1+4n\sqrt{2})$-quadruples, by *Lemma 2*. □

Proposition 2. *If $z\in\mathbb{Z}[\sqrt{2}]$ is of the form $4m+2+4n\sqrt{2}$, then there exist infinitely many $D(z)$-quadruples.*

Proof. According to *Proposition 1*, for given $m,n\in\mathbb{Z}$, there exists a Diophantine quadruple $\{c_1,c_2,c_3,c_4\}$ with property $D(2m+1+2n\sqrt{2})$. *Lemma 2* implies that the set $\{c_1\sqrt{2},c_2\sqrt{2},c_3\sqrt{2},c_4\sqrt{2}\}$ represents a Diophantine quadruple with property $D(4m+2+4n\sqrt{2})$. Obviously, there exist infinitely many such quadruples, according to *Proposition 1*. □

Proposition 3. *If an integer $z\in\mathbb{Z}[\sqrt{2}]$ is of the form $4m+4n\sqrt{2}$, then there exist infinitely many $D(z)$-quadruples.*

Proof. We will prove the statement of this Proposition in four steps.

1) According to *Proposition 1*, there exist infinitely many Diophantine quadruples $\{c_1,c_2,c_3,c_4\}$ with property $D(2m+1+2n\sqrt{2})$. Further, by *Lemma 2*, the set $\{2c_1,2c_2,2c_3,2c_4\}$ is a $D(8m+4+8n\sqrt{2})$-quadruple.

2) Let $m,n\in\mathbb{Z}$ and $l=\alpha+\beta$ be a solution of Pell equation $x^2-2y^2=1$. We will show that there exists $k\in\mathbb{Z}[\sqrt{2}]$ such that the following equation holds

$$2m+2n\sqrt{2}=(2k+1)l+1. \tag{11}$$

Equation (11) splits into

$$\begin{aligned}2\alpha\gamma+4\beta\delta&=2m-\alpha-1,\\ 2\beta\gamma+2\alpha\delta&=2n-\beta,\end{aligned} \tag{12}$$

where $k=\gamma+\delta\sqrt{2}$. By solving system (12) in unknowns γ and δ, we obtain

$$\gamma=((2m-\alpha-1)\alpha-2(2n-\beta)\beta)/2, \tag{13}$$

$$\delta=((2n-\beta)\alpha-(2m-\alpha-1)\beta)/2. \tag{14}$$

Obviously, γ and δ are integers, because α is odd and β is even.

By *Lemma 1*, the set

$$\left\{\frac{l}{2}, (3k+1)^2\frac{l}{2}+2k, (3k+2)^2\frac{l}{2}+2k+2, 9(2k+1)^2\frac{l}{2}+8k+4\right\} \tag{15}$$

has the property $D((2k+1)l+1)$. According to *Lemma 2*, we can obtain the set with the property $D(4((2k+1)l+1))$ by multiplying the elements of (15) by 2. In fact, the $D(4((2k+1)l+1))$-quadruple was obtained in this way, because the elements of (15) multiplied by 2 are integers. Since different solutions of the Pell's equation induce different Diophantine quadruples, we conclude that there exist infinitely many $D(8m+8n\sqrt{2})$-quadruples.

3) Let $m, n \in \mathbb{Z}$. Similarly as in case 2, we have that

$$2m+1+(2n+1)\sqrt{2} = (2k+1)l+1, \tag{16}$$

for $k, l \in \mathbb{Z}[\sqrt{d}]$ such that $l = \alpha+\beta\sqrt{2}$ is a solution of the Pellian equation $x^2 - 2y^2 = \pm 2$. Indeed, as before equation (16) can be understood as a linear system in unknowns γ and δ, where $k = \gamma+\delta\sqrt{2}$. Solutions are given by

$$\gamma = \pm((2m-\alpha)\alpha - 2(2n+1-\beta)\beta)/4, \tag{17}$$
$$\delta = \pm((2n+1-\beta)\alpha - (2m-\alpha)\beta)/4. \tag{18}$$

Since, α is even and β is odd, it is clear that the numerator in (17) is divisible by 4, but the numerator in (17) is congruent to 2 modulo 4. Therefore, if m is odd we take $l = \alpha+\beta\sqrt{2}$ to be a solution of the equation $x^2-2y^2 = 2$, because in this case $\alpha \equiv 2 \pmod 4$, and if m is even then we take l to be a solution of $x^2-2y^2 = -2$.

Now, as in case 2, we obtain the $D(8m+4+(8n+4)\sqrt{2})$-quadruple by multiplying the elements of set (15) by 2, because (15) represents a set with the property $D(2m+1+(2n+1)\sqrt{2})$, for corresponding k and l. Also, there are infinitely many such quadruples because there are infinitely many solutions of the Pellian equation $x^2-2y^2 = \pm 2$.

4) In this last case, we have to prove that there exist infinitely many $D(z)$-quadruples, where z is of the form $8m+(8n+4)\sqrt{2}$. In the same manner as in cases 2 and 3, we can show that $z \in \mathbb{Z}[\sqrt{2}]$ of the form $2m+(2n+1)\sqrt{2}$ can be represented in the form of $(2k+1)l+1$, where $l = \alpha+\beta\sqrt{2}$ is a solution of the Pellian equation $x^2-2y^2 = -1$ and $k = \gamma+\delta\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. Indeed, we solve the equation

$$2m+(2n+1)\sqrt{2} = (2k+1)l+1. \tag{19}$$

as a linear system in γ and δ and we obtain

$$\gamma = ((2m-\alpha-1)\alpha - 2(2n+1-\beta)\beta)/2, \tag{20}$$
$$\delta = ((2n+1-\beta)\alpha - (2m-\alpha-1)\beta)/2. \tag{21}$$

Since α and β are odd, γ and δ are integers.

The rest of the proof goes as in cases 2 and 3. □

Proposition 4. *If $z \in \mathbb{Z}[\sqrt{2}]$ is of the form $4m+2+(4n+2)\sqrt{2}$, then there exist infinitely many $D(z)$-quadruples.*

Proof. In the proof of *Proposition 3*, case 3, we showed that set (15) has the property $D(2m+1+(2n+1)\sqrt{2})$, for corresponding $k, l \in \mathbb{Z}[\sqrt{2}]$, where l is a solution of the Pellian equation $x^2 - 2y^2 = \pm 2$. By multiplying the elements of (15) by $\sqrt{2}$, we obtain the set of integers with the property $D(4m+2+(4n+2)\sqrt{2})$, i.e. the Diophantine quadruple. Indeed, $\frac{l}{2}\sqrt{2} = \frac{\alpha}{2}\sqrt{2} + \beta$ is an integer, because α is even. □

4. The nonexistence of $D(z)$-quadruples

In this section, we will prove that there is no Diophantine quadruple with the property $D(z)$ for $z \in \mathbb{Z}[\sqrt{2}]$ such that z is not representable as a difference of two squares of integers, i.e. such that z is of the form $m+(2n+1)\sqrt{2}$ or $4m+(4n+2)\sqrt{2}$. In fact, we can prove that if $d \equiv 2 \pmod 4$ and if $z \in \mathbb{Z}[\sqrt{d}]$ is of the form $m+(2n+1)\sqrt{d}$ or $4m+(4n+2)\sqrt{d}$, then there is no $D(z)$-quadruple. This improves the corresponding results from [1].

Proposition 5. *Let $d \in \mathbb{Z}$ such that $d \equiv 2 (mod\ 4)$. If $z \in \mathbb{Z}[\sqrt{d}]$ is of the form $m+(2n+1)\sqrt{d}$ or $4m+(4n+2)\sqrt{d}$, then there is no $D(z)$-quadruple.*

Proof. If z is of the form $m+(2n+1)\sqrt{d}$ the proof is analogous to the proof of [1, Proposition 1].

Let z be of the form $4m+(4n+2)\sqrt{d}$. Suppose that the set $\{z_1, z_2, z_3, z_4\}$ is a $D(z)$-quadruple. If $z_i = x_i + y_i\sqrt{d}$, for $i = 1, 2, 3, 4$, then

$$(x_i + y_i\sqrt{d})(x_j + y_j\sqrt{d}) + z = (\xi_{ij} + \eta_{ij}\sqrt{d})^2, \tag{22}$$

for all $1 \le i < j \le 4$ and for some $\xi_{ij}, \eta_{ij} \in \mathbb{Z}$. Equation (22) splits into

$$x_i x_j + y_i y_j d + 4m = \xi_{ij}^2 + d\eta_{ij}^2, \tag{23}$$

$$x_i y_j + x_j y_i d + 4n + 2 = 2\xi_{ij}\eta_{ij}. \tag{24}$$

Let us discuss values of the right-hand sides of (23) and (24) in the set of remainders modulo 4. The following cases may appear:

i) if $2\xi_{ij}\eta_{ij} \equiv 0 \pmod 4$, then $(\xi_{ij}^2 + d\eta_{ij}^2) \bmod 4 \in \{0, 1, 2\}$

ii) if $2\xi_{ij}\eta_{ij} \equiv 2 \pmod 4$, then $\xi_{ij}^2 + d\eta_{ij}^2 \equiv 3 \pmod 4$.

That means that if $x_i + y_i\sqrt{d}$ and $x_j + y_j\sqrt{d}$ are the elements of a Diophantine quadruple with property $D(4m+(4n+2)\sqrt{d})$, then

$$(x_i x_j + y_i y_j d, x_i y_j + x_j y_i + 2) \bmod 4 \in \{(0,0), (1,0), (2,0), (3,2)\} \tag{25}$$

Our intention is to show that there is no $D(4m+(4n+2)\sqrt{d})$-quadruple which satisfies condition (25) for all $1 \le i < j \le 4$.

For example, let $x_1 \equiv 0 \pmod 4$ and $y_1 \equiv 1 \pmod 4$. If $x_2 + y_2\sqrt{d}$ is the element of the $D(4m+(4n+2)\sqrt{d})$-quadruple, then conditions (25), for $i = 1$ and $j = 2$, imply that

$$(2y_2, x_2 + 2) \bmod 4 \in S,$$

where $S = \{(0,0), (1,0), (2,0), (3,2)\}$, i.e. that

$$(x_2, y_2) \bmod 4 \in \{(2,0), (2,1), (2,2), (2,3)\}.$$

Suppose that $x_2 \equiv 2 \pmod 4$ and $y_2 \equiv 0 \pmod 4$. Now, we must find the conditions on the third element $x_3 + y_3\sqrt{d}$ of the $D(4m + (4n+2)\sqrt{d})$-quadruple. As before, for $i = 1, 2$ and $j = 3$, (25) gives us the following conditions

$$(2y_3, x_3 + 2) \bmod 4 \ \in S \text{ and } (2x_3, 2y_3 + 2) \bmod 4 \ \in S.$$

It turns out that these conditions are fulfilled only for $x_3 \equiv 2 \pmod 4$ and $y_3 \equiv 1 \pmod 4$. Finally, let $x_4 + y_4\sqrt{d}$ be the last element of the quadruple. Then (25) must be satisfied for $i = 1, 2, 3$ and $j = 4$. From the previous case it follows that

$$x_4 \equiv 2 \pmod 4,\ y_4 \equiv 1 \pmod 4 \text{ and } (2x_4 + 2y_4, x_4 + 2y_4 + 2) \bmod 4 \ \in S.$$

But, these conditions are contradictory, because $2x_4+2y_4 \equiv 2 \pmod 4$, $x_4+2y_4+2 \equiv 2 \pmod 4$ and $(2, 2) \notin S$.

We have checked all of the other possibilities by using the programme written in FORTRAN. □

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