Stability of Mann and Ishikawa iterative processes with errors for a class of nonlinear variational inclusions problem*

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Abstract. *Under the lack of the condition , some new convergence and stability theorems of Mann and Ishikawa iterative processes with errors for solutions to variational inclusions involving accretive mappings in real reflexive Banach spaces are established. The main results of this paper extend and improve the corresponding results obtained by Chang, Ding, Hassouni and Moudafi, Huang, Kazmi, Noor, Siddiqi and Ansari and Zeng.*

Key words: *variational inclusion, variational inequality, accretive mapping, Ishikawa iterative process with errors, stability*

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1. Introduction and preliminaries

Let X an arbitrary real Banach space with norm $\|\cdot\|$ and dual X^*, and J denotes the normalized duality map from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|f\|^2 = \|x\|^2\},\ \forall x \in X$$

where $\langle\cdot,\cdot\rangle$ is the generalized duality pairing. In the sequel, I denotes the identity operator on X, $D(T)$ and $R(T)$ denote the domain and the range of T, respectively.

An operator $T : D(T) \subset X \to X$ is said to be accretive, if for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0.$$

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If T is accretive and $(I+\lambda T)(D(T)) = X$ for any $\lambda > 0$, then T is called m-accretive.

Suppose that T is an operator on X. Let x_0 be a point in X and let $x_{n+1} = f(T, x_n)$ denote an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in. Assume that $F(T) = \{x : Tx = x \in X\} \neq \emptyset$ and that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x^* \in F(T)$. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X, and set $\epsilon_n = \|x_{n+1} - f(T, y_n)\|$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T.

Let $T, A : X \to X, g : X \to X^*$ be three mappings and $\varphi : X^* \to R \cup +\infty$ be a proper convex lower semicontinuous function.

In 1999, Chang [1] introduced and studied the existence and iterative approximation problem of solutions for a class of nonlinear variational inclusions with accretive mappings in uniformly smooth Banach space as follows:

For any given $f \in X$, to find a $u \in X$ such that

$$\begin{cases} g(u) \in D(\partial\varphi), \\ \langle Tu - Au - f, v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v), \ \forall v \in X^*, \end{cases} \tag{1}$$

where $\partial\varphi$ denotes the subdifferential of φ.

Observe that, if X is a Hilbert space H, then (1) is equivalent to find a $u \in H$ for given $f \in H$ such that

$$\begin{cases} g(u) \in D(\partial\varphi), \\ \langle Tu - Au - f, v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v), \ \forall v \in H. \end{cases} \tag{2}$$

(2) is said to be a variational inclusion problem in Hilbert space which has been studied in Chang [3], Deng [6,7], Hassouni and Moudafi [8], Kazmi [12] and Zeng [17].

The purpose of this paper is further to study the existence and uniqueness of solutions, and the convergence and stability problem of the Mann and Ishikawa iterative process with errors for a class of accretive type variational inclusions in real reflexive Banach spaces. The results presented in this paper not only extend and improve the main results in Chang [1], but also extend and improve the corresponding results in Chang [2-4], Chang, Cho and Lee et al [5], Ding [6,7], Hassouni and Moudafi [8], Huang [9-11], Kazmi [13], Noor [14,15], Siddiqi and Ansari [16], Siddiqi, Ansari and Kazmi [17] and Zeng [18].

The following results will be needed in the sequel.

Proposition 1 [19]. *Suppose that X is an arbitrary real Banach space, $T : D(T) \subset X \to X$ is accretive and continuous, and $D(T) = X$. Then T is m-accretive.*

Lemma 1 [20]. *Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq \gamma a_n + b_n, \ n \geq 0,$$

where $\gamma \in [0, 1)$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 [21]. *Suppose that X is an arbitrary real Banach space, $T : D(T) \subset X \to X$ is an m-accretive mapping. Then the equation $x + Tx = f$ has a unique solution in $D(T)$ for any $f \in X$.*

Lemma 3 [1]. *Suppose that X is a real reflexive Banach space, $\partial\varphi\circ g : X \to 2^X$ is a mapping, then the following conclusions are equivalent:*

(i) $x^* \in X$ is a solution of variational inclusion problem (1);

(ii) $x^* \in X$ is a fixed point of the mapping $S: X \to 2^X$:

$$S(x) = f - (Tx - Ax + \partial\varphi(g(x))) + x;$$

(iii) $x^* \in X$ is a solution of the equation $f \in Tx - Ax + \partial\varphi(g(x))$.

2. Main results

We now state the main result of this section.

Theorem 1. *Suppose that X is a real reflexive Banach space, $T, A: X \to X$, $g: X \to X^*$ are three mappings, and $\varphi: X^* \to R \cup \{+\infty\}$ is a function with a Gâteaux differential $\partial\varphi$. For any given $f \in X$, define a mapping $S : X \to X$ by*

$$Sx = f - (Tx - Ax + \partial\varphi(g(x))) + x.$$

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Ishikawa iterative process with errors defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S z_n + u_n, \ n \geq 0, \\ z_n = (1-\beta_n)x_n + \beta_n S x_n + v_n, \ n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$, and $\{u_n\}$, $\{v_n\}$ are two sequences in X satisfying the following conditions:

(i) $T - A + \partial\varphi \circ g - I : X \to X$ is an accretive operator;

(ii) $T - A + \partial\varphi \circ g : X \to X$ is a Lipschitz operator with the Lipschitz constant L;

(iii) $L_*(1+L_*)(\alpha_n + \beta_n) + L_*(L_*^2 - 1)\alpha_n\beta_n \leq 1 - t$, $n \geq 0$, where $L_* = 1 + L$, $t \in (0,1)$;

(iv) $\alpha_n \geq \alpha > 0$, $n \geq 0$, where α is a constant;

(v) $\|u_n\| \to 0$ and $\|v_n\| \to 0$ $(n \to \infty)$.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and let us define $\{\epsilon_n\}_{n=0}^{\infty}$ by

$$\begin{cases} w_n = (1-\beta_n)y_n + \beta_n S y_n + v_n, \ n \geq 0, \\ \epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S w_n - u_n\|, \ n \geq 0. \end{cases} \tag{4}$$

Then the following conclusions hold:

(I) The nonlinear variational inclusion problem (1) has a unique solution $x^* \in X$.

(II) The Ishikawa iterative sequence $\{x_n\}$ with errors converges strongly to the unique solution $x^* \in X$ of the variational inclusion problem (1); moreover,

$$\|x_n - x^*\| \leq (1-t\alpha)^n \|x_0 - x^*\| + \frac{1-(1-t\alpha)^n}{t\alpha} \cdot C, \ n \geq 0,$$

where $C = \sup\{L_*(1+L_*)\|v_n\| + (1+L_*)\|u_n\| : \ n \geq 0\}$.

(III) $\|y_{n+1}-x^*\| \le (1-t\alpha)\|y_n-x^*\|+\epsilon_n+\|u_n\|+L_*(1+L_*)\|v_n\|,\ n\ge 0.$

(IV) $\lim_{n\to\infty} y_n = x^*$ *if and only if* $\lim_{n\to\infty}\epsilon_n = 0$.

Proof. (I) From conditions (i) and (ii), the mapping $T-A+\partial\varphi\circ g-I : X\to X$ is continuous and accretive. By *Proposition 1* we know that $T-A+\partial\varphi\circ g-I : X\to X$ is m-accretive. Therefore, by *Lemma 2*, for any given $f\in X$, the equation $f = x+(T-A+\partial\varphi\circ g-I)(x)$ has a unique solution $x^*\in X$. Hence, by *Lemma 3* we know that x^* is a unique solution of the variational inclusions problem (1), and it is also a unique fixed point of S, i.e., $Sx^*=x^*$.

(II) Since $T-A+\partial\varphi\circ g-I$ is an accretive operator, therefore, for any $x,y\in X$, there exists $j(x-y)\in J(x-y)$ such that

$$\langle Sx-Sy, j(x-y)\rangle = -\langle (T-A+\partial\varphi\circ g-I)x-(T-A+\partial\varphi\circ g-I)y, j(x-y)\rangle \le 0.$$

It follows from Lemma 1.1 of Kato [12] that

$$\|x-y\| \le \|x-y-r(Sx-Sy)\| \tag{5}$$

for all $x,y\in X$ and $r>0$. Using (3), we have

$$(x_{n+1}-x^*)-\alpha_n(Sx_{n+1}-Sx^*) = (1-\alpha_n)(x_n-x^*)-\alpha_n(Sx_{n+1}-Sz_n)+u_n. \tag{6}$$

From (5) and (6), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \|x_{n+1}-x^*-\alpha_n(Sx_{n+1}-Sx^*)\|\\ &\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|Sx_{n+1}-Sz_n\|+\|u_n\|.\end{aligned} \tag{7}$$

Since $T-A+\partial\varphi\circ g$ is a Lipschitz mapping with the constant L, it is easy to verify that S is also a Lipschitz mapping with the constant $L_*=1+L$. Furthermore, we have the following estimates:

$$\begin{aligned}\|x_n-z_n\| &\le \beta_n\|x_n-Sx_n\|+\|v_n\|\\ &\le (1+L_*)\beta_n\|x_n-x^*\|+\|v_n\|,\end{aligned} \tag{8}$$

and

$$\begin{aligned}\|Sz_n-z_n\| &\le (1+L_*)\|z_n-x^*\|\\ &\le (1+L_*)[(1-\beta_n)\|x_n-x^*\|+\beta_n\|Sx_n-x^*\|+\|v_n\|]\\ &\le (1+L_*)[1+(L_*-1)\beta_n]\|x_n-x^*\|+(1+L_*)\|v_n\|.\end{aligned} \tag{9}$$

By (3), (8), (9) and condition (iii), it yields that

$$\begin{aligned}\|Sx_{n+1}-Sz_n\| &\le L_*\|x_{n+1}-z_n\|\\ &\le L_*[(1-\alpha_n)\|x_n-z_n\|+\alpha_n\|Sz_n-z_n\|+\|u_n\|]\\ &\le \{L_*(1+L_*)(1-\alpha_n)\beta_n+L_*(1+L_*)[1+(L_*-1)\beta_n]\alpha_n\}\|x_n-x^*\|\\ &\quad +L_*(1+L_*\alpha_n)\|v_n\|+L_*\|u_n\|\\ &\le [L_*(1+L_*)(\alpha_n+\beta_n)+L_*(L_*^2-1)\alpha_n\beta_n]\|x_n-x^*\|\\ &\quad +L_*(1+L_*)\|v_n\|+L_*\|u_n\|\\ &\le (1-t)\|x_n-x^*\|+L_*(1+L_*)\|v_n\|+L_*\|u_n\|.\end{aligned} \tag{10}$$

Substituting (10) into (7), and by condition (iv) we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le [(1-\alpha_n)+(1-t)\alpha_n]\|x_n-x^*\|\\ &\quad + L_*(1+L_*)\alpha_n\|v_n\|+(1+L_*\alpha_n)\|u_n\|\\ &\le (1-t\alpha_n)\|x_n-x^*\|+L_*(1+L_*)\|v_n\|+(1+L_*)\|u_n\|\\ &\le (1-t\alpha)\|x_n-x^*\|+L_*(1+L_*)\|v_n\|+(1+L_*)\|u_n\|. \qquad (11)\end{aligned}$$

Letting

$$\gamma = 1-t\alpha,\ \ a_n=\|x_n-x^*\|,\ \ b_n=L_*(1+L_*)\|v_n\|+(1+L_*)\|u_n\|,\ \ n\ge 0.$$

Then (11) can be written as follows

$$a_{n+1}\le \gamma a_n+b_n,\ n\ge 0.$$

In view of *Lemma 1*, conditions (iii)–(v), we deduce that $a_n\to 0$ $(n\to\infty)$, that is, $x_n\to x^*$ $(n\to\infty)$. Furthermore, using (11), we have

$$\begin{aligned}\|x_n-x^*\| &\le (1-t\alpha)\|x_{n-1}-x^*\|+C\\ &\le (1-t\alpha)^n\|x_0-x^*\|+\frac{1-(1-t\alpha)^n}{t\alpha}\cdot C\end{aligned}$$

for all $n\ge 0$, where $C=\sup\{L_*(1+L_*)\|v_n\|+(1+L_*)\|u_n\|:\ \ n\ge 0\}$. This completes the proof of (II).

(III) Put $p_n=(1-\alpha_n)y_n+\alpha_n Sw_n$ for all $n\ge 0$. Note that

$$(p_n-x^*)-\alpha_n(Sp_n-Sx^*)=(1-\alpha_n)(y_n-x^*)-\alpha_n(Sp_n-Sw_n). \qquad (12)$$

From (5) and (12), we have

$$\begin{aligned}\|p_n-x^*\| &\le \|p_n-x^*-\alpha_n(Sp_n-Sx^*)\|\\ &\le (1-\alpha_n)\|y_n-x^*\|+\alpha_n\|Sp_n-Sw_n\|. \qquad (13)\end{aligned}$$

for all $n\ge 0$. Similar to (8) and (9), we have also the following

$$\|y_n-w_n\|\le (1+L_*)\beta_n\|y_n-x^*\|+\|v_n\|, \qquad (14)$$

and

$$\|Sw_n-w_n\|\le (1+L_*)[1+(L_*-1)\beta_n]\|y_n-x^*\|+(1+L_*)\|v_n\|. \qquad (15)$$

By (4), (14), (15) and condition (iii), we obtain that

$$\begin{aligned}\|Sp_n-Sw_n\| &\le L_*\|p_n-w_n\|\\ &\le L_*[(1-\alpha_n)\|y_n-w_n\|+\alpha_n\|Sw_n-w_n\|]\\ &\le \{L_*(1+L_*)(1-\alpha_n)\beta_n+L_*(1+L_*)[1+(L_*-1)\beta_n]\alpha_n\}\|y_n-x^*\|\\ &\quad +L_*(1+L_*\alpha_n)\|v_n\|\\ &\le [L_*(1+L_*)(\alpha_n+\beta_n)+L_*(L_*^2-1)\alpha_n\beta_n]\|y_n-x^*\|+L_*(1+L_*)\|v_n\|\\ &\le (1-t)\|y_n-x^*\|+L_*(1+L_*)\|v_n\|. \qquad (16)\end{aligned}$$

Substituting (16) into (13), we obtain

$$\begin{aligned}\|p_n - x^*\| &\le [(1-\alpha_n) + (1-t)\alpha_n]\|y_n - x^*\| + L_*(1+L_*)\alpha_n\|v_n\| \\ &\le (1-t\alpha_n)\|y_n - x^*\| + L_*(1+L_*)\|v_n\| \\ &\le (1-t\alpha)\|y_n - x^*\| + L_*(1+L_*)\|v_n\|. \qquad (17)\end{aligned}$$

Note that

$$\begin{aligned}\|y_{n+1} - x^*\| &\le \|y_{n+1} - p_n - u_n\| + \|u_n\| + \|p_n - x^*\| \\ &= \epsilon_n + \|u_n\| + \|p_n - x^*\|. \qquad (18)\end{aligned}$$

Substituting (17) into (18), we have

$$\|y_{n+1} - x^*\| \le (1-t\alpha)\|y_n - x^*\| + \epsilon_n + \|u_n\| + L_*(1+L_*)\|v_n\|, \quad n \ge 0. \qquad (19)$$

That is, (III) holds.

(IV) Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Note that $t\alpha \in (0,1)$, it follows from the condition (v), (19), and *Lemma 1* that $\lim_{n\to\infty} y_n = x^*$.

Suppose that $\lim_{n\to\infty} y_n = x^*$. Using (17) and (v), we immediately conclude that

$$\begin{aligned}\epsilon_n &= \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S w_n - u_n\| \\ &= \|y_{n+1} - p_n - u_n\| \\ &\le \|y_{n+1} - x^*\| + \|p_n - x^*\| + \|u_n\| \\ &\le \|y_{n+1} - x^*\| + (1-t\alpha)\|y_n - x^*\| + L_*(1+L_*)\|v_n\| + \|u_n\| \to 0 \ (n \to \infty).\end{aligned}$$

That is, $\lim_{n\to\infty} \epsilon_n = 0$. Hence (IV) holds. This completes the proof. □

Remark 1. Theorem 1 *improves and extends Theorem 3.1 of Chang [1] in its five aspects:*

(1) "The X is uniformly smooth" is replaced by the "X is reflexive".

(2) The Ishikawa iterative process is replaced by the more general Ishikawa iterative process with errors.

(3) It abolishes the condition that the range $R(S)$ of S is bounded.

(4) Sequences $\{\alpha_n\}$ and $\{\beta_n\}$ need not converge to zero.

(5) It is proved that the Ishikawa iterative process with errors is S-stable in Theorem 1*, too.*

Remark 2. Theorem 1 *extends and improves Theorem 2.1 of Chang [2] in the following ways:*

(1) Sequences $\{\alpha_n\}$ and $\{\beta_n\}$ need not converge to zero.

(2) It abolishes the condition that $\{Sx_n\}$ and $\{Sy_n\}$ are bounded.

(3) *It is proved that the Ishikawa iterative process with errors is* S*-stable in* Theorem 1*, too.*

Remark 3. Theorem 1 *also extends and improves the corresponding results of Chang [3, 4], Chang, Cho and Lee et al [5], Ding [6,7], Hassouni and Moudafi [8], Huang [9-11], Kazmi [12], Noor [13,14], Siddiqi and Ansari [15], Siddiqi, Ansari and Kazmi [16] and Zeng [17].*

In *Theorem 1*, if $\beta_n \equiv 0,\ v_n \equiv 0,\ \forall n \geq 0$, then $z_n = x_n,\ w_n = y_n$, hence we have the following result.

Theorem 2. *Suppose that* X *is a real reflexive Banach space,* $T, A : X \to X$, $g : X \to X^*$ *are three mappings, and* $\varphi : X^* \to R \cup \{+\infty\}$ *is a function with a Gâteaux differential* $\partial\varphi$*. For any given* $f \in X$*, define a mapping* $S : X \to X$ *by*

$$Sx = f - (Tx - Ax + \partial\varphi(g(x))) + x.$$

Let $x_0 \in X$ *be any given point and* $\{x_n\}$ *the Mann iterative process with errors defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S x_n + u_n,\ n \geq 0, \tag{20}$$

where $\{\alpha_n\}$ *is a real sequence in* $[0,1]$*, and* $\{u_n\}$ *is a sequence in* X *satisfying the following conditions:*

(i) $T - A + \partial\varphi \circ g - I : X \to X$ *is an accretive operator;*

(ii) $T - A + \partial\varphi \circ g : X \to X$ *is a Lipschitz operator with the Lipschitz constant* L*;*

(iii) $\alpha_n \leq \frac{1-t}{L_*(1+L_*)}$, $n \geq 0$, *where* $L_* = 1 + L$, $t \in (0,1)$*;*

(iv) $\alpha_n \geq \alpha > 0$, $n \geq 0$, *where* α *is a constant;*

(v) $\|u_n\| \to 0\ (n \to \infty)$.

Let $\{y_n\}_{n=0}^{\infty}$ *be any sequence in* X *and let us define* $\{\epsilon_n\}_{n=0}^{\infty}$ *by*

$$\epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S y_n - u_n\|,\ n \geq 0. \tag{21}$$

Then the following conclusions hold:

(I) *The nonlinear variational inclusion problem (1) has a unique solution* $x^* \in X$*.*

(II) *The Mann iterative sequence* $\{x_n\}$ *with errors converges strongly to the unique solution* $x^* \in X$ *of the variational inclusion problem (1); moreover,*

$$\|x_n - x^*\| \leq (1-t\alpha)^n \|x_0 - x^*\| + \frac{1-(1-t\alpha)^n}{t\alpha} \cdot C,\ n \geq 0,$$

where $C = \sup\{(1+L_*)\|u_n\| :\ n \geq 0\}$.

(III) $\|y_{n+1} - x^*\| \leq (1-t\alpha)\|y_n - x^*\| + \epsilon_n + \|u_n\|,\ n \geq 0.$

(IV) $\lim_{n\to\infty} y_n = x^*$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$.

In case $\varphi \equiv 0$, *Theorem 1* reduces to the following result.

Theorem 3. *Suppose that X is a real Banach space, $T, A : X \to X$, $g : X \to X^*$ are three mappings. For any given $f \in X$, define a mapping $S : X \to X$ by*

$$Sx = f - (Tx - Ax) + x.$$

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Ishikawa iterative process with errors defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S z_n + u_n, \ n \geq 0, \\ z_n = (1-\beta_n)x_n + \beta_n S x_n + v_n, \ n \geq 0, \end{cases} \tag{22}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$, and $\{u_n\}$, $\{v_n\}$ are two sequences in X satisfying the following conditions:

(i) $T - A - I : X \to X$ is an accretive operator;

(ii) $T - A : X \to X$ is a Lipschitz operator with the Lipschitz constant L;

(iii) $L_*(1+L_*)(\alpha_n+\beta_n) + L_*(L_*^2-1)\alpha_n\beta_n \leq 1-t, \ n \geq 0$, where $L_* = 1+L$, $t \in (0,1)$;

(iv) $\alpha_n \geq \alpha > 0, \ n \geq 0$, where α is a constant;

(v) $\|u_n\| \to 0$ and $\|v_n\| \to 0$ $(n \to \infty)$.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and let us define $\{\epsilon_n\}_{n=0}^{\infty}$ by

$$\begin{cases} w_n = (1-\beta_n)y_n + \beta_n S y_n + v_n, \ n \geq 0, \\ \epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S w_n - u_n\|, \ n \geq 0. \end{cases} \tag{23}$$

Then the following conclusions hold:

(I) The following variational inequality (24) has a unique solution $x^* \in X$.

$$\langle Tx - Ax - f, v - g(x)\rangle \geq 0, \quad \forall v \in X^*. \tag{24}$$

(II) The Ishikawa iterative sequence $\{x_n\}$ with errors converges strongly to the unique solution $x^* \in X$ of the variational inequality (24); moreover,

$$\|x_n - x^*\| \leq (1-t\alpha)^n \|x_0 - x^*\| + \frac{1-(1-t\alpha)^n}{t\alpha} \cdot C, \ n \geq 0,$$

where $C = \sup\{L_*(1+L_*)\|v_n\| + (1+L_*)\|u_n\| : \ n \geq 0\}$.

(III) $\|y_{n+1} - x^*\| \leq (1-t\alpha)\|y_n - x^*\| + \epsilon_n + \|u_n\| + L_*(1+L_*)\|v_n\|, \ n \geq 0.$

(IV) $\lim_{n\to\infty} y_n = x^*$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

In *Theorem 2*, if $\varphi \equiv 0$, then we have the following result.

Theorem 4. *Suppose that X is a real Banach space, $T, A : X \to X$, $g : X \to X^*$ are three mappings. For any given $f \in X$, define a mapping $S : X \to X$ by*

$$Sx = f - (Tx - Ax) + x.$$

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Mann iterative process with errors defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S x_n + u_n,\ n \geq 0, \tag{25}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$, and $\{u_n\}$ is a sequence in X satisfying the following conditions:

(i) $T - A - I : X \to X$ *is an accretive operator;*

(ii) $T - A : X \to X$ *is a Lipschitz operator with the Lipschitz constant* L*;*

(iii) $\alpha_n \leq \frac{1-t}{L_*(1+L_*)}$, $n \geq 0$, *where* $L_* = 1 + L$, $t \in (0,1)$*;*

(iv) $\alpha_n \geq \alpha > 0$, $n \geq 0$, *where* α *is a constant;*

(v) $\|u_n\| \to 0$ $(n \to \infty)$.

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and let us define $\{\epsilon_n\}_{n=0}^{\infty}$ by

$$\epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S y_n - u_n\|,\ n \geq 0. \tag{26}$$

Then the following conclusions hold:

(I) The variational inequality (24) has a unique solution $x^* \in X$.

(II) The Mann iterative sequence $\{x_n\}$ with errors converges strongly to the unique solution $x^* \in X$ of the variational inequality (24); moreover,

$$\|x_n - x^*\| \leq (1-t\alpha)^n \|x_0 - x^*\| + \frac{1-(1-t\alpha)^n}{t\alpha} \cdot C,\ n \geq 0,$$

where $C = \sup\{(1+L_*)\|u_n\| :\ n \geq 0\}$.

(III) $\|y_{n+1} - x^*\| \leq (1-t\alpha)\|y_n - x^*\| + \epsilon_n + \|u_n\|,\ n \geq 0.$

(IV) $\lim_{n\to\infty} y_n = x^*$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$.

Remark 4. *The following example reveals that* Theorem 1 *extends properly Theorem 3.1 of Chang [1] and Theorem 2.1 of Chang [2].*

Example 1. *Let $X, T, A, g, f, S, \varphi$ be as in* Theorem 1 *and*

$$t = \frac{1}{2},\ \ \alpha_n = \frac{1}{4L_*(1+L_*)+L_*},\ \ \beta_n = \frac{1}{4L_*(1+L_*)},\ \ \|u_n\| = \|v_n\| = \frac{1}{(n+1)}$$

for all $n \geq 0$. Then the conditions of Theorem 1 *are satisfied. But Theorem 3.1 in [1] and Theorem 2.1 in [2] are not applicable since $\{\alpha_n\}$ and $\{\beta_n\}$ do not converge to* 0.

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