Stammler's circles, Stammler's triangle and Morley's triangle of a given triangle

Ružica Kolar-Šuper* and Vladimir Volenec†

Abstract. *By means of complex coordinates shorter proofs of the results of* L. Stammler *[1], [2] will be given, plus several statements connected with them.*

Key words: *Stammler's circles, Stammler's triangle, Morley's triangle*

AMS subject classifications: 51M04, 51N25

Received February 6, 2004 Accepted December 1, 2004

In the Gauss plane any point Z can be uniquely represented by the determined complex number z, its complex coordinate; then we shall write $Z = (z)$. If ABC is any triangle, then we can choose the origin of the coordinate system and the measure unity so that the circumscribed circle of the triangle ABC is a unit circle with the equation $z\bar{z} = 1$ and radius $R = 1$. If $A = (a)$, $B = (b)$, $C = (c)$, then $a\bar{a} = b\bar{b} = c\bar{c} = 1$, i.e.

$$\bar{a} = \frac{1}{a}, \qquad \bar{b} = \frac{1}{b}, \qquad \bar{c} = \frac{1}{c}. \tag{1}$$

Proposition 1. *Point $H = (h)$ given by*

$$h = a + b + c \tag{2}$$

is the orthocenter of the triangle ABC.

Proof. For the number

$$k = \frac{h-a}{b-c} = \frac{b+c}{b-c}$$

because of (1) we get

$$\bar{k} = \frac{\bar{b}+\bar{c}}{\bar{b}-\bar{c}} = \frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{b}-\frac{1}{c}} = -\frac{b+c}{b-c} = -k,$$

*Teacher Training College, University of Osijek, L. Jägera 9, HR-31 000 Osijek, Croatia, e-mail: `rkolar@vusos.hr`

†Department of Mathematics, University of Zagreb, Bijenička c. 30, HR-10 000 Zagreb, Croatia, e-mail: `volenec@math.hr`

so k becomes a purely imaginary number, which means that the line AH is perpendicular to the line BC. In the same way $BH \perp CA$ and $CH \perp AB$. □

The lines have got linear equations on variables z and $\bar{z}$. The most general equation can be written in the form $z + t\bar{z} = s$. It is obvious that the equations of the form $z + t\bar{z} = s$ and $z + t\bar{z} = s'$ present parallel lines because there is no point $Z = (z)$ that goes with $s \neq s'$ and that would satisfy both equations.

Proposition 2. *Line BC has the equation*

$$z + bc\bar{z} = b + c, \tag{3}$$

and the line perpendicular to that line through point $P = (p)$ has the equation

$$z - bc\bar{z} = p - bc\bar{p}. \tag{4}$$

The feet $Q = (q)$ of the perpendicular line from point P to the line BC is given by the equation

$$2q = b + c + p - bc\bar{p}. \tag{5}$$

Proof. Points $B = (b)$ and $C = (c)$ satisfy equation (3) as we get $b + bc\bar{b} = b + c$ and $c + bc\bar{c} = b + c$ because of (1). The altitude AH has got the equation

$$z - bc\bar{z} = a - \frac{bc}{a} \tag{6}$$

as because of (1) and (2) we get

$$a - bc\bar{a} = a - \frac{bc}{a}, \quad h - bc\bar{h} = a + b + c - bc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = a - \frac{bc}{a}.$$

Lines (4) and (6) are parallel, and point P obviously lies on line (4), so that the line is perpendicular from point P to the line BC. For the intersection $Z = (z)$ of lines (3) and (4) from (3) and (4) follows $2z = b + c + p - bc\bar{p}$. That proves the last statement of the proposition. □

Corollary 1. *The line joining two points $Z_1 = (z_1)$ and $Z_2 = (z_2)$ on the unit circle has the equation $z + z_1 z_2 \bar{z} = z_1 + z_2$.*

Proposition 3. *The distance of the point $P = (p)$ from the line BC with the equation (3) is given by the formula*

$$|PQ|^2 = \frac{1}{4bc}(p + bc\bar{p} - b - c)^2, \tag{7}$$

and the side BC has the length given by the formula

$$|BC|^2 = -\frac{1}{bc}(b - c)^2. \tag{8}$$

Proof. Because of (5) and (1) we get

$$\begin{aligned} 4|PQ|^2 &= (2q - 2p)(2\bar{q} - 2\bar{p}) = (b + c - p - bc\bar{p})(\bar{b} + \bar{c} - \bar{p} - \bar{b}\bar{c}p) \\ &= (b + c - p - bc\bar{p})\left(\frac{1}{b} + \frac{1}{c} - \bar{p} - \frac{p}{bc}\right) = \frac{1}{bc}(b + c - p - bc\bar{p})^2, \end{aligned}$$

$$|BC|^2 = (c-b)(\bar{c}-\bar{b}) = (c-b)\left(\frac{1}{c}-\frac{1}{b}\right) = -\frac{1}{bc}(b-c)^2.$$

□

We can choose the real axis so that

$$abc = 1, \tag{9}$$

and owing to (1)

$$\bar{a} = bc, \qquad \bar{b} = ca, \qquad \bar{c} = ab. \tag{10}$$

Because of that choice points $B_1 = (\varepsilon)$, $B_2 = (\varepsilon^2)$, $B_3 = (1)$, where

$$\varepsilon^3 = 1, \quad \varepsilon^2 + \varepsilon + 1 = 0, \quad \bar{\varepsilon} = \varepsilon^2, \quad \overline{\varepsilon^2} = \varepsilon, \tag{11}$$

are the so-called Boutin's points of the triangle ABC. Tangents of the circle ABC at points B_1, B_2, B_3 form the triangle $S_1S_2S_3$ with the vertices $S_1 = (-2\varepsilon)$, $S_2 = (-2\varepsilon^2)$, $S_3 = (-2)$, which becomes from the triangle $B_1B_2B_3$ by homothecy with the center in the center $O = (0)$ of the circle ABC and the coefficient -2 (*Figure 2*). The triangles $B_1B_2B_3$ and $S_1S_2S_3$ are equilateral. We shall name the points S_1, S_2, S_3 Stammler's points of the triangle ABC, and the triangle $S_1S_2S_3$ Stammler's triangle of the triangle ABC.

Points S_1, S_2, S_3 cannot be constructed by the compass and ruler (Stammler, [1]). The same statement is also valid for Boutin's points. If the parallels with lines BC, CA, AB through points A, B, C intersect circle ABC again at points A', B', C', then the points which trisect the arcs $\overset{\frown}{AA'}$, $\overset{\frown}{BB'}$, $\overset{\frown}{CC'}$ of that circle are exactly Boutin's points of the triangle ABC.

Proposition 4. *If D_i, E_i, F_i are the feet of the perpendicular lines from Stammler's point S_i of the triangle ABC on the lines BC, CA, AB, then these equations are valid*

$$|S_1D_1|^2 + \frac{1}{4}|BC|^2 = |S_1E_1|^2 + \frac{1}{4}|CA|^2 = |S_1F_1|^2 + \frac{1}{4}|AB|^2 = 3 + h\varepsilon^2 + \bar{h}\varepsilon, \tag{12}$$

$$|S_2D_2|^2 + \frac{1}{4}|BC|^2 = |S_2E_2|^2 + \frac{1}{4}|CA|^2 = |S_2F_2|^2 + \frac{1}{4}|AB|^2 = 3 + h\varepsilon + \bar{h}\varepsilon^2, \tag{13}$$

$$|S_3D_3|^2 + \frac{1}{4}|BC|^2 = |S_3E_3|^2 + \frac{1}{4}|CA|^2 = |S_3F_3|^2 + \frac{1}{4}|AB|^2 = 3 + h + \bar{h}. \tag{14}$$

Proof. If $S_1 = (s_1) = (-2\varepsilon)$, according to *Proposition 3*, because of (9) and (11) we get

$$|S_1D_1|^2 = \frac{1}{4bc}(s_1 + bc\bar{s}_1 - b - c)^2 = \frac{a}{4}(-2\varepsilon - 2bc\varepsilon^2 - b - c)^2$$

and $|BC|^2 = -a(b-c)^2$, therefore because of (9), (11), (10) and (2) follows

$$\begin{aligned}|S_1D_1|^2 + \frac{1}{4}|BC|^2 &= \frac{a}{4}[(2\varepsilon + 2bc\varepsilon^2 + b + c)^2 - (b-c)^2] \\ &= \frac{a}{4}(2\varepsilon + 2bc\varepsilon^2 + 2b)(2\varepsilon + 2bc\varepsilon^2 + 2c) \\ &= a(b + \varepsilon + bc\varepsilon^2)(c + \varepsilon + bc\varepsilon^2) \\ &= abc + a(b+c)\varepsilon + abc\varepsilon^2(b+c) + a\varepsilon^2 + 2abc\varepsilon^3 + ab^2c^2\varepsilon^4 \\ &= 3 + (a+b+c)\varepsilon^2 + (bc+ca+ab)\varepsilon \\ &= 3 + (a+b+c)\epsilon^2 + (\bar{a}+\bar{b}+\bar{c})\varepsilon \\ &= 3 + h\varepsilon^2 + \bar{h}\varepsilon.\end{aligned}$$

Because of the symmetry on a, b, c it follows that the other equations (12) are valid. By substitutions $\varepsilon \to \varepsilon^2$ and $\varepsilon \to 1$ from equation (12) follow the equations (13) and (14) owing to (11). □

From *Proposition 4* it follows:

Theorem 1. *Circles $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ with the centers at Stammler's points of the triangle ABC and the radii ρ_1, ρ_2, ρ_3 given by the equations*

$$\rho_1^2 = 3 + h\varepsilon^2 + \bar{h}\varepsilon, \;\; \rho_2^2 = 3 + h\varepsilon + \bar{h}\varepsilon^2, \;\; \rho_3^2 = 3 + h + \bar{h} \tag{15}$$

cut off segments $|BC|$, $|CA|$, $|AB|$ on lines BC, CA, AB, respectively . Centers of these circles form an equilateral triangle (Figure 1).

We can find the statement of *Theorem 1* (except formula (15)) in Stammler [1], so we shall name circles $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ in that theorem Stammler's circles of the triangle ABC. See also Stärk [3].

Because of (11) and (15) we get

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = 9 + (h + \bar{h})(\varepsilon^2 + \varepsilon + 1) = 9,$$

i.e. there holds:

Corollary 2. *For the radii ρ_1, ρ_2, ρ_3 of Stammler's circles and the radius R of the circumscribed circle of the triangle ABC the equation*

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = 9R^2 \tag{16}$$

is valid.

Because of the inequality of the squared, arithmetic and geometric means from (16) we get the following inequalities

$$R\sqrt{3} = \sqrt{\frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{3}} \geq \frac{\rho_1 + \rho_2 + \rho_3}{3} \geq \sqrt[3]{\rho_1\rho_2\rho_3},$$

where the equalities holds if and only if $\rho_1 = \rho_2 = \rho_3$, i.e.

$$h\varepsilon^2 + \bar{h}\varepsilon = h\varepsilon + \bar{h}\varepsilon^2 = h + \bar{h}. \tag{17}$$

However, the first equation (17) can be written in the form $(h - \bar{h})(\varepsilon^2 - \varepsilon) = 0$, that owing to $\varepsilon^2 - \varepsilon \neq 0$ gives $h = \bar{h}$. Because of that from (17) we get the equation

$h(\varepsilon^2+\varepsilon-2)=0$, i.e. because of (11) we get $-3h=0$ or finally $h=0$. The equation $h=0$, i.e. $H=O$ is valid if and only if the triangle ABC is equilateral.

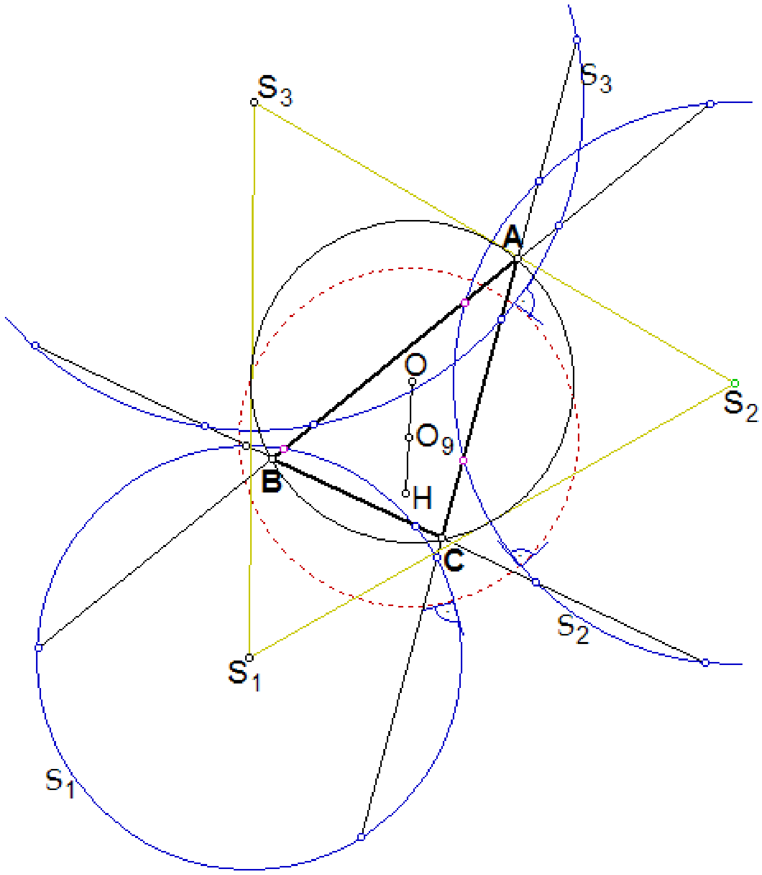

Figure 1.

Theorem 2. *The potential center of Stammler's circles of the triangle ABC is the center $O_9=(\frac{h}{2})$ of his Euler's circle (Figure 1).*

Proof. Point O_9 has with respect to the circle $\mathcal{S}_1$ the power which owing to (11) is equal to

$$p_{O_9,\mathcal{S}_1}=|O_9S_1|^2-\rho_1^2=\left(\frac{h}{2}+2\varepsilon\right)\left(\frac{\bar{h}}{2}+2\varepsilon^2\right)-(3+h\varepsilon^2+\bar{h}\varepsilon)$$
$$=\frac{h\bar{h}}{4}+4\varepsilon^3-3=\frac{h\bar{h}}{4}+1=R^2+\frac{1}{4}|OH|^2,$$

and by substitutions $\varepsilon\to\varepsilon^2$, i.e. $\varepsilon\to 1$ the result does not change, so the point O_9 has the same powers with respect to the circles $\mathcal{S}_2$ i $\mathcal{S}_3$. $\square$

The orthogonal circle of the circles $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ has the center O_9 and radius ρ so that it gives $\rho^2 = R^2 + \frac{1}{4}|OH|^2$ (*Figure 1*).

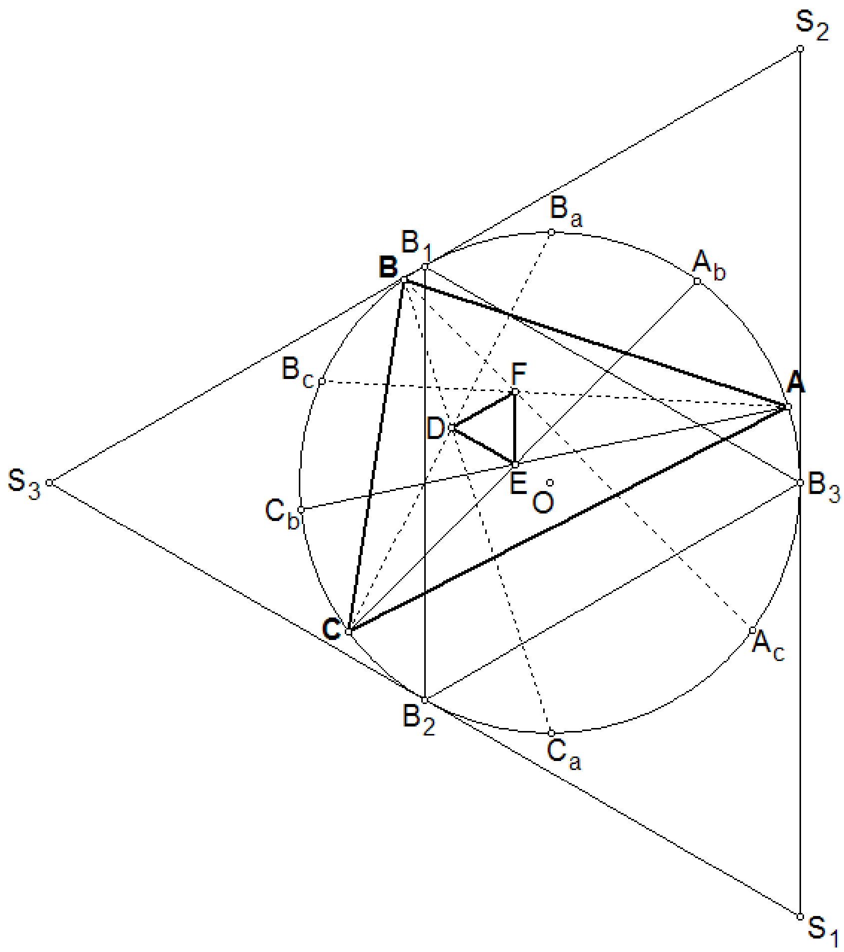

Figure 2.

Let the order of the points on a unit circle and the unity point B_3 be chosen so that in positive sense the points come in the sequence B_3, A, B, C. For the sake of simpler calculation let follow this $A = (a^3) = (a^3\varepsilon^3)$, $B = (b^3)$, $C = (c^3)$, while $a^3b^3c^3 = 1$, so

$$abc = \varepsilon \qquad \text{or} \qquad abc = \varepsilon^2. \tag{18}$$

In that case points A_b, B_a; B_c, C_b; C_a, A_c which divide the arcs in the following order $\widehat{AB}$, $\widehat{BC}$, $\widehat{CA}$ into three equal parts (*Figure 2*), having this form

$$A_b = (a^2b),\ B_a = (ab^2),\ B_c = (b^2c),\ C_b = (bc^2),\ C_a = (c^2a\varepsilon),\ A_c = (ca^2\varepsilon^2).$$

According to *Corollary 1*, lines CA_b and AC_b have the following equations

$$z + a^2bc^3\bar{z} = a^2b + c^3,$$

$$z + a^3bc^2\bar{z} = a^3 + bc^2.$$

Multiplying these equations by a and c and by substracting the obtained equations we get for the intersection $E = (e)$ of these lines the equation

$$(a - c)z = a^3b - c^3b + ac^3 - a^3c = (a - c)(abc + a^2b + bc^2 - a^2c - ac^2)$$

with the result

$$e = abc + a^2b + bc^2 - a^2c - ac^2. \tag{19}$$

By substitutions $a \to b \to c \to a\varepsilon$ for the points $F = AB_c \cap BA_c = (f)$ and $D = BC_a \cap CB_a = (d)$ we get equations

$$\begin{aligned} f &= abc\varepsilon + b^2c + a^2c\varepsilon^2 - ab^2\varepsilon - a^2b\varepsilon^2, \\ d &= abc\varepsilon^2 + ac^2\varepsilon + ab^2 - bc^2\varepsilon - b^2c\varepsilon^2. \end{aligned} \tag{20}$$

Because of (11) from (19) and (20) it follows e.g.

$$\begin{aligned} f - e &= abc(\varepsilon - 1) + b^2c + a^2c(\varepsilon^2 + 1) - ab^2\varepsilon - a^2b(\varepsilon^2 + 1) - bc^2 + ac^2 \\ &= abc(\varepsilon - 1) + b^2c - a^2c\varepsilon - ab^2\varepsilon + a^2b\varepsilon - bc^2 + ac^2 \\ &= (b^2 + ac - ab - bc)(c - a\varepsilon) = -(a - b)(b - c)(c - a\varepsilon), \end{aligned}$$

and then by means of substitutions $a \to b \to c \to a\varepsilon$ we have the equations

$$\begin{aligned} f - e &= -(a - b)(b - c)(c - a\varepsilon), \\ d - f &= -\varepsilon(a - b)(b - c)(c - a\varepsilon) = \varepsilon(f - e), \\ e - d &= -\varepsilon^2(a - b)(b - c)(c - a\varepsilon) = \varepsilon^2(f - e). \end{aligned} \tag{21}$$

From these equations it immediately follows that DEF is a positively oriented equilateral triangle, which is the matter of the famous Morley's theorem.

If the first possibility in (18) is valid, then from (21) because of (11) we get

$$\begin{aligned} \bar{f} - \bar{e} &= -\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{c} - \frac{1}{a\varepsilon}\right) = \frac{1}{a^2b^2c^2\varepsilon}(a - b)(b - c)(c - a\varepsilon) \\ &= (a - b)(b - c)(c - a\varepsilon) = -(f - e). \end{aligned}$$

That means that the number $f - e$ is purely imaginary, i.e. the line EF is perpendicular to real axis OB_3, namely it is parallel with the lines B_1B_2 and S_1S_2. Therefore the triangle EFD is homothetic with the triangles $B_1B_2B_3$ and $S_1S_2S_3$.

If the second possibility in (18) is valid, then from (21) we get

$$\begin{aligned} \bar{e} - \bar{d} &= -\varepsilon\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{c} - \frac{1}{a\varepsilon}\right) = \frac{1}{a^2b^2c^2}(a - b)(b - c)(c - a\varepsilon) \\ &= \varepsilon^2(a - b)(b - c)(c - a\varepsilon) = -(e - d), \end{aligned}$$

so now the line DE is perpendicular to the real axis OB_3, i.e. parallel with the lines B_1B_2 and S_1S_2. Therefore the triangle DEF is homothetic with the triangles $B_1B_2B_3$ and $S_1S_2S_3$.

So we have proved:

Theorem 3. *Stammler's triangle of the given triangle is homothetic with its Morley's triangle (Stammler [2]).*

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