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An existence theorem concerning strong shape of Cartesian products

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Abstract. The paper is devoted to the question is the Cartesian product $X \times P$ of a compact Hausdorff space X and a polyhedron P a product in the strong shape category SSh of topological spaces. The question consists of two parts. The existence part, which asks whether, for a topological space Z, for a strong shape morphism $F\colon Z \to X$ and a homotopy class of mappings $[g]\colon Z \to P$, there exists a strong shape morphism $H\colon Z \to X \times P$, whose compositions with the canonical projections of $X \times P$ equal F and $[g]$, respectively. The uniqueness part asks if H is unique. The main result of the paper asserts that H exists, whenever Z is either metrizable or has the homotopy type of a polyhedron. If X is a metric compactum, H exists for all topological spaces Z. The proofs use resolutions of spaces and coherent homotopies of inverse systems. It is known that, in the ordinary shape category Sh, H need not be unique, even in the case when Z is a metrizable space or a polyhedron.
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1. Introduction

In an arbitrary category the (direct) product of two objects is well defined. It may not exist, but if it does, it is unique up to natural isomorphism. It is well-known that in the category of topological spaces Top the product of two spaces X and Y exists and consists of the Cartesian product $X \times Y$ and the canonical projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$. Similarly, in the homotopy category of topological spaces H the Cartesian product $X \times Y$ and the homotopy classes $[\pi_X]$, $[\pi_Y]$ of the canonical projections π_X, π_Y form the product of X and Y. Since shape is a modification of homotopy, it is natural to ask if products exist in the ordinary shape category Sh and the strong shape category SSh. The answer is known only in some cases when the Cartesian product $X \times Y$, together with morphisms induced by the canonical projections, is a product. It is long known that, in general, the Cartesian product is not a product in Sh. Such an example for metric spaces X, Y was given in [13]. A more subtle example, where X is compact metric (in fact, the dyadic solenoid) and Y is a polyhedron (in fact, the pointed sum of a sequence of 1-spheres S^1) was given in [8]. Other results concerning the Cartesian product in Sh can be found in [14] and [18].

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In strong shape, the following question is still open.

Question 1. *Is the Cartesian product $X \times Y$ of topological spaces X, Y, together with the strong shape morphisms $\overline{S}[\pi_X]\colon X \times Y \to X$ and $\overline{S}[\pi_Y]\colon X \times Y \to Y$, induced by canonical projections, a product in the strong shape category* SSh*?*

Here $\overline{S}\colon \mathrm{H} \to \mathrm{SSh}$ denotes the strong shape functor, which keeps objects (spaces) fixed and associates with every homotopy class of mappings the corresponding strong shape morphism (see [15, 8.2]).

In the present paper we are primarily interested in a special case of Question 1, which is also open and reads as follows.

Question 2. *Let X be a compact Hausdorff space and P a polyhedron* (CW*-topology*)*. Is the Cartesian product $X \times P$, together with the strong shape morphisms $\overline{S}[\pi_X]$ and $\overline{S}[\pi_Y]$, a product in the strong shape category* SSh*?*

Even in the simple case, when X is the Hawaiian earring and Y is the pointed sum of a sequence of copies of the 1-sphere S^1, this author does not know if $X \times Y$ is a product in SSh. There are two cases when it is known that the answer to Question 2 is affirmative. The first one is when P is compact, because the Cartesian product of two compact Hausdorff spaces is a product in SSh [17, Theorem 12]. The second case is when X is an FANR and P is finite-dimensional, because the Cartesian product of an FANR and a finitistic space is a product in SSh [17, Theorem 14].

The *universal property* which makes $(X \times P, \overline{S}[\pi_X], \overline{S}[\pi_P])$ a product in SSh is the conjunction of two properties, an *existence property* for all topological spaces Z (abbreviated $(\mathrm{ESS})_Z$) and a *uniqueness property* for all topological spaces Z (abbreviated $(\mathrm{USS})_Z$). Since strong shape morphisms of a topological space Z to a polyhedron P coincide with homotopy classes $[g]$ of mappings $g\colon Z \to P$, these properties assume the following form.

> $(\mathrm{ESS})_Z$ *For every strong shape morphism $F\colon Z \to X$ and every homotopy class of mappings $[g]\colon Z \to P$, there exists a strong shape morphism $H\colon Z \to X \times P$ such that $\overline{S}[\pi_X]H = F$ and $\overline{S}[\pi_P]H = \overline{S}[g]$.*
>
> $(\mathrm{USS})_Z$ *If $H_i\colon Z \to X \times P$, $i = 1, 2$, are two strong shape morphisms such that $\overline{S}[\pi_X]H_1 = \overline{S}[\pi_X]H_2$ and $\overline{S}[\pi_P]H_1 = \overline{S}[\pi_P]H_2$, then $H_1 = H_2$.*

The results of this paper refer to the existence property $(\mathrm{ESS})_Z$ (see Theorems 1 and 2). Unfortunately, up to now, the author was unable to obtain relevant results concerning the uniqueness property $(\mathrm{USS})_Z$.

The main result of the paper is the following theorem.

Theorem 1. *If X is a compact Hausdorff space and P is a polyhedron* (CW*-topology*)*, then the existence property $(\mathrm{ESS})_Z$ holds for every metrizable space Z.*

From Theorem 1 we will derive the following result for compact metric spaces X.

Theorem 2. *If X is a compact metric space and P is a polyhedron* (CW*-topology*)*, then the existence property $(\mathrm{ESS})_Z$ holds for every topological space Z.*

Remark 1. *In the referee's report of an early version of the present paper (which did not contain Theorem 1) an outline of a short proof of a version of Theorem 2 was given. It was based on a result of J. Dydak and S. Nowak [7, Theorem 3.10]. However, the techniques of that proof do not apply to non-metric compact spaces X and cannot be used to prove Theorem 1.*

The two main technical tools used in the proof of Theorem 1 are *resolutions* and coherent homotopy mappings (or shorter, *coherent mappings*) $\boldsymbol{h}\colon Z \to \boldsymbol{Y}$ from a space Z to an inverse system $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$. Coherent mappings are collections of mappings $h_{\boldsymbol{\mu}}\colon Z \times \Delta^n \to Y_{\mu_0}$, satisfying appropriate *coherence conditions* (see Section 2). Here $\boldsymbol{\mu}$ are multiindices in M, i.e., increasing sequences $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ of elements in M, $\mu_0 \le \dots \le \mu_n$. We refer to n as to the length $|\boldsymbol{\mu}|$ of $\boldsymbol{\mu}$. By $\Delta^n \subseteq \mathbb{R}^{n+1}$ we denote the standard n-simplex spanned by the vertices $e_0 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$. If $n = 1$, the coherence conditions imply that $h_{\mu_0\mu_1}\colon Z \times \Delta^1 = Z \times I \to Y_{\mu_0}$, $\mu_0 \le \mu_1$, is a homotopy connecting the mappings h_{μ_0} and $q_{\mu_0\mu_1} h_{\mu_1}$. A natural definition of homotopy of coherent mappings yields homotopy classes $[\boldsymbol{h}]\colon Z \to \boldsymbol{Y}$. Chapter I of the book [15] (also see Section 2 of this paper) can serve as general reference for coherent homotopy.

It is well known that, for a topological space Z and a compact Hausdorff space Y, there is a bijection between strong shape morphisms $H\colon Z \to Y$ and homotopy classes $[\boldsymbol{h}]$ of coherent mappings $\boldsymbol{h}\colon Z \to \boldsymbol{Y}$, where $\boldsymbol{Y}$ is an inverse system $\boldsymbol{Y}$ of compact polyhedra with a limit $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$. The analogous result for arbitrary topological spaces Y assumes the following form.

Proposition 1. *Let Z and Y be topological spaces and let $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$ be an* HPol*-resolution of Y. Then there is a bijection $\Gamma_{\boldsymbol{q}}$ from the set* $\mathrm{SSh}(Z, Y)$ *of strong shape morphisms $H\colon Z \to Y$ to the set* $\mathrm{CH}(Z, \boldsymbol{Y})$ *of homotopy classes $[\boldsymbol{h}]$ of coherent mappings $\boldsymbol{h}\colon Z \to \boldsymbol{Y}$. If $[\boldsymbol{h}] = \Gamma_{\boldsymbol{q}}(H)$, we say that H and $[\boldsymbol{h}]$ are associated with each other.*

A mapping $\boldsymbol{q}\colon Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is a collection of mappings $q_\mu\colon Z \to Y_\mu$, $\mu \in M$, such that $q_\mu = q_{\mu\mu'} q_{\mu'}$, for $\mu \le \mu'$. A resolution of Y is a mapping $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$, which satisfies certain conditions, named (R1) and (R2) (see [21, I. 6.3] or [15, II. 6.1]). If Y is topologically complete, e.g., if it is paracompact, and all Y_μ are Tychonoff spaces, then resolutions are inverse limits [15, Theorem 6.16]. Conversely, the limit of an inverse system of compact Hausdorff spaces is always a resolution (see [15, Theorem 6.20]). An HPol-resolution is a resolution $\boldsymbol{q}\colon Y \to \boldsymbol{Y}$, where all Y_μ belong to the class HPol of spaces having the homotopy type of polyhedra.

If $\boldsymbol{Y}$ is a cofinite system, i.e., every element of the index set M has a finite number of predecessors, then Proposition 1 is an immediate consequence of the definition of a strong shape morphism, as described in [15, 8.2]. That Proposition 1 also holds in the case when $\boldsymbol{Y}$ is not cofinite was proved in [20, Theorem 1].

In the case when $Y = X \times P$, it is convenient to use a particular HPol-resolution $\boldsymbol{q}\colon X \times P \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, called the *standard resolution* of $X \times P$, introduced in [16] (there it was called the basic construction). It is determined by a limit $\boldsymbol{p}\colon X \to \boldsymbol{X}$, where $\boldsymbol{X}$ is a cofinite inverse system of compact polyhedra and by a triangulation K of P. Moreover, the canonical projections $\pi_X\colon X \times P \to X$ and

$\pi_P\colon X\times P\to P$ induce mappings of systems $\boldsymbol{\pi}_X\colon \boldsymbol{Y}\to\boldsymbol{X}$ and $\boldsymbol{\pi}_P\colon\boldsymbol{Y}\to P$ (see Section 2). Note that the standard resolution of $X\times P$ is not cofinite.

We now state an existence property for coherent homotopy $(\mathrm{ECH})_Z$, which is the analogue of property $(\mathrm{ESS})_Z$ for strong shape.

$(\mathrm{ECH})_Z$ *For every homotopy class of coherent mappings* $[\boldsymbol{f}]\colon Z\to\boldsymbol{X}$ *and every homotopy class of mappings* $[g]\colon Z\to P$*, there exists a homotopy class of coherent mappings* $[\boldsymbol{h}]\colon Z\to\boldsymbol{Y}$ *such that* $[C(\boldsymbol{\pi}_X)][\boldsymbol{h}]=[\boldsymbol{f}]$ *and* $[C(\boldsymbol{\pi}_P)][\boldsymbol{h}]=[C(g)]$*, where* C *denotes the coherence operator* (*see Section* 3).

The following proposition was proved in [20], as Theorem 2. It relates conditions $(\mathrm{ESS})_Z$ and $(\mathrm{ECH})_Z$ and establishes an important property of standard resolutions $\boldsymbol{q}\colon X\times P\to\boldsymbol{Y}$.

Proposition 2. *Let* $\boldsymbol{X}$ *be a cofinite inverse system of compact polyhedra with limit* $\boldsymbol{p}\colon X\to\boldsymbol{X}$ *and let* K *be a simplicial complex with carrier* $P=|K|$*. Let* $\boldsymbol{q}\colon X\times P\to\boldsymbol{Y}$ *be the standard resolution of* $X\times P$ *associated with* $\boldsymbol{p}$ *and* K *and let* $\boldsymbol{\pi}_X\colon\boldsymbol{Y}\to\boldsymbol{X}$, $\boldsymbol{\pi}_P\colon\boldsymbol{Y}\to P$ *be mappings of systems, induced by the canonical projections* π_X,π_P*. For every topological space* Z*, the properties* $(\mathrm{ESS})_Z$ *for* X,P *and* $(\mathrm{ECH})_Z$ *for* $\boldsymbol{X}$, K *are equivalent.*

In view of Proposition 2, Theorem 1 is an immediate consequence of the following Theorem 3, which is the main technical result of the present paper, proved in Sections 4–9.

Theorem 3. *Let* $\boldsymbol{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ *be a cofinite inverse system of compact polyhedra with limit* $\boldsymbol{p}\colon X\to\boldsymbol{X}$ *and let* K *be a simplicial complex with carrier* $P=|K|$*. Let* $\boldsymbol{q}\colon X\times P\to\boldsymbol{Y}$ *be the standard resolution associated with* $\boldsymbol{X}$ *and* K *and let* $\boldsymbol{\pi}_X\colon\boldsymbol{Y}\to\boldsymbol{X}$ *and* $\boldsymbol{\pi}_P\colon\boldsymbol{Y}\to P$ *be mappings of systems induced by the canonical projections* π_X,π_P*. Then, for every metrizable space* Z*, property* $(\mathrm{ECH})_Z$ *holds.*

2. The standard resolution of $X\times P$

2.1. Let X be a compact Hausdorff space and let P be a polyhedron (CW-topology). Let $\boldsymbol{p}=(p_\lambda)\colon X\to\boldsymbol{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ be the inverse limit of an inverse system of compact polyhedra and let K be a triangulation of P. According to [16] (also see [20]), the standard resolution $\boldsymbol{q}=(q_\mu)\colon X\times P\to\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$ of the Cartesian product $X\times P$ is defined as follows.

Let M be the set of all increasing functions $\mu\colon K\to\Lambda$, i.e., functions such that $\sigma\le\sigma'$ implies $\mu(\sigma)\le\mu(\sigma')$. Endow M with the natural ordering, i.e., put $\mu\le\mu'$ provided $\mu(\sigma)\le\mu'(\sigma)$, for every $\sigma\in K$. It is easy to see that $(M,\le)$ is a directed ordered set, but in general, M fails to be cofinite.

In order to define the spaces Y_μ, one first associates with every $\sigma\in K$ and $\mu\in M$ the Cartesian product $X_{\mu(\sigma)}\times\sigma$. Then one considers the coproduct (disjoint sum)

$$\tilde{Y}_\mu=\coprod_{\sigma\in K}(X_{\mu(\sigma)}\times\sigma). \tag{1}$$

By definition, Y_μ is the quotient space

$$Y_\mu=\tilde{Y}_\mu/\sim_\mu, \tag{2}$$

where $\sim_\mu$ denotes the equivalence relation determined by considering points $(x,t) \in X_{\mu(\sigma)} \times \sigma \subseteq \tilde{Y}_\mu$ and $(x',t') \in X_{\mu(\sigma')} \times \sigma' \subseteq \tilde{Y}_\mu$ equivalent, provided $\sigma \leq \sigma'$, $x = p_{\mu(\sigma)\mu(\sigma')}(x')$ and $t' = i_{\sigma\sigma'}(t)$, where $i_{\sigma\sigma'}\colon \sigma \to \sigma'$ is the inclusion mapping (we shall usually simplify the notation and write $t' = t$ instead of $t' = i_{\sigma\sigma'}(t)$). The corresponding quotient mapping is denoted by $\phi_\mu\colon \tilde{Y}_\mu \to Y_\mu$.

In order to define the mappings $q_{\mu\mu'}\colon Y_{\mu'} \to Y_\mu$, one first defines mappings $\tilde{q}_{\mu\mu'}\colon \tilde{Y}_{\mu'} \to \tilde{Y}_\mu$, by putting

$$\tilde{q}_{\mu\mu'}(x,t) = (p_{\mu(\sigma)\mu'(\sigma)}(x), t), \tag{3}$$

for $(x,t) \in X_{\mu'(\sigma)} \times \sigma \subseteq \tilde{Y}_\mu$. It is readily seen that there exist unique mappings $q_{\mu\mu'}\colon Y_{\mu'} \to Y_\mu$ such that

$$q_{\mu\mu'}\phi_{\mu'} = \phi_\mu \tilde{q}_{\mu\mu'}. \tag{4}$$

Moreover, $q_{\mu\mu'}q_{\mu'\mu''} = q_{\mu\mu''}$, for $\mu \leq \mu' \leq \mu''$, so that $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is an inverse system.

$\boldsymbol{q}\colon X \times P \to \boldsymbol{Y}$ consists of mappings $q_\mu\colon X \times P \to Y_\mu$, $\mu \in M$, defined as follows. With every $\sigma \in K$ and $\mu \in M$ one associates the mapping $p_{\mu(\sigma)} \times 1_\sigma\colon X \times \sigma \to X_{\mu(\sigma)} \times \sigma$, where $p_\lambda\colon X \to X_\lambda$, $\lambda \in \Lambda$, are the projections forming $\boldsymbol{p}\colon X \to \boldsymbol{X}$. Put

$$\tilde{Y} = \coprod_{\sigma\in K} (X \times \sigma) = X \times \coprod_{\sigma\in K} \sigma \tag{5}$$

and define mappings $\tilde{q}_\mu\colon \tilde{Y} \to \tilde{Y}_\mu$, by putting

$$\tilde{q}_\mu(x,t) = (p_{\mu(\sigma)}(x), t), \tag{6}$$

for $(x,t) \in X \times \sigma \subseteq \tilde{Y}$. We also consider the quotient mapping $\phi = 1_X \times u\colon \tilde{Y} \to X \times P$, where $u\colon \coprod_{\sigma\in K} \sigma \to P$ is the quotient mapping defined by the requirement that the restrictions $u|\sigma\colon \sigma \to P$ are inclusion mappings $\sigma \hookrightarrow P$. It is readily seen that there exist unique mappings $q_\mu\colon X \times P \to Y_\mu$ such that

$$\phi_\mu \tilde{q}_\mu = q_\mu \phi. \tag{7}$$

Moreover, $q_\mu = q_{\mu\mu'}q_{\mu'}$, for $\mu \leq \mu'$.

We also consider two mappings of systems $\boldsymbol{\pi}_X\colon \boldsymbol{Y} \to \boldsymbol{X}$ and $\boldsymbol{\pi}_P\colon \boldsymbol{Y} \to P$, defined as follows. With every $\lambda \in \Lambda$ one associates the constant function $\sigma \mapsto \lambda$, for $\sigma \in K$, denoted by $\overline{\lambda}$. Clearly, $\overline{\lambda}$ belongs to M. By (1), $\tilde{Y}_{\overline{\lambda}} = X_\lambda \times (\coprod_{\sigma\in K} \sigma)$. Moreover, if $(x,t) \in X_{\overline{\lambda}(\sigma)} \times \sigma = X_\lambda \times \sigma \subseteq \tilde{Y}_{\overline{\lambda}}$, $(x',t') \in X_{\overline{\lambda}(\sigma')} \times \sigma' = X_\lambda \times \sigma' \subseteq \tilde{Y}_{\overline{\lambda}}$ and $(x,t) \sim_{\overline{\lambda}} (x',t')$, then $x = x'$ and $u(t) = u(t')$. To verify this assertion, it suffices to consider the case when $\sigma \leq \sigma'$. In that case, $x = p_{\overline{\lambda}(\sigma)\overline{\lambda}(\sigma')}(x') = p_{\lambda\lambda}(x') = x'$ and $t' = i_{\sigma\sigma'}(t)$, hence also $u(t) = u(t')$. All this shows that $Y_{\overline{\lambda}} = X_\lambda \times P$ and the quotient mapping $\phi_{\overline{\lambda}}\colon \tilde{Y}_{\overline{\lambda}} \to Y_{\overline{\lambda}}$ is the mapping $1_{X_\lambda} \times u\colon X_\lambda \times (\coprod_{\sigma\in K} \sigma) \to X_\lambda \times P$.

By definition, the mapping $\boldsymbol{\pi}_X$ is given by the increasing function $\lambda \mapsto \overline{\lambda}$ and by the first projections $\pi_\lambda\colon Y_{\overline{\lambda}} = X_\lambda \times P \to X_\lambda$. Since $q_{\overline{\lambda\lambda'}} = p_{\lambda\lambda'} \times 1_P$, one has $\pi_\lambda q_{\overline{\lambda\lambda'}} = p_{\lambda\lambda'}\pi_{\lambda'}$ and thus, $\boldsymbol{\pi}_X\colon \boldsymbol{Y} \to \boldsymbol{X}$ is a mapping. Since P is a polyhedron, the mapping $\boldsymbol{\pi}_P\colon \boldsymbol{Y} \to P$ is determined (up to equivalence), by any index $\lambda \in \Lambda$ and by the second projection $\pi_P\colon Y_{\overline{\lambda}} = X_\lambda \times P \to P$. It is readily seen that

$$\boldsymbol{\pi}_X \boldsymbol{q} = \boldsymbol{p}\pi_X, \quad \boldsymbol{\pi}_P \boldsymbol{q} = \pi_P, \tag{8}$$

where $\pi_X\colon X \times P \to X$ and $\pi_P\colon X \times P \to P$ are the canonical projections.

2.2. In [16], it was proved that the spaces Y_μ are (Hausdorff) paracompact spaces, belonging to the class HPol of spaces having the homotopy type of polyhedra. Consequently, the standard resolution $\boldsymbol{q}\colon X \times P \to \boldsymbol{Y}$ is a non-cofinite HPol-resolution. Recently, the author showed that the spaces Y_μ are (Hausdorff) stratifiable k-spaces (see [19, Lemmas 4 and 5]). Recall that stratifiable spaces were introduced in 1961 by J. Ceder [4] as a generalization of metrizable spaces. Ceder proved that polyhedra (even CW-complexes), which in general are non-metrizable, belong to the class of stratifiable spaces. Moreover, he proved that stratifiable spaces are (Hausdorff) paracompact and perfectly normal spaces.

In some situations the spaces Y_μ are (non-compact) polyhedra and it was proved in 1952 by J. Dugundji [6] that polyhedra are absolute neighborhood extensors, abbreviated ANEs, for metrizable spaces. It is known that, in general, polyhedra are not ANEs for Lindelöf spaces [5], let alone ANEs for paracompact spaces [1]. Therefore, the spaces Y_μ cannot be ANEs for these two classes of spaces. In the present paper we will use the following lemma, established in [19] as Theorem 1 and Lemma 7.

Lemma 1. *The spaces Y_μ in the standard resolution $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ of $X \times P$ are* ANE*s for metrizable spaces.*

R. Cauty proved that polyhedra (even CW-complexes) are ANEs for stratifiable spaces ([3, Theorem 2.3 and Corollary 1.5]; for CW-complexes see [2, Theorem 8]). This opens the question, whether the spaces Y_μ are ANEs for stratifiable spaces. We do not know the answer.

A pair of spaces (A, B), where B is a closed subset of A, is said to have the *homotopy extension property* (abbreviated (HEP)) with respect to a space Y, provided every mapping $f\colon (A \times 1) \cup (B \times I) \to Y$ admits an extension $h\colon A \times I \to Y$. Recall the following elementary fact.

Lemma 2. *If Y is an* ANE *for metrizable spaces, then every metrizable pair of spaces (A, B), B closed in A, has the homotopy extension property with respect to Y.*

Proof. The well-known Dowker lemma (see [12, Lemma IV.2.1]) asserts that a pair of spaces (A, B), B closed in A, has the (HEP) with respect to a space Y, provided the spaces A and $A \times I$ are normal and every mapping $f\colon (A \times 1) \cup (B \times I) \to Y$ admits a neighborhood U of B in A such that f can be extended to a mapping $\overline{f}\colon (A \times 1) \cup (U \times I) \to Y$. If A is metrizable and Y is an ANE for metrizable spaces, then f admits an extension $\overline{\overline{f}}$ to a neighborhood V of $(A \times 1) \cup (B \times I)$ in $A \times I$. Using compactness of I, it is easy to find a neighborhood U of B in A such that $(A \times 1) \cup (U \times I) \subseteq V$. Clearly, the restriction $\overline{f}$ of $\overline{\overline{f}}$ to $(A \times 1) \cup (U \times I)$ has the required property. $\square$

3. Preliminaries on coherent mappings

3.1 The general reference for this section is [15]. However, in distinction to [15], when considering mappings of system (abbreviated as mappings) and coherent mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ between inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, unless explicitly stated, we do not assume that M is cofinite. A mapping consists of an increasing function $f\colon M \to \Lambda$ (the index function) and of a collection of mappings $f_\mu\colon X_{f(\mu)} \to Y_\mu$ such that

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}, \quad \mu \le \mu'. \tag{9}$$

If $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}\colon \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$ are mappings, given by index functions f, g and by mappings f_μ, g_ν, the composition $\boldsymbol{g}\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Z}$ is the mapping $\boldsymbol{h}\colon \boldsymbol{X} \to \boldsymbol{Z}$, given by the index function $h = fg$ and by the mappings $h_\nu = g_\nu f_{g(\nu)}$.

A coherent mapping $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing function $f\colon M \to \Lambda$ and of a collection of mappings $f_{\boldsymbol{\mu}} = f_{\mu_0\ldots\mu_n}\colon X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$, where $\Delta^n = [e_0, \ldots, e_n]$ is the standard n-simplex and $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ is a multiindex in M of length $|\boldsymbol{\mu}| = n \ge 0$. One requires that the following two *coherence conditions* are fulfilled. The *boundary condition*

$$f_{\boldsymbol{\mu}}(x, d_j t) = \begin{cases} q_{\mu_0\mu_1} f_{d^0\boldsymbol{\mu}}(x,t), & j = 0,\\ f_{d^j\boldsymbol{\mu}}(x,t), & 1 \le j \le n-1,\\ f_{d^n\boldsymbol{\mu}}(p_{f(\mu_{n-1})f(\mu_n)}(x), t), & j = n, \end{cases} \tag{10}$$

where $d_j\colon \Delta^{n-1} \to \Delta^n$ are the standard boundary operators and d^j are the operators which omit μ_j from $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$, i.e., $d^j\boldsymbol{\mu} = (\mu_0, \ldots, \widehat{\mu_j}, \ldots, \mu_n)$. Condition (10) makes sense only when $n > 0$.

The *degeneracy condition*

$$f_{\boldsymbol{\mu}}(x, s_j t) = f_{s^j\boldsymbol{\mu}}(x,t), \quad 0 \le j \le n, \tag{11}$$

where $s_j\colon \Delta^{n+1} \to \Delta^n$ are the standard degeneracy operators and s^j is the operator which repeats μ_j, i.e., $s^j\boldsymbol{\mu} = (\mu_0, \ldots, \mu_j, \mu_j, \ldots, \mu_n)$. The composition $\boldsymbol{g}\boldsymbol{f}$ of two coherent mappings is given by a rather complicated formula (see Section 1.3 of [15]), which we do not need in this paper.

If $\boldsymbol{X}$ consists of a single space X, formula (10) assumes the simpler form

$$f_{\boldsymbol{\mu}}(x, d_j t) = \begin{cases} q_{\mu_0\mu_1} f_{d^0\boldsymbol{\mu}}(x,t), & j = 0,\\ f_{d^j\boldsymbol{\mu}}(x,t), & 1 \le j \le n. \end{cases} \tag{12}$$

Coherent mappings can be viewed as generalizations of mappings, because with every mapping $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ one can associate a coherent mapping $C(\boldsymbol{f})\colon \boldsymbol{X} \to \boldsymbol{Y}$ which consists of the index function f of $\boldsymbol{f}$ and of mappings $f_{\boldsymbol{\mu}}\colon X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$, where $f_{\boldsymbol{\mu}}(x,t) = f_{\mu_0} p_{f(\mu_0)f(\mu_n)}(x)$. For mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}\colon \boldsymbol{Y} \to \boldsymbol{Z}$ one has $C(\boldsymbol{g}\boldsymbol{f}) = C(\boldsymbol{g})C(\boldsymbol{f})$ ([15, Lemma 1.17]).

3.2. Two mappings $\boldsymbol{f}, \boldsymbol{f}'\colon \boldsymbol{X} \to \boldsymbol{Y}$, given by increasing index functions f, f' and mappings f_μ, f'_μ, $\mu \in M$, are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, if there exists an increasing function

$F\colon M \to \Lambda$, $F \geq f, f'$, such that

$$f_\mu p_{f(\mu_n)F(\mu_n)} \simeq f'_\mu p_{f'(\mu_n)F(\mu_n)}. \tag{13}$$

Two coherent mappings $\boldsymbol{f}, \boldsymbol{f}'\colon X \to \boldsymbol{Y}$, given by mappings $f_{\boldsymbol{\mu}}, f'_{\boldsymbol{\mu}}$ are homotopic, $\boldsymbol{f} \simeq \boldsymbol{f}'$, provided there exists a coherent mapping $\boldsymbol{F}\colon X \times I \to \boldsymbol{Y}$, given by mappings $F_{\boldsymbol{\mu}}\colon X \times I \times \Delta^n \to Y_{\mu_0}$, which satisfy the corresponding coherence conditions and

$$F_{\boldsymbol{\mu}}(x, 0, t) = f_{\boldsymbol{\mu}}(x, t), \qquad F_{\boldsymbol{\mu}}(x, 1, t) = f'_{\boldsymbol{\mu}}(x, t). \tag{14}$$

If for given $\boldsymbol{X}$ and $\boldsymbol{Y}$ the homotopy relation $\simeq$ for mappings (coherent mappings) $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an equivalence relation, the homotopy class $[\boldsymbol{f}]$ of $\boldsymbol{f}$ is well defined. There are two simple cases when this is the case. The first one is when $\boldsymbol{X} = X$ is a single space and the second one is when $\boldsymbol{Y}$ is cofinite (see the proofs of Lemmas 1.2 and 2.1 of [15]). Moreover, if $\mathrm{Coh}(\boldsymbol{X}, \boldsymbol{Y})$ denotes the set of all coherent mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$, then the proofs of Lemmas 2.4 and 2.5 of [15] show that, whenever $\simeq$ is an equivalence relation on $\mathrm{Coh}(\boldsymbol{X}, \boldsymbol{Y})$, $\mathrm{Coh}(\boldsymbol{Y}, \boldsymbol{Z})$ and $\mathrm{Coh}(\boldsymbol{X}, \boldsymbol{Z})$, then the homotopy classes $[\boldsymbol{f}]\colon \boldsymbol{X} \to \boldsymbol{Y}$, $[\boldsymbol{g}]\colon \boldsymbol{Y} \to \boldsymbol{Z}$ and $[\boldsymbol{gf}]\colon \boldsymbol{X} \to \boldsymbol{Z}$ are well defined and $[\boldsymbol{gf}]$ depends only on $[\boldsymbol{f}]$ and $[\boldsymbol{g}]$. Therefore, one defines the composition $[\boldsymbol{g}][\boldsymbol{f}]$ by putting $[\boldsymbol{g}][\boldsymbol{f}] = [\boldsymbol{gf}]$ (for more details see [20, Lemma 1]).

3.3. In the proof of Theorem 3 we will also need the following lemma on coherent mappings (see [15, Lemma 2.12]).

Lemma 3. *Let $\boldsymbol{f} = (f, f_{\boldsymbol{\mu}})\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a coherent mapping and let $\boldsymbol{g} = (g, g_\nu)\colon \boldsymbol{Y} \to \boldsymbol{Z}$ be a mapping. Then the composition $C(\boldsymbol{g})\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Z}$ is homotopic to the coherent mapping $\boldsymbol{h} = (h, h_{\boldsymbol{\nu}})\colon \boldsymbol{X} \to \boldsymbol{Z}$, where $h = fg$ and $h_{\boldsymbol{\nu}}\colon X_{h(\nu_n)} \times \Delta^n \to Z_{\nu_0}$ is given by*

$$h_{\boldsymbol{\nu}}(x, t) = g_{\nu_0} f_{g(\nu_0),\dots,g(\nu_n)}(x, t). \tag{15}$$

4. Structure of the proof of Theorem 3

4.1. Let $\boldsymbol{p}\colon X \to \boldsymbol{X}$, K and $\boldsymbol{\pi}_X$ and $\boldsymbol{\pi}_P$ be as in Theorem 3. To prove that $(\mathrm{ECH})_Z$ holds for metrizable spaces Z, one considers a homotopy class of coherent mappings $[\boldsymbol{f}]\colon Z \to \boldsymbol{X}$ and a homotopy class of mappings $[g]\colon Z \to P$. We will construct a homotopy class of coherent mappings $[\boldsymbol{h}]\colon Z \to \boldsymbol{Y}$ such that

$$[C(\boldsymbol{\pi}_X)][\boldsymbol{h}] = [\boldsymbol{f}], \tag{16}$$

$$[C(\boldsymbol{\pi}_P)][\boldsymbol{h}] = [C(g)]. \tag{17}$$

Let $\boldsymbol{f}$ consist of mappings $f_{\boldsymbol{\lambda}}\colon Z \times \Delta^n \to X_{\lambda_0}$, where $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_n)$ is a multiindex in Λ of length $|\boldsymbol{\lambda}| = n$, satisfying the coherence conditions

$$f_{\boldsymbol{\lambda}}(z, d_j t) = \begin{cases} p_{\lambda_0\lambda_1} f_{d^0\boldsymbol{\lambda}}(z, t), & j = 0, \\ f_{d^j\boldsymbol{\lambda}}(z, t), & 1 \leq j \leq n, \end{cases} \tag{18}$$

$$f_{\boldsymbol{\lambda}}(z, s_j t) = f_{s^j\boldsymbol{\lambda}}(z, t), \quad 0 \leq j \leq n. \tag{19}$$

Note that condition (18) makes sense only when $n > 0$.

The coherent mapping $\boldsymbol{h}\colon Z \to \boldsymbol{Y}$, which we will construct, will consist of mappings $h_{\boldsymbol{\mu}}\colon Z \times \Delta^n \to Y_{\mu_0}$, where $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ is a multiindex in M of length $|\boldsymbol{\mu}| = n$, $n \geq 0$, having the following three properties.

$$h_{\boldsymbol{\mu}}(z, d_j t) = \begin{cases} q_{\mu_0\mu_1} h_{d^0\boldsymbol{\mu}}(z,t), & j = 0, \\ h_{d^j\boldsymbol{\mu}}(z,t), & 1 \leq j \leq n, \end{cases} \tag{20}$$

$$h_{\boldsymbol{\mu}}(z, s_j t) = h_{s^j\boldsymbol{\mu}}(z,t), \quad 0 \leq j \leq n, \tag{21}$$

$$h_{\overline{\lambda_0}\dots\overline{\lambda_n}}(z,t) = (f_{\lambda_0\dots\lambda_n}(z,t), g(z)), \tag{22}$$

for $(z,t) \in Z \times \Delta^n$. We refer to (20) as to the *boundary condition* for $h_{\boldsymbol{\mu}}$. It makes sense only when $n > 0$. We refer to (21) as to the *degeneracy condition* for $h_{\boldsymbol{\mu}}$. We refer to multiindices $\boldsymbol{\mu}$ of the form $\overline{\boldsymbol{\lambda}} = (\overline{\lambda_0}, \dots, \overline{\lambda_n})$ as to *special multiindices* and we refer to (22) as to the *special condition* for $h_{\overline{\boldsymbol{\lambda}}}$. Note that $h_{\overline{\lambda_0}\dots\overline{\lambda_n}}(z,t)$ belongs to $Y_{\overline{\lambda_0}} = X_{\lambda_0} \times P$ just as $(f_{\lambda_0\dots\lambda_n}(z,t), g(z))$ does.

Special condition (22) insures that conditions (16) and (17) are fulfilled. Indeed, since $\boldsymbol{h}$ is a coherent mapping and $\boldsymbol{\pi}_X$ is a mapping, Lemma 3 shows that the composition $C(\boldsymbol{\pi}_X)\boldsymbol{h}$ is homotopic to a coherent mapping $\boldsymbol{h}'\colon Z \to \boldsymbol{X}$, given by mappings $h'_{\lambda_0\dots\lambda_n}: Z \times \Delta^n \to X_{\lambda_0}$, where $h'_{\lambda_0\dots\lambda_n} = \pi_{\lambda_0} h_{\overline{\lambda_0}\dots\overline{\lambda_n}}$. Since $\pi_{\lambda_0}\colon X_{\lambda_0} \times P \to X_{\lambda_0}$ is the first projection, (22) shows that $h'_{\lambda_0\dots\lambda_n} = f_{\lambda_0\dots\lambda_n}$ and thus, $\boldsymbol{h}' = \boldsymbol{f}$. Consequently, $C(\boldsymbol{\pi}_X)\boldsymbol{h} \simeq \boldsymbol{f}$. Again by Lemma 3, the composition $C(\boldsymbol{\pi}_P)\boldsymbol{h}$ is homotopic to a coherent mapping $\boldsymbol{h}''\colon Z \to P$, given by mappings $h''_{\nu\dots\nu}: Z \times \Delta^n \to P$, where $h''_{\nu\dots\nu}(z,t) = \pi_P h_{\overline{\lambda_0}\dots\overline{\lambda_0}}(z,t)$; here ν is the index of the only element P of the rudimentary system P. Since $\pi_P\colon X_{\lambda_0} \times P \to P$ is the second projection, (22) shows that $h''_{\nu\dots\nu}(z,t) = g(z)$. On the other hand, the coherent mapping $C(g)$ consists of the mappings $g_{\nu\dots\nu}\colon Z \times \Delta^n \to P$, where $g_{\nu\dots\nu}(z,t) = g(z)$. Consequently, $\boldsymbol{h}'' = C(g)$ and thus, $C(\boldsymbol{\pi}_P)\boldsymbol{h} \simeq C(g)$.

4.2. In order to construct the mappings $h_{\boldsymbol{\mu}}$, we consider the subsets $Z^\sigma \times \Delta^n$ of $Z \times \Delta^n$, where $\sigma \in K$ and

$$Z^\sigma = g^{-1}(\sigma) \subseteq Z. \tag{23}$$

Note that, whenever τ is a face of σ, i.e., $\tau \leq \sigma$, one has $Z^\tau \subseteq Z^\sigma$. We will define mappings $h^\sigma_{\boldsymbol{\mu}}\colon Z^\sigma \times \Delta^n \to Y_{\mu_0}$, which satisfy the boundary and the degeneracy conditions and thus, form a coherent mapping $\boldsymbol{h}^\sigma = (h^\sigma_{\boldsymbol{\mu}})$. Moreover, the mappings $h^\sigma_{\boldsymbol{\mu}}$ will satisfy the special condition and the following *additional condition*

$$h^\sigma_{\boldsymbol{\mu}}|(Z^\tau \times \Delta^n) = h^\tau_{\boldsymbol{\mu}}, \tag{24}$$

whenever, $\tau \leq \sigma$. Clearly, (24) holds in general if it holds in the case when $\dim\sigma = \dim\tau + 1$.

Note that for some $\sigma \in K$, the set Z^σ can be empty. In that case we define $h^\sigma_{\boldsymbol{\mu}}$ to be the empty function. Clearly, $\{Z^\sigma : \sigma \in K\}$ is a closed covering of Z and $\{Z^\sigma \times \Delta^n : \sigma \in K\}$ is a closed covering of $Z \times \Delta^n$. Because of (24), there is a unique function $h_{\boldsymbol{\mu}}\colon Z \times \Delta^n \to Y_{\mu_0}$ such that, for every $\sigma \in K$,

$$h_{\boldsymbol{\mu}}|(Z^\sigma \times \Delta^n) = h^\sigma_{\boldsymbol{\mu}}. \tag{25}$$

Indeed, for every $(z,t) \in Z \times \Delta^n$, there is a $\sigma \in K$ such that $(z,t) \in Z^\sigma \times \Delta^n$. Put $h_{\boldsymbol{\mu}}(z,t) = h^\sigma_{\boldsymbol{\mu}}(z,t)$. If also $(z,t) \in Z^{\sigma'} \times \Delta^n$, for some $\sigma' \in K$, then (24), for $\tau = \sigma \cap \sigma' \in K$, shows that $h^\sigma_{\boldsymbol{\mu}}(z,t) = h^\tau_{\boldsymbol{\mu}}(z,t) = h^{\sigma'}_{\boldsymbol{\mu}}(z,t)$. Consequently, $h_{\boldsymbol{\mu}} \colon Z \times \Delta^n \to Y_{\mu_0}$ is a well-defined function, which satisfies (25). Uniqueness of $h_{\boldsymbol{\mu}}$ is an obvious consequence of (25). The functions $h_{\boldsymbol{\mu}}$ satisfy the coherence conditions and special condition (22), because the mappings $h^\sigma_{\boldsymbol{\mu}}$ satisfy these conditions. It remains to prove continuity of $h_{\boldsymbol{\mu}}$. Since the restrictions $h_{\boldsymbol{\mu}}|(Z^\sigma \times \Delta^n) = h^\sigma_{\boldsymbol{\mu}}$ are continuous, the continuity of $h_{\boldsymbol{\mu}}$ is an immediate consequence of the following lemma.

Lemma 4. *$Z\times\Delta^n$ has the weak topology determined by its closed covering $\{Z^\sigma\times\Delta^n : \sigma \in K\}$.*

Proof. Let B be a subset of $Z \times \Delta^n$ such that $B \cap (Z^\sigma \times \Delta^n)$ is closed in $Z^\sigma \times \Delta^n$, for every $\sigma \in K$. By the definition of weak topology, we must prove that B is closed in $Z \times \Delta^n$. Being a metrizable space, $Z \times \Delta^n$ is a k-space (see [10], Theorems 3.3.18 and 3.3.20). Therefore, it suffices to prove that $B \cap C$ is a closed subset of C, for every compact subset C of $Z \times \Delta^n$. Since C is compact, so is $g\pi(C) \subseteq P$, where π denotes the projection $\pi \colon Z \times \Delta^n \to Z$. Consequently, there is a finite subcomplex $L \subseteq K$ such that $g\pi(C) \subseteq |L|$ and thus, $C \subseteq \pi^{-1}g^{-1}(|L|)$. Since $|L|$ is the union of finitely many simplices $\sigma_1, \dots, \sigma_m \in K$, it follows that $\pi^{-1}g^{-1}(|L|) = (\pi^{-1}g^{-1}(\sigma_1)) \cup \ldots \cup (\pi^{-1}g^{-1}(\sigma_m))$. However, $\pi^{-1}g^{-1}(\sigma_i) = \pi^{-1}(Z^{\sigma_i}) = Z^{\sigma_i} \times \Delta^n$ and thus, $\pi^{-1}g^{-1}(|L|) = (Z^{\sigma_1}\times\Delta^n)\cup\ldots\cup(Z^{\sigma_m}\times\Delta^n)$. Therefore, $B\cap\pi^{-1}g^{-1}(|L|) = (B \cap (Z^{\sigma_1} \times \Delta^n)) \cup \ldots \cup (B \cap (Z^{\sigma_m} \times \Delta^n))$. By the assumption, $B \cap (Z^{\sigma_i} \times \Delta^n)$ is a closed subset of $Z^{\sigma_i} \times \Delta^n$ and since $Z^{\sigma_i} \times \Delta^n$ is a closed subset of $Z \times \Delta^n$, it is also a closed subset of $\pi^{-1}g^{-1}(|L|)$. It follows that $B \cap (Z^{\sigma_i} \times \Delta^n)$ is a closed subset of $\pi^{-1}g^{-1}(|L|)$. Since $C \subseteq g^{-1}(|L|)$, we conclude that indeed, $B \cap C$ is a closed subset of C. $\square$

4.3. We will denote simplices $\sigma \in K$ of dimension i by σ^i. Therefore, we need mappings $h^{\sigma^i}_{\boldsymbol{\mu}} : Z^{\sigma^i} \times \Delta^n \to Y_{\mu_0}$, $i = 0, 1, \dots$, which satisfy the *boundary*, the *degeneracy* and the *special condition*

$$h^{\sigma^i}_{\boldsymbol{\lambda}}(z,t) = (f_{\lambda_0\ldots\lambda_n}(z,t), g(z)), \quad z \in Z^{\sigma^i},\ t \in \Delta^n, \tag{26}$$

as well as the *additional condition*

$$h^{\sigma^{i+1}}_{\boldsymbol{\mu}}(z,t) = h^{\sigma^i}_{\boldsymbol{\mu}}(z,t), \quad \text{for } \sigma^i \le \sigma^{i+1},\ z \in Z^{\sigma^i},\ t \in \Delta^n. \tag{27}$$

In order to define the coherent mappings $\boldsymbol{h}^{\sigma^i} = (h^{\sigma^i}_{\boldsymbol{\mu}}) \colon Z \to \boldsymbol{Y}$ we need some auxiliary coherent mappings. We distinguish two types. The ones of the first type are defined by explicit formulae in Sections 5, 6 and 7. They are of the form

$$\begin{aligned}
\overline{\boldsymbol{h}}^{\sigma^i} &= (\overline{h}^{\sigma^i}_{\boldsymbol{\mu}}) \colon Z^{\sigma^i} \to \boldsymbol{Y}, \quad 0 \le i,\\
\overline{\boldsymbol{h}}^{\sigma^i\sigma^j} &= (\overline{h}^{\sigma^i\sigma^j}_{\boldsymbol{\mu}}) \colon Z^{\sigma^i} \times \Delta^1 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^j,\ 0 \le i < j,\\
\overline{\boldsymbol{h}}^{\sigma^i\sigma^j\sigma^k} &= (\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}) \colon Z^{\sigma^i} \times \Delta^2 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^j \le \sigma^k,\ 0 \le i < j < k,
\end{aligned}$$

and satisfy appropriate special and additional conditions. The *special conditions* are the analogues of (26) and read as follows.

$$\overline{h}^{\sigma^i}_{\boldsymbol{\lambda}}(z,t) = \overline{h}^{\sigma^i\sigma^j}_{\boldsymbol{\lambda}}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\lambda}}(z,s,t) = (f_{\lambda_0\ldots\lambda_n}(z,t), g(z)). \tag{28}$$

The *additional conditions* are the following.

$$\overline{h}^{\sigma^i\sigma^j}_{\boldsymbol{\mu}}(z,e_0,t) = \overline{h}^{\sigma^i}_{\boldsymbol{\mu}}(z,t), \quad z \in Z^{\sigma^i}, \tag{29}$$

$$\overline{h}^{\sigma^i\sigma^j}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{h}^{\sigma^j}_{\boldsymbol{\mu}}(z,t), \quad z \in Z^{\sigma^i}, \tag{30}$$

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z,d_l s,t) = \begin{cases} \overline{h}^{\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z,s,t), & l=0, \\ \overline{h}^{\sigma^i\sigma^k}_{\boldsymbol{\mu}}(z,s,t), & l=1, \\ \overline{h}^{\sigma^i\sigma^j}_{\boldsymbol{\mu}}(z,s,t), & l=2, \end{cases} \tag{31}$$

where $z \in Z^{\sigma^i}$, $s \in \Delta^1$, $t \in \Delta^n$.

Auxiliary coherent mappings of the second type will be defined in the course of an induction process, using also auxiliary coherent mappings of the first type, some explicit formulae and two coherent homotopy extension properties (CHEP) (see Section 8). The auxiliary coherent mappings of the second type are of the form

$$\begin{aligned}
\boldsymbol{H}^{\sigma^i} &= (H^{\sigma^i}_{\boldsymbol{\mu}})\colon Z^{\sigma^i} \times \Delta^1 \to \boldsymbol{Y}, \\
\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+1}} &= (\overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}})\colon Z^{\sigma^i} \times \Delta^2 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^{i+1}, \\
\boldsymbol{H}^{\sigma^i\sigma^{i+1}} &= (H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}})\colon Z^{\sigma^i} \times \Delta^1 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^{i+1}, \\
\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+2}} &= (\overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}})\colon Z^{\sigma^i} \times \Delta^2 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^{i+2}, \\
\boldsymbol{H}^{\sigma^i\sigma^{i+2}} &= (H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}})\colon Z^{\sigma^i} \times \Delta^1 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^{i+2}, \\
\boldsymbol{H}^{\sigma^i\sigma^{i+1}\sigma^{i+2}} &= (H^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}})\colon Z^{\sigma^i} \times \Delta^2 \to \boldsymbol{Y}, \quad \sigma^i \le \sigma^{i+1} \le \sigma^{i+2}.
\end{aligned}$$

The corresponding special conditions are

$$\begin{aligned}
H^{\sigma^i}_{\boldsymbol{\lambda}}(z,s,t) &= \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\lambda}}(z,s,t) = H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\lambda}}(z,s,t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\lambda}}(z,s,t) \\
&= H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\lambda}}(z,s,t) = H^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\lambda}}(z,s,t) \\
&= (f_{\lambda_0\ldots\lambda_n}(z,t), g(z)), \quad z \in Z^{\sigma^i}.
\end{aligned} \tag{32}$$

The corresponding additional conditions are

$$H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_0,t) = h^{\sigma^i}_{\boldsymbol{\mu}}(z,t), \quad z \in Z^{\sigma^i}, \tag{33}$$

$$H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{h}^{\sigma^i}_{\boldsymbol{\mu}}(z,t), \quad z \in Z^{\sigma^i}, \tag{34}$$

$$H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = H^{\sigma^{i-1}\sigma^i}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^{i-1}}, \tag{35}$$

$$\overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_j s,t) = \begin{cases} \overline{h}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t), & z \in Z^{\sigma^i},\ s \in \Delta^1,\ j=0, \\ H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t), & z \in Z^{\sigma^i},\ s \in \Delta^1,\ j=1, \\ H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t), & z \in Z^{\sigma^i},\ s \in \Delta^1,\ j=2, \end{cases} \tag{36}$$

$$\overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z,s,t) = H_{\boldsymbol{\mu}}^{\sigma^{i-1}\sigma^i\sigma^{i+1}}(z,s,t), \quad z \in Z^{\sigma^{i-1}},\ s \in \Delta^2. \tag{37}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z,e_0,t) = h_{\boldsymbol{\mu}}^{\sigma^i}(z,t), \quad z \in Z^{\sigma^i}, \tag{38}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z,e_1,t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z,t), \quad z \in Z^{\sigma^i}, \tag{39}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z,s,t) = H_{\boldsymbol{\mu}}^{\sigma^{i-1}\sigma^{i+1}}(z,s,t), \quad z \in Z^{\sigma^{i-1}}. \tag{40}$$

$$\overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,d_js,t) = \begin{cases} \overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,s,t), & z \in Z^{\sigma^i},\ s \in \Delta^1,\ j=0, \\ H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,s,t), & z \in Z^{\sigma^i},\ s \in \Delta^1,\ j=1, \\ H_{\boldsymbol{\mu}}^{\sigma^i}(z,s,t), & z \in Z^{\sigma^i},\ s \in \Delta^1,\ j=2, \end{cases} \tag{41}$$

$$\overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,s,t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^0\sigma^i\sigma^{i+2}}(z,s,t), \quad z \in Z^{\sigma^0},\ s \in \Delta^2. \tag{42}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,e_0,t) = h_{\boldsymbol{\mu}}^{\sigma^i}(z,t), \quad z \in Z^{\sigma^i}, \tag{43}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,e_1,t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+2}}(z,t), \quad z \in Z^{\sigma^i}, \tag{44}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,s,t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^0\sigma^{i+2}}(z,s,t), \quad z \in Z^{\sigma^0}, \tag{45}$$

$$H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z,d_js,t) = \begin{cases} \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+1}\sigma^{i+2}}(z,s,t), & j=0, \\ H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z,s,t), & j=1, \\ H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z,s,t), & j=2, \end{cases} \tag{46}$$

where $z \in Z^{\sigma^i}$, $s \in \Delta^2$, $t \in \Delta^n$. Note that in (42) and (45) z is restricted to Z^{σ^0}.

4.4. We will define, by induction, a sequence $\boldsymbol{H}^0, \boldsymbol{H}^1, \dots$ of coherent mappings of the following form.

$$\begin{array}{llllll} \boldsymbol{h}^{\sigma^0}, & \boldsymbol{H}^{\sigma^0\sigma^1}, & & \boldsymbol{H}^{\sigma^0\sigma^2}, & \boldsymbol{H}^{\sigma^0\sigma^1\sigma^2}, & \boldsymbol{H}^{\sigma^1}, \\ \boldsymbol{h}^{\sigma^1}, \overline{\boldsymbol{H}}^{\sigma^1\sigma^2}, & \boldsymbol{H}^{\sigma^1\sigma^2}, & \overline{\boldsymbol{H}}^{\sigma^1\sigma^3}, & \boldsymbol{H}^{\sigma^1\sigma^3}, & \boldsymbol{H}^{\sigma^1\sigma^2\sigma^3}, & \boldsymbol{H}^{\sigma^2}, \\ \dots\dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots \\ \boldsymbol{h}^{\sigma^i}, \overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+1}}, & \boldsymbol{H}^{\sigma^i\sigma^{i+1}}, & \overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+2}}, & \boldsymbol{H}^{\sigma^i\sigma^{i+2}}, & \boldsymbol{H}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}, & \boldsymbol{H}^{\sigma^{i+1}} \\ \dots\dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots \end{array} \tag{47}$$

All terms of the sequence will satisfy the above given special and additional conditions. Since the sequence (47) contains all $\boldsymbol{h}^{\sigma^i}$, $i \geq 0$, this will complete the proof of Theorem 3.

To obtain the sequence (47), we will show how one defines $\boldsymbol{H}^j$, assuming that we already have $\boldsymbol{H}^0, \dots, \boldsymbol{H}^{j-1}$ and know that the latter coherent mappings satisfy the corresponding special and additional conditions. We begin the induction process by defining explicitly the four initial terms of (47) by putting

$$h_{\boldsymbol{\mu}}^{\sigma^0} = \overline{h}_{\boldsymbol{\mu}}^{\sigma^0},\ H_{\boldsymbol{\mu}}^{\sigma^0\sigma^1} = \overline{h}_{\boldsymbol{\mu}}^{\sigma^0\sigma^1},\ H_{\boldsymbol{\mu}}^{\sigma^0\sigma^2} = \overline{h}_{\boldsymbol{\mu}}^{\sigma^0\sigma^2},\ H_{\boldsymbol{\mu}}^{\sigma^0\sigma^1\sigma^2} = \overline{h}_{\boldsymbol{\mu}}^{\sigma^0\sigma^1\sigma^2}. \tag{48}$$

It is readily seen that these coherent mappings do satisfy the corresponding special and additional properties.

Terms of the form $\boldsymbol{h}^i_{\boldsymbol{\mu}}$, $\boldsymbol{H}^{ii+1}_{\boldsymbol{\mu}}$ and $\boldsymbol{H}^{ii+2}_{\boldsymbol{\mu}}$ are easily obtained from their immediate predecessors by explicit formulae, described in the Subsection 4.5. The remaining terms are obtained from their predecessors using one of the four constructions (C1)–(C4), described in Section 9. These constructions are based on a coherent homotopy extension lemma (Lemma 12), which guarantees that we do obtain coherent mappings, satisfying the special and the corresponding additional properties. This will complete the proof of Theorem 3.

4.5. *The coherent mappings* $\boldsymbol{h}^{\sigma^i}$, $\boldsymbol{H}^{\sigma^i\sigma^{i+1}}$, $\boldsymbol{H}^{\sigma^i\sigma^{i+2}}$.

4.5.1. Given $H^{\sigma^i}_{\boldsymbol{\mu}}$, we define $h^{\sigma^i}_{\boldsymbol{\mu}}$, $i \geq 1$, by formula (33). To verify (27) (for $i-1$) note that, for $z \in Z^{\sigma^{i-1}}$, (35) and (38) (for $i-1$) imply

$$h^{\sigma^i}_{\boldsymbol{\mu}}(z,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_0,t) = H^{\sigma^{i-1}\sigma^i}_{\boldsymbol{\mu}}(z,e_0,t) = h^{\sigma^{i-1}}_{\boldsymbol{\mu}}(z,t).$$

4.5.2. Given $\overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}$, we define $H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}$ by formula (36), for $j=1$. That $H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}$ has the additional properties (38), (39) and (40) is verified as follows. Since $d_1e_0 = e_0 = d_2e_0$, (41) and (33) imply

$$H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,e_0,t) = \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_1e_0,t) = \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_2e_0,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_0,t) = h^{\sigma^i}_{\boldsymbol{\mu}}(z,t),$$

which is (38). Similarly, since $d_1e_1 = e_2 = d_0e_1$, (36) and (30) imply

$$H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_1e_1,t) = \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_0e_1,t) = \overline{h}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{h}^{\sigma^{i+1}}_{\boldsymbol{\mu}}(z,t),$$

which is (39). Finally, using (37) and (46), we see that for $z \in Z^{\sigma^{i-1}}$, one has

$$H^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t) = \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_1s,t) = H^{\sigma^{i-1}\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_1s,t) = H^{\sigma^{i-1}\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t),$$

which is (46).

4.5.3. Given $\overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}$, we define $H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}$ by formula (41), for $j=1$. That $H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}$ has the additional properties (43), (44) and (45) now easily follows. Indeed, since $e_0 = d_1e_0 = d_2e_0$, (41) and (33) imply

$$H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,e_0,t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_1e_0,t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_2e_0,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_0,t) = h^{\sigma^i}_{\boldsymbol{\mu}}(z,t),$$

which is (43). Similarly, since $e_2 = d_1e_1 = d_0e_1$, (41) and (30) imply

$$H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_1e_1,t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_0e_1,t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{h}^{\sigma^{i+2}}_{\boldsymbol{\mu}}(z,t),$$

which is (44). Finally, using (42) and (31), we see that for $z \in Z^{\sigma^0}$, one has

$$H^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_1s,t) = \overline{h}^{\sigma^0\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_1s,t) = \overline{h}^{\sigma^0\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t),$$

which is (45).

4.6. The four constructions, (C1)–(C4) have the following form.

(C1) *associates with* $\boldsymbol{H}^{\sigma^i}$ *and* $\boldsymbol{H}^{\sigma^{i-1}\sigma^i\sigma^{i+1}}$ *the coherent mapping* $\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+1}}$.

(C2) *associates with* $\boldsymbol{H}^{\sigma^i}$ *the coherent mapping* $\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+2}}$.

(C3) *associates with* $\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+1}}$ *and* $\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+2}}$ *the coherent mapping* $\boldsymbol{H}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}$.

(C4) *associates with the set of coherent mappings* $\boldsymbol{H}^{\sigma^i\sigma^{i+1}}$, *where* σ^i *ranges over all i-dimensional faces of* σ^{i+1}, *the coherent mapping* $\boldsymbol{H}^{\sigma^{i+1}}$.

It is readily seen that the input coherent mappings in any one of the constructions (C1)–(C4) precede the output coherent mapping in the sequence $(\boldsymbol{H}^j)$.

5. Construction of mappings $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}$

For every $i \geq 0$ and every multiindex $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$, we define mappings $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i} : Z^{\sigma^i} \times \Delta^n \to Y_{\mu_0}$, by the natural formula

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z,t) = \phi_{\mu_0}(f_{\boldsymbol{\mu}(\sigma^i)}(z,t), g(z)), \tag{49}$$

where $\boldsymbol{\mu}(\sigma^i) = (\mu_0(\sigma^i), \dots, \mu_n(\sigma^i))$. Note that $z \in Z^{\sigma^i}$ implies $g(z) \in \sigma^i$ and thus, $(f_{\boldsymbol{\mu}(\sigma^i)}(z,t), g(z)) \in X_{\mu_0(\sigma^i)} \times \sigma^i \subseteq \tilde{Y}_{\mu_0}$. Therefore, $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z,t)$ is a well-defined point of Y_{μ_0}.

Lemma 5. *Mappings* $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i} : Z^{\sigma^i} \times \Delta^n \to Y_{\mu_0}$ *form a coherent mapping* $\overline{\boldsymbol{h}}^{\sigma^i} : Z^{\sigma^i} \to \boldsymbol{Y}$, *which satisfies special condition* (28).

Proof.
5.1. *Verification of the boundary condition*

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z, d_j t) = \begin{cases} q_{\mu_0\mu_1}\overline{h}_{d^0\boldsymbol{\mu}}^{\sigma^i}(z,t), & j = 0, \\ \overline{h}_{d^j\boldsymbol{\mu}}^{\sigma^i}(z,t), & 1 \leq j \leq n. \end{cases} \tag{50}$$

If $j = 0$, (49) shows that $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z, d_0 t) = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_n(\sigma^i)}(z, d_0 t), g(z))$. Using (18), we see that $f_{\mu_0(\sigma^i)\dots\mu_n(\sigma^i)}(z, d_0 t) = p_{\mu_0(\sigma^i)\mu_1(\sigma^i)} f_{\mu_1(\sigma^i)\dots\mu_n(\sigma^i)}(z,t)$ and thus, $(f_{\mu_0(\sigma^i)\dots\mu_n(\sigma^i)}(z, d_0 t), g(z)) = (p_{\mu_0(\sigma^i)\mu_1(\sigma^i)} \times 1)(f_{\mu_1(\sigma^i)\dots\mu_n(\sigma^i)}(z,t), g(z))$. Since $\tilde{q}_{\mu_0\mu_1} = p_{\mu_0(\sigma^i)\mu_1(\sigma^i)} \times 1$ and $\phi_{\mu_0}\tilde{q}_{\mu_0\mu_1} = q_{\mu_0\mu_1}\phi_{\mu_1}$, we conclude that $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z, d_0 t)$ $= q_{\mu_0\mu_1}\phi_{\mu_1}(f_{\mu_1(\sigma^i)\dots\mu_n(\sigma^i)}(z,t), g(z))$. However, $(\mu_1(\sigma^i), \dots, \mu_n(\sigma^i)) = d^0\boldsymbol{\mu}(\sigma^i)$ and thus, $\phi_{\mu_1}(f_{\mu_1(\sigma^i)\dots\mu_n(\sigma^i)}(z,t), g(z)) = \overline{h}_{d^0\boldsymbol{\mu}}^{\sigma^i}(z,t)$. We omit verification of formula (50), when $1 \leq j \leq n$, because that case is similar to the case $j = 0$ and is simpler.

5.2. *Verification of the degeneracy condition*

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z, s_j t) = \overline{h}_{s^j\boldsymbol{\mu}}^{\sigma^i}(z,t). \tag{51}$$

By definition $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z, s_j t) = \phi_{\mu_0}(f_{\boldsymbol{\mu}(\sigma^i)}(z, s_j t), g(z)) = \phi_{\mu_0}(f_{s^j\boldsymbol{\mu}(\sigma^i)}(z,t), g(z))$ and also $\overline{h}_{s^j\boldsymbol{\mu}}^{\sigma^i}(z,s,t) = \phi_{\mu_0}(f_{s^j\boldsymbol{\mu}(\sigma^i)}(z,t), g(z))$.

5.3. *Verification of special condition* (28). If $\boldsymbol{\mu} = (\overline{\lambda_0}, \dots, \overline{\lambda_n})$, then $\boldsymbol{\mu}(\sigma^i) = (\lambda_0, \dots, \lambda_n)$ and we see that $\overline{h}^{\sigma^i}_{\overline{\lambda_0} \dots \overline{\lambda_n}}(z,t) = \phi_{\overline{\lambda_0}}(f_{\lambda_0 \dots \lambda_n}(z,t), g(z))$. If we put $(x,s) = (f_{\lambda_0 \dots \lambda_n}(z,t), g(z)) \in X_{\lambda_0} \times \sigma^i = X_{\overline{\lambda_0(\sigma^i)}} \times \sigma^i$ and note that the quotient mapping $\phi_{\overline{\lambda_0}}$ maps $X_{\lambda_0} \times \sigma$ to $X_{\lambda_0} \times P$ by inclusion, we conclude that $\phi_{\overline{\lambda_0}}$ maps (x,s) to itself and thus, $\overline{h}^{\sigma^i}_{\boldsymbol{\lambda}}$ satisfies (28).

Note that mappings $\overline{h}^{\sigma^i}_{\boldsymbol{\mu}}$ do not satisfy the analogue of condition (27). E.g., if $\sigma^0 \leq \sigma^1$ and $z \in Z^{\sigma^0}$ and thus, $g(z) = \sigma^0 = e_0$, then $\overline{h}^{\sigma^0}_{\boldsymbol{\mu}}(z,t) = \phi_{\mu_0}(f_{\boldsymbol{\mu}(\sigma^0)}(z,t), e_0)$ and $\overline{h}^{\sigma^1}_{\boldsymbol{\mu}}(z,t) = \phi_{\mu_0}(f_{\boldsymbol{\mu}(\sigma^1)}(z,t), e_0)$. However, $(f_{\boldsymbol{\mu}(\sigma^0)}(z,t), e_0) \sim_\mu (f_{\boldsymbol{\mu}(\sigma^1)}(z,t), e_0)$ implies $p_{\mu_0(\sigma^0)\mu_0(\sigma^1)} f_{\mu_0(\sigma^1)}(z) = f_{\mu_0(\sigma^0)}(z)$ (apply Lemma 1 of [19]), which does not hold in general, because the coherent mapping $\boldsymbol{f}$ need not be a mapping. This is the reason why we introduced homotopies $\overline{h}^{\sigma^i \sigma^j}_{\boldsymbol{\mu}}$, which connect $\overline{h}^{\sigma^i}_{\boldsymbol{\mu}}$ to $\overline{h}^{\sigma^j}_{\boldsymbol{\mu}} | (Z^{\sigma^i} \times \Delta^n)$. □

6. Construction of the homotopies $\overline{h}^{\sigma^i \sigma^j}_{\boldsymbol{\mu}}$

6.1. For $0 \leq i < j$ and every multiindex $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$, we will define the mapping $\overline{h}^{\sigma^i \sigma^j}_{\boldsymbol{\mu}} : Z^{\sigma^i} \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$, though in the proof of Theorem 3 we need only the cases when (i,j) is of the form $(i, i+1)$, $(i, i+2)$ and $(0, i)$. To state the definition, we need the standard triangulation $\Delta^{1,n}$ of the Cartesian product $\Delta^1 \times \Delta^n = |\Delta^{1,n}|$. Recall that $\Delta^1 = [e_0, e_1] \subseteq \mathbb{R}^2$ and $\Delta^n = [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$. For the points $(e_u, e_v) \in \mathbb{R}^{n+3}$, where $0 \leq u \leq 1$, $0 \leq v \leq n$, we will use the abbreviation e_{uv}. We consider $(n+1)$-simplices

$$\Delta^{1,n}_k = [e_{00}, \dots, e_{0k}, e_{1k}, \dots, e_{1n}], \tag{52}$$

where $0 \leq k \leq n$. Note that the points $e_{00}, \dots, e_{0k}, e_{1k}, \dots, e_{1n}$ are in general position. The simplices $\Delta^{1,n}_k$, $k \in \{0, \dots, n\}$, and their faces form the simplicial complex $\Delta^{1,n}$ (see [11]). Note that $n = 0$ implies $k = 0$ and thus, $\Delta^{1,n}_0 = [e_{00}, e_{10}] = [e_0, e_1] \times e_0$ is the only 1-simplex of $\Delta^{1,0}$.

For $k \leq l$,

$$\Delta^{1,n}_k \cap \Delta^{1,n}_{l+1} = [e_{00}, \dots, e_{0k}, e_{1l+1}, \dots, e_{1n}]. \tag{53}$$

In particular,

$$\Delta^{1,n}_k \cap \Delta^{1,n}_{k+1} = [e_{00}, \dots, e_{0k}, e_{1k+1}, \dots, e_{1n}]. \tag{54}$$

Comparing (53), for k and $k+1$, we see that, for $0 \leq k < k+1 \leq l < n$,

$$\Delta^{1,n}_k \cap \Delta^{1,n}_{l+1} \subseteq \Delta^{1,n}_{k+1} \cap \Delta^{1,n}_{l+1}. \tag{55}$$

The following relations are easily verified.

$$(1 \times d_l)\,(\Delta^{1,n}_k) \subseteq \begin{cases} \Delta^{1,n+1}_{k+1}, & l \leq k, \\ \Delta^{1,n+1}_k, & k < l, \end{cases} \tag{56}$$

$$(1 \times s_l)\,(\Delta_k^{1,n+1}) \subseteq \begin{cases} \Delta_k^{1,n}, & k \leq l, \\ \Delta_{k-1}^{1,n}, & l < k. \end{cases} \tag{57}$$

We will also need the simplicial mapping $\varepsilon^{1,n}\colon \Delta^{1,n} \to \Delta^{n+1}$, given by $\varepsilon^{1,n}(e_{0v}) = e_v$ and $\varepsilon^{1,n}(e_{1v}) = e_{v+1}$, for $0 \leq v \leq n$. The induced mapping $\Delta^1 \times \Delta^n \to \Delta^{n+1}$ will also be denoted by $\varepsilon^{1,n}$. Note that $\varepsilon^{1,0}(e_{00}) = e_0$ and $\varepsilon^{1,0}(e_{10}) = e_1$ and therefore, $\varepsilon^{1,0}\colon [e_0, e_1] \times e_0$ coincides with the first projection $[e_0, e_1] \times e_0 \to [e_0, e_1]$, i.e., $\varepsilon^{1,0}(s, e_0) = s$, for $s \in [e_0, e_1]$.

The following formulae are easily verified, because it suffices to verify their validity at the vertices.

$$\varepsilon^{1,n}(\Delta_k^{1,n} \cap \Delta_{k+1}^{1,n}) = [e_0, \ldots, e_k, e_{k+2}, \ldots, e_{n+1}] = d_{k+1}(\Delta^n). \tag{58}$$

$$\varepsilon^{1,n+1}(1 \times d_l)\,|\,\Delta_k^{1,n} = \begin{cases} d_l \varepsilon^{1,n}|\,\Delta_k^{1,n}, & l \leq k, \\ d_{l+1}\varepsilon^{1,n}|\,\Delta_k^{1,n}, & k < l. \end{cases} \tag{59}$$

$$\varepsilon^{1,n-1}(1 \times s_l)\,|\,\Delta_k^{1,n} = \begin{cases} s_{l+1}\varepsilon^{1,n}|\,\Delta_k^{1,n}, & k \leq l, \\ s_l \varepsilon^{1,n}|\,\Delta_k^{1,n}, & l < k. \end{cases} \tag{60}$$

6.2. For $0 \leq i < j$, $z \in Z^{\sigma^i}$, $0 \leq k \leq n$ and $(s,t) \in \Delta_k^{1,n} \subseteq \Delta^1 \times \Delta^n$, we put

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z, s, t) = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_k(\sigma^i)\mu_k(\sigma^j)\ldots\mu_n(\sigma^j)}(z, \varepsilon^{1,n}(s,t)), g(z)). \tag{61}$$

Since $\varepsilon^{1,n}(s,t) \in \Delta^{n+1}$ and $(\mu_0(\sigma^i), \ldots, \mu_k(\sigma^i)\mu_k(\sigma^j), \ldots, \mu_n(\sigma^j))$ is a multiindex of length $n+1$, it follows that $x = f_{\mu_0(\sigma^i)\ldots\mu_k(\sigma^i)\mu_k(\sigma^j)\ldots\mu_n(\sigma^j)}(z, \varepsilon^{1,n}(s,t))$ is a well-defined point of $X_{\mu_0(\sigma^i)}$. However, $g(z) \in \sigma^i$ and thus, $(x, g(z)) \in X_{\mu_0(\sigma^i)} \times \sigma^i \subseteq \tilde{Y}_{\mu_0}$. Consequently, $\phi_{\mu_0}(x, g(z))$ is a well-defined point of Y_{μ_0}. If $n = 0$, i.e., $\boldsymbol{\mu} = \mu_0$, formula (61) assumes the form

$$\overline{h}_{\mu_0}^{\sigma^i\sigma^j}(z, s, e_0) = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\mu_0(\sigma^j)}(z, s), g(z)), \tag{62}$$

because $\varepsilon^{1,0}(s, e_0) = s$, for $s \in [e_0, e_1]$.

Lemma 6. *For $0 \leq i < j \leq n$ and $z \in Z^{\sigma^i}$, formula* (61) *determines a well-defined mapping $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}\colon Z^{\sigma^i} \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$.*

Proof. To prove the lemma denote by $\overline{h}_{\boldsymbol{\mu}\,k}^{\sigma^i\sigma^j}$ the mapping $Z^{\sigma^i} \times \Delta_k^{1,n} \to Y_{\mu_0}$, given by the right-hand side of (61). We must prove that $(s,t) \in \Delta_k^{1,n} \cap \Delta_l^{1,n}$ implies $\overline{h}_{\boldsymbol{\mu}\,k}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\,l}^{\sigma^i\sigma^j}(z,s,t)$, for $z \in Z^{\sigma^i}$, $0 \leq k, l \leq n$. This is obvious if $k = l$. If $k \neq l$, we can assume that $k < l$. We will first prove the assertion in the special case, when $l = k+1$, i.e., we will prove that $(s,t) \in \Delta_k^{1,n} \cap \Delta_{k+1}^{1,n}$ implies $\overline{h}_{\boldsymbol{\mu}\,k}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\,k+1}^{\sigma^i\sigma^j}(z,s,t)$.

By (58), $\varepsilon^{1,n}(s,t) = d_{k+1}v$, for some point $v \in \Delta^n$. Therefore, $\overline{h}_{\boldsymbol{\mu}\,k}^{\sigma^i\sigma^j}(z,s,t) = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_k(\sigma^i)\mu_k(\sigma^j)\ldots\mu_n(\sigma^j)}(z, d_{k+1}v), g(z))$. We also have $\overline{h}_{\boldsymbol{\mu}\,k+1}^{\sigma^i\sigma^j}(z,s,t) = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\ldots\mu_n(\sigma^j)}(z, d_{k+1}v), g(z))$. However, these two values coincide, because by the boundary condition (18),

$$f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_n(\sigma^j)}(z, d_{k+1}v) = f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_n(\sigma^j)}(z, v) \qquad (63)$$
$$= f_{\mu_0(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_n(\sigma^j)}(z, d_{k+1}v).$$

If $k+1 < n$, (55) for $l = k+1$, yields $\Delta_k^{1,n} \cap \Delta_{k+2}^{1,n} \subseteq \Delta_{k+1}^{1,n} \cap \Delta_{k+2}^{1,n}$ and thus, $(s,t) \in \Delta_k^{1,n} \cap \Delta_{k+2}^{1,n}$ implies $(s,t) \in \Delta_{k+1}^{1,n} \cap \Delta_{k+2}^{1,n}$. By the assertion in the special case, we see that $\overline{h}_{\boldsymbol{\mu}\, k+1}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\, k+2}^{\sigma^i\sigma^j}(z,s,t)$. Since also $(s,t) \in \Delta_k^{1,n} \cap \Delta_{k+1}^{1,n}$, we conclude, using again the assertion in the special case, that $\overline{h}_{\boldsymbol{\mu}\, k}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\, k+1}^{\sigma^i\sigma^j}(z,s,t)$. Consequently, $\overline{h}_{\boldsymbol{\mu}\, k}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\, k+2}^{\sigma^i\sigma^j}(z,s,t)$. If $k+2 < l$, we repeat the argument and conclude that also $\overline{h}_{\boldsymbol{\mu}\, k}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\, k+3}^{\sigma^i\sigma^j}(z,s,t)$, etc. By induction, we obtain the desired conclusion that $\overline{h}_{\boldsymbol{\mu}\, k}^{\sigma^i\sigma^j}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\, l}^{\sigma^i\sigma^j}(z,s,t)$. □

6.3. Our next goal is to prove the following lemma.

Lemma 7. *The mappings* $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j} : Z^{\sigma^i} \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$ *form a coherent mapping* $\overline{\boldsymbol{h}}^{\sigma^i\sigma^j} : Z^{\sigma^i} \times \Delta^1 \to \boldsymbol{Y}$*, which satisfies special and additional conditions* (28)*,* (29) *and* (30)*.*

Proof.
6.3.1. *Verification of the boundary condition.* Let $(s,t) \in \Delta_k^{1,n}$. In determining $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,d_l t)$ we distinguish two cases, when $l \le k$ and when $k < l$. In the first case, by (56), $(s, d_l t) \in \Delta_{k+1}^{1,n+1}$. Therefore, by (61),

$$\begin{aligned}
\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,d_l t) &= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z, \varepsilon^{1,n+1}(s, d_l t)), g(z)) \\
&= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z, d_l \varepsilon^{1,n}(s,t)), g(z)) \qquad (64)\\
&= \phi_{\mu_0}(p \times 1)(f_{d^l(\mu_0(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_n(\sigma^j))}(z, \varepsilon^{1,n}(s,t)), g(z)),
\end{aligned}$$

where $p \times 1$ stands for $p_{\mu_0(\sigma^i)\mu_1(\sigma^i)} \times 1$ if $l = 0$ and should be omitted if $0 < l \le k$. Since $\phi_{\mu_0}(p_{\mu_0(\sigma^i)\mu_1(\sigma^i)} \times 1) = q_{\mu_0\mu_1}\phi_{\mu_1}$, we see that

$$\begin{aligned}
&\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,d_0 t) \qquad (65)\\
&\quad = q_{\mu_0\mu_1}\phi_{\mu_1}(f_{\mu_1(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j))}(z, \varepsilon^{1,n}(s,t)), g(z)),
\end{aligned}$$

which, for $l = 0$, coincides with $q_{\mu_0\mu_1}\overline{h}_{d^0\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,t)$, as required by the boundary condition. If $0 < l \le k$, by (64),

$$\begin{aligned}
&\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,d_l t) \qquad (66)\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{l-1}(\sigma^i)\mu_{l+1}(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z, \varepsilon^{1,n}(s,t)), g(z)).
\end{aligned}$$

Now put $d^l\boldsymbol{\mu} = \boldsymbol{\nu} = (\nu_0, \dots, \nu_l, \dots, \nu_n)$ and note that $(\nu_0, \dots, \nu_{l-1}) = (\mu_0, \dots, \mu_{l-1})$ and $(\nu_l, \dots, \nu_n) = (\mu_{l+1}, \dots, \mu_{n+1})$. Since $(s,t) \in \Delta_k^{1,n}$, we see that

$$\begin{aligned}
&\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j}(z,s,t)\\
&\quad = \phi_{\nu_0}(f_{\nu_0(\sigma^i)\dots\nu_{l-1}(\sigma^i)\nu_l(\sigma^i)\dots\nu_k(\sigma^i)\nu_k(\sigma^j)\dots\nu_n(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)) \qquad (67)\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{l-1}(\sigma^i)\mu_{l+1}(\sigma^i)\dots\mu_{k+1}(\sigma^i)\mu_{k+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)).
\end{aligned}$$

(66) and (67) imply the desired boundary condition.

In the second case, i.e., when $k < l$, $(s, d_l t) \in \Delta_k^{1,n+1}$ and therefore,

$$\begin{aligned}
&\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,d_lt)\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z,\varepsilon^{1,n}(s,d_lt)),g(z))\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z,d_{l+1}\varepsilon^{1,n}(s,t)),g(z)) \qquad (68)\\
&\quad = \phi_{\mu_0}(f_{d^{l+1}(\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{n+1}(\sigma^j))}(z,\varepsilon^{1,n}(s,t)),g(z))\\
&\quad = \phi_{\mu_0}(f_{(\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{l-1}(\sigma^j)\mu_{l+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j))}(z,\varepsilon^{1,n+1}(s,t)),g(z)).
\end{aligned}$$

Since $(s,t) \in \Delta_k^{1,n}$, we see that

$$\begin{aligned}
\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j}(z,s,t) &= \phi_{\nu_0}(f_{\nu_0(\sigma^i)\dots\nu_k(\sigma^i)\nu_k(\sigma^j)\dots\nu_{l-1}(\sigma^j)\dots\nu_n(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)) \qquad (69)\\
&= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{l-1}(\sigma^j)\mu_{l+1}(\sigma^j)\dots\mu_{n+1}(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)).
\end{aligned}$$

Clearly, (68) and (69) yield the desired boundary condition.

6.3.2. *Verifying the degeneracy condition.* Let $(s,t) \in \Delta_k^{1,n}$. In order to determine $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,s_lt)$, we distinguish two cases, when $k \le l$ and when $l < k$. In the first case, by (57), $(s, s_lt) \in \Delta_k^{1,n-1}$. Therefore, by (60),

$$\begin{aligned}
\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,s_lt) &= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{n-1}(\sigma^j)}(z,\varepsilon^{1,n-1}(s,s_lt)),g(z))\\
&= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{n-1}(\sigma^j)}(z,s_{l+1}\varepsilon^{1,n}(s,t)),g(z)) \qquad (70)\\
&= \phi_{\mu_0}(f_{s^{l+1}(\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_{n-1}(\sigma^j))}(z,\varepsilon^{1,n}(s,t)),g(z))\\
&= \phi_{\mu_0}(f_{(\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_l(\sigma^j)\mu_l(\sigma^j)\dots\mu_{n-1}(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)).
\end{aligned}$$

Put $s^l(\boldsymbol{\mu}) = \boldsymbol{\nu} = (\nu_0,\dots,\nu_l,\nu_{l+1},\dots,\nu_n)$. Clearly, $(\nu_0,\dots,\nu_l) = (\mu_0,\dots,\mu_l)$ and $(\nu_{l+1},\dots,\nu_n) = (\mu_l,\dots,\mu_{n-1})$. Since $(s,t) \in \Delta_k^{1,n}$, we see that

$$\begin{aligned}
\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j}(z,s,t) &= \phi_{\nu_0}(f_{\nu_0(\sigma^i)\dots\nu_k(\sigma^i)\nu_k(\sigma^j)\dots\nu_l(\sigma^j)\nu_{l+1}(\sigma^j)\dots\nu_n(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)) \qquad (71)\\
&= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_k(\sigma^i)\mu_k(\sigma^j)\dots\mu_l(\sigma^j)\mu_l(\sigma^j)\dots\mu_{n-1}(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)).
\end{aligned}$$

Now (70) and (71) imply the desired degeneracy condition $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,s_lt) = \overline{h}_{s^l\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,t)$.

In the second case, i.e. when $l < k$, (57) shows that $(s, s_lt) \in \Delta_{k-1}^{1,n-1}$ and thus,

$$\begin{aligned}
\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,s_lt) &= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{k-1}(\sigma^i)\mu_{k-1}(\sigma^j)\dots\mu_{n-1}(\sigma^j)}(z,\varepsilon^{1,n-1}(s,s_lt)),g(z))\\
&= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{k-1}(\sigma^i)\mu_{k-1}(\sigma^j)\dots\mu_{n-1}(\sigma^j)}(z,s_l\varepsilon^{1,n}(s,t)),g(z)) \qquad (72)\\
&= \phi_{\mu_0}(f_{s^l(\mu_0(\sigma^i)\dots\mu_{k-1}(\sigma^i)\mu_{k-1}(\sigma^j)\dots\mu_{n-1}(\sigma^j))}(z,\varepsilon^{1,n}(s,t)),g(z))\\
&= \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_l(\sigma^i)\mu_l(\sigma^i)\dots\mu_{k-1}(\sigma^i)\mu_{k-1}(\sigma^j)\dots\mu_{n-1}(\sigma^j))}(z,\varepsilon^{1,n}(s,t)),g(z)).
\end{aligned}$$

Since $(s,t)\in\Delta_k^{1,n}$, we see that

$$\begin{aligned}\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j}&(z,s,t)\\ &=\phi_{\nu_0}(f_{\nu_0(\sigma^i)\dots\nu_l(\sigma^i)\nu_{l+1}(\sigma^i)\dots\nu_k(\sigma^i)\nu_k(\sigma^j)\dots\nu_n(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)) \qquad (73)\\ &=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_l(\sigma^i)\mu_l(\sigma^i)\dots\mu_{k-1}(\sigma^i)\mu_{k-1}(\sigma^j)\dots\mu_{n-1}(\sigma^j))}(z,\varepsilon^{1,n}(s,t)),g(z)).\end{aligned}$$

Now (72) and (73) imply the desired degeneracy condition $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,s_lt)=\overline{h}_{s^l\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,s,t)$.

6.3.3. *Verifying the special condition.* If $z\in Z^{\sigma^i}$ and $(s,t)\in\Delta_k^{1,n}$, then (61) shows that

$$\overline{h}_{\overline{\lambda_0}\dots\overline{\lambda_n}}^{\sigma^i\sigma^j}(z,s,t)=\phi_{\overline{\lambda_0}}(f_{\lambda_0\dots\lambda_k\lambda_k\dots\lambda_n}(z,\varepsilon^{1,n}(s,t)),g(z)). \qquad (74)$$

Since $(\lambda_0,\dots,\lambda_k,\lambda_k,\dots,\lambda_n)=s^k(\lambda_0,\dots,\lambda_k,\dots,\lambda_n)$, one concludes by (19) that

$$\overline{h}_{\overline{\lambda_0}\dots\overline{\lambda_n}}^{\sigma^i\sigma^j}(z,s,t)=\phi_{\overline{\lambda_0}}(f_{\lambda_0\dots\lambda_n}(z,s_k\,\varepsilon^{1,n}(s,t)),g(z)). \qquad (75)$$

The restriction of $\phi_{\overline{\lambda_0}}$ to $X_{\lambda_0}\times\sigma^i$ is the inclusion mapping $X_{\lambda_0}\times\sigma^i\to X_{\lambda_0}\times P$. Therefore, in formula (75) one can omit $\phi_{\overline{\lambda_0}}$. Consequently, to prove (28), it suffices to show that, for $(s,t)\in\Delta_k^{1,n}$,

$$s_k\,\varepsilon^{1,n}(s,t)=t, \qquad (76)$$

i.e., the restriction $s_k\,\varepsilon^{1,n}|\Delta_k^{1,n}$ coincides with the corresponding restriction of the second projection $\Delta^1\times\Delta^n\to\Delta^n$ to $\Delta_k^{1,n}$. Since both mappings are simplicial mappings of $\Delta_k^{1,n}$, the assertion follows from the fact that $s_k\,\varepsilon^{1,n}$ maps the vertices $e_{00},\dots,e_{0k},e_{1k},\dots e_{1n}$ to $e_0,\dots,e_k,e_k,\dots e_n$, respectively, and the second projection does the same.

6.3.4. *Verifying the additional conditions.* For $t\in\Delta^n$, we have $(e_0,t)\in[e_{00},\dots,e_{0n}]\subseteq[e_{00},\dots,e_{0n}e_{1n}]=\Delta_n^{1,n}$ and $\varepsilon^{1,n}(e_0,t)=d_{n+1}(t)$, because $\varepsilon^{1,n}(e_{0v})=e_v=d_{n+1}(e_v)$, for $0\le v\le n$. Consequently, for $z\in Z^{\sigma^i}$,

$$\begin{aligned}\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,e_0,t)&=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_n(\sigma^i)\mu_n(\sigma^j)}(z,d_{n+1}t),g(z))\\ &=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_n(\sigma^i)}(z,t),g(z))=\overline{h}_{\boldsymbol{\mu}}^{\sigma^i}(z,t). \qquad (77)\end{aligned}$$

Similarly, $(e_1,t)\in[e_{10},\dots,e_{1n}]\subseteq[e_{00},e_{10},\dots,e_{1n}]=\Delta_0^n$ and $\varepsilon^{1,n}(e_1,t)=d_0(t)$, because $\varepsilon^{1,n}(e_{1v})=e_{v+1}=d_0(e_v)$, for $0\le v\le n$. Therefore,

$$\begin{aligned}\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,e_1,t)&=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\mu_0(\sigma^j)\dots\mu_n(\sigma^j)}(z,d_0t),g(z))\\ &=\phi_{\mu_0}(p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}f_{\mu_0(\sigma^j)\dots\mu_n(\sigma^j)}(z,t),g(z)). \qquad (78)\end{aligned}$$

Since $x=f_{\mu_0(\sigma^j)\dots\mu_n(\sigma^j)}(z,t)\in X_{\mu_0(\sigma^j)}$ and $g(z)\in\sigma^i\le\sigma^j$, we see that the points $(x,g(z))\in X_{\mu_0(\sigma^j)}\times\sigma$ and $(p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}(x),g(z))\in X_{\mu_0(\sigma^i)}\times\sigma^i$ are $\sim_{\mu_0}$-equivalent and thus, $\phi_{\mu_0}(p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}(x),g(z))=\phi_{\mu_0}(x,g(z))$. Consequently, by (78),

$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j}(z,e_1,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^j)\dots\mu_n(\sigma^j)}(z,t),g(z))=\overline{h}_{\boldsymbol{\mu}}^{\sigma^j}(z,t).$ □

7. Construction of the 2-homotopies $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}$

7.1. For $0 \le i < j < k$ and every multiindex $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$, we will define the mapping $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k} : Z^{\sigma^i} \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$, though in the proof of Theorem 3 we need only the cases when (i, j, k) is of the form $(i, i+1, i+2)$ or $(0, i, i+1)$. To state the definition, we need the standard triangulation $\Delta^{2,n}$ of $\Delta^2 \times \Delta^n$ (see [11]). Recall that $\Delta^2 = [e_0, e_1, e_2] \subseteq \mathbb{R}^3$, $\Delta^n = [e_0, \dots, e_n] \subseteq \mathbb{R}^{n+1}$. The vertices of $\Delta^{2,n}$ are the points $(e_{uv}) = (e_u, e_v) \in \mathbb{R}^{n+3}$, where $0 \le u \le 2$, $0 \le v \le n$. For $0 \le k' \le k'' \le n$, the points $e_{00}, \dots, e_{0k'}, e_{1k'}, \dots, e_{1k''}, e_{2k''}, \dots, e_{2n}$ are in general position and span an $(n+2)$-simplex

$$\Delta^{2,n}_{k'k''} = [e_{00}, \dots, e_{0k'}, e_{1k'}, \dots, e_{1k''}, e_{2k''}, \dots, e_{2n}]. \tag{79}$$

The simplices $\Delta^{2,n}_{k'k''}$ and their faces form the complex $\Delta^{2,n}$. Note that $n = 0$ implies $k' = k'' = 0$ and thus, $\Delta^{2,0}_{00} = [e_{00}, e_{10}, , e_{20}] = [e_0, e_1, e_2] \times e_0$ is the only 2-simplex of $\Delta^{2,0}$.

It is readily seen that, for $k' + 1 \le k' + r \le k''$,

$$\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k'+r,k''} = [e_{00}, \dots, e_{0k'}, e_{1k'+r}, \dots, e_{1k''}, e_{2k''}, \dots, e_{2n}] \tag{80}$$

and for $k'' + 1 \le k'' + r \le n$,

$$\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k',k''+r} = [e_{00}, \dots, e_{0k'}, e_{1k'}, \dots, e_{1k''}, e_{2k''+r}, \dots, e_{2n}]. \tag{81}$$

Comparing (80) with (80) for $r+1$, we see that

$$\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k'+r+1,k''} \subseteq \Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k'+r,k''}. \tag{82}$$

Similarly, comparing (81) with (81) for $r+1$, we see that

$$\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k',k''+r+1} \subseteq \Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k',k''+r}. \tag{83}$$

The following relations are easily verified.

$$(1 \times d_l)\,(\Delta^{2,n}_{k'k''}) \subseteq \begin{cases} \Delta^{2,n+1}_{k'+1,k''+1}, & l \le k', \\ \Delta^{2,n+1}_{k',k''+1}, & k' < l \le k'', \\ \Delta^{2,n+1}_{k'k''}, & k'' < l. \end{cases} \tag{84}$$

$$(1 \times s_l)\,(\Delta^{2,n+1}_{k'k''}) \subseteq \begin{cases} \Delta^{2,n}_{k'k''}, & k'' \le l, \\ \Delta^{2,n}_{k',k''-1}, & k' \le l < k'', \\ \Delta^{2,n}_{k'-1,k''-1}, & l < k'. \end{cases} \tag{85}$$

We also need the simplicial mapping $\varepsilon^{2,n} \colon \Delta^{2,n} \to \Delta^{n+2}$, given by $\varepsilon^{2,n}(e_{0v}) = e_v$, $\varepsilon^{2,n}(e_{1v}) = e_{v+1}$ and $\varepsilon^{2,n}(e_{2v}) = e_{v+2}$. A straightforward verification establishes the following relations.

$$\varepsilon^{2,n}(\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k'+1k''}) = [e_0, \dots, e_{k'}, e_{k'+2}, \dots, e_{n+2}] = d_{k'+1}(\Delta^{n+1}), \tag{86}$$

$$\varepsilon^{2,n}(\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k'k''+1}) = [e_0, \dots, e_{k''+1}, e_{k''+3}, \dots, e_{n+2}] = d_{k''+2}(\Delta^{n+1}). \tag{87}$$

$$\varepsilon^{2,n+1}(1 \times d_l)\,|\,\Delta^{2,n}_{k'k''} = \begin{cases} d_l \varepsilon^{2,n}|\,\Delta^{2,n}_{k'k''}, & l \le k', \\ d_{l+1}\varepsilon^{2,n}|\,\Delta^{2,n}_{k'k''}, & k' < l \le k'' \\ d_{l+2}\varepsilon^{2,n}|\,\Delta^{2,n}_{k'k''}, & k'' < l, \end{cases} \tag{88}$$

$$\varepsilon^{2,n-1}(1 \times s_l)\,|\,\Delta^{2,n}_{k'k''} = \begin{cases} s_{l+2} \varepsilon^{2,n}|\,\Delta^{2,n}_{k'k''}, & k'' \le l, \\ s_{l+1}\varepsilon^{2,n}|\,\Delta^{2,n}_{k'k''}, & k' \le l < k'' \\ s_{l}\varepsilon^{2,n}|\,\Delta^{2,n}_{k'k''}, & l < k'. \end{cases} \tag{89}$$

Note that $\varepsilon^{2,0}(e_{00}) = e_0$, $\varepsilon^{2,0}(e_{10}) = e_1$ and $\varepsilon^{2,0}(e_{20}) = e_2$. Therefore, $\varepsilon^{2,0}\colon [e_0, e_1, e_2] \times e_0$ coincides with the first projection $[e_0, e_1, e_2] \times e_0 \to [e_0, e_1, e_2]$, i.e., $\varepsilon^{2,0}(s, e_0) = s$, for $s \in [e_0, e_1, e_2]$.

7.2. For $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$, $z \in Z^{\sigma^i}$, $0 \le k' \le k'' \le n$ and $(s,t) \in \Delta^{2,n}_{k'k''} \subseteq \Delta^2 \times \Delta^n$, we put

$$\begin{aligned} &\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,t) \\ &\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\dots\mu_n(\sigma^k)}(z, \varepsilon^{2,n}(s,t)), g(z)). \end{aligned} \tag{90}$$

If $n = 0$, i.e., $\boldsymbol{\mu} = \mu_0$, formula (90) assumes the form

$$\overline{h}_{\mu_0}^{\sigma^i\sigma^j\sigma^k}(z,s,e_0) = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\mu_0(\sigma^j)\mu_0(\sigma^k)}(z,s), g(z)), \tag{91}$$

because $\varepsilon^{2,0}(s, e_0) = s$, for $s \in [e_0, e_1, e_2]$.

Lemma 8. *For $0 \le i < j < k \le n$ and $z \in Z^{\sigma^i}$, formula* (90) *determines a well-defined mapping $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k} \colon Z^{\sigma^i} \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$.*

Proof. To prove the lemma, denote by $\overline{h}_{\boldsymbol{\mu}\,k'k''}^{\sigma^i\sigma^j\sigma^k}$ the mapping $Z^{\sigma^i} \times \Delta^{2,n}_{k'k''} \to Y_{\mu_0}$, given by the right-hand side of (90). We must show that, for $z \in Z^{\sigma^i}$, $0 \le k' \le k'' \le n$, $0 \le l' \le l'' \le n$ and $(s,t) \in \Delta^{2,n}_{k'k''} \cap \Delta^{1,n}_{l'l''}$, one has $\overline{h}_{\boldsymbol{\mu}\,k'k''}^{\sigma^i\sigma^j\sigma^k}(z,s,t) = \overline{h}_{\boldsymbol{\mu}\,l'l''}^{\sigma^i\sigma^j\sigma^k}(z,s,t)$. We will first prove the assertion in four special cases (i)–(iv).

Case (i), $(l', l'') = (k'+1, k'')$, $k'+1 \le k''$. We must show that $(s,t) \in \Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k'+1k''}$ implies $\overline{h}_{\boldsymbol{\mu}k'k''}^{\sigma^i\sigma^j\sigma^k}(z,s,t) = \overline{h}_{\boldsymbol{\mu}k'+1k''}^{\sigma^i\sigma^j\sigma^k}(z,s,t)$. By (84), $\varepsilon^{2,n}(s,t) = d_{k+1}v$, for some point $v \in \Delta^{n+1}$. Therefore,

$$\begin{aligned} &\overline{h}_{\boldsymbol{\mu}\,k'k''}^{\sigma^i\sigma^j\sigma^k}(z,s,t) \\ &\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\dots\mu_n(\sigma^k)}(z, d_{k+1}v), g(z)). \end{aligned} \tag{92}$$

We also have

$$\begin{aligned}
&\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'+1,k''}(z,s,t) \qquad (93)\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k+1}v),g(z)).
\end{aligned}$$

However, the values given by (92) and (93) coincide, because by the boundary condition (18),

$$\begin{aligned}
&f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k'+1}v)\\
&\quad = f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'+1}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,v) \qquad (94)\\
&\quad = f_{\mu_0(\sigma^i)\ldots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k'+1}v).
\end{aligned}$$

Case (ii), $(l',l'')=(k'+r,k'')$, $k'+1\le k'+r\le k''$. We must show that $(s,t)\in\Delta^{2,n}_{k'k''}\cap\Delta^{2,n}_{k'+rk''}$ implies $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t)=\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'+rk''}(z,s,t)$. If $r=1$, this is just case (i). Let us show that the assertion holds for $r+1$, if it holds for r. By (82), $(s,t)\in\Delta^{2,n}_{k'k''}\cap\Delta^{2,n}_{k'+r+1,k''}$ implies $(s,t)\in\Delta^{2,n}_{k'k''}\cap\Delta^{2,n}_{k'+r,k''}$ and the induction hypothesis shows that $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t)=\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'+r,k''}(z,s,t)$. Moreover, since $(s,t)\in\Delta^{2,n}_{k'+r,k''}\cap\Delta^{2,n}_{k'+r+1,k''}$, (i) yields $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'+r,k''}(z,s,t)=\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'+r+1,k''}(z,s,t)$. Consequently, $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t)=\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'+r+1,k''}(z,s,t)$, which is assertion (ii) for $(k'+r+1,k'')$.

Case (iii), $(l',l'')=(k',k''+1)$, $k''+1\le n$. We must show that $(s,t)\in\Delta^{2,n}_{k'k''}\cap\Delta^{2,n}_{k'k''+1}$ implies $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t)=\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k',k''+1}(z,s,t)$. By (84), $\varepsilon^{2,n}(s,t)=d_{k''+2}v$, for some point $v\in\Delta^{n+1}$. Therefore,

$$\begin{aligned}
&\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t) \qquad (95)\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k''+2}v),g(z)).
\end{aligned}$$

We also have

$$\begin{aligned}
&\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k',k''+1}(z,s,t) \qquad (96)\\
&\quad = \phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k''+2}v),g(z)).
\end{aligned}$$

However, the values given by (95) and (96) coincide, because by the boundary condition (18),

$$\begin{aligned}
&f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\mu_{k''+1}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k''+2}v)\\
&\quad = f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''+1}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k''+2}v) \qquad (97)\\
&\quad = f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\ldots\mu_n(\sigma^k)}(z,d_{k''+2}v).
\end{aligned}$$

Case (iv), $(l',l'')=(k',k''+r)$, $k''+r\le n$. We must show that $(s,t)\in\Delta^{2,n}_{k',k''}\cap\Delta^{2,n}_{k',k''+r}$ implies $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t)=\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''+r}(z,s,t)$. If $r=1$, this is just assertion (iii). Let us show that the assertion holds for $r+1$, if it holds for r. By (83), $(s,t)\in$

$\Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k',k''+r+1}$ implies $(s,t) \in \Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{k',k''+r}$ and the induction hypothesis shows that $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k',k''+r}(z,s,t)$. Moreover, since $(s,t) \in \Delta^{2,n}_{k',k''+r} \cap \Delta^{2,n}_{k',k''+r+1}$, (iii) yields $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k',k''+r}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k',k''+r+1}(z,s,t)$. It now follows that $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k',k''+r+1}(z,s,t)$, which is assertion (iv) for $(k', k''+r+1)$.

General case. Let $(s,t) \in \Delta^{2,n}_{k'k''} \cap \Delta^{2,n}_{l'l''}$. We must show that

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,l'l''}(z,s,t). \tag{98}$$

This is obvious if $(k',k'') = (l',l'')$. Therefore, we assume that $(k',k'') \neq (l',l'')$. There is no loss of generality in assuming that $k'' \leq l''$. We distinguish three cases: (a), when $k' = l'$, (b), when $k' < l'$ and (c), when $k' > l'$.

Case (a). Since $(k',k'') \neq (l',l'')$ and $k' = l'$, we must have $k'' \neq l''$ and thus, $k'' < l''$. Consequently, l'' is of the form $l'' = k'' + r$ and (iv) shows that (98) holds.

Case (b). In this case, l' is of the form $l' = k' + r \leq l''$ and (ii) shows that $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'l''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,l'l''}(z,s,t)$. If $k'' = l''$, the latter equality becomes (98). If $k'' < l''$, l'' is of the form $l'' = k'' + r$ and (iv) for (k',k'') yields $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'l''}(z,s,t)$. This and the previously obtained relation prove that again (98) holds.

Case (c). In this case, k' is of the form $k' = l' + r \leq k'' \leq l''$ and (ii) shows that $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,l'l''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'l''}(z,s,t)$. If $k'' = l''$, the latter equality becomes (98). If $k'' < l''$, l'' is of the form $l'' = k''+r$ and (iv) yields $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'k''}(z,s,t) = \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}\,k'l''}(z,s,t)$. This and the previously obtained relation prove that again (98) holds. □

7.3. Our next goal is to prove the following lemma.

Lemma 9. *The mappings* $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}} : Z^{\sigma^i} \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$ *form a coherent mapping* $\overline{\boldsymbol{h}}^{\sigma^i\sigma^j\sigma^k} : Z^{\sigma^i} \times \Delta^2 \to \boldsymbol{Y}$, *which satisfies special and additional conditions* (28) *and* (31).

Proof.
7.3.1. *Verification of the boundary condition.* Let $(s,t) \in \Delta^{2,n}_{k'k''}$. In determining $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z,s,d_lt)$ we distinguish three cases, when $l \leq k'$, when $k' < l \leq k''$ and when $k'' < l$. In the first case, by (84), $(s,d_lt) \in \Delta^{2,n+1}_{k'+1,k''+1}$ and thus,

$$\begin{aligned}\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z,s,d_lt) = \ &\phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\ldots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\ldots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)}\\ &(z,\varepsilon^{2,n+1}(s,d_lt)),g(z)\big).\end{aligned} \tag{99}$$

By (88), $\varepsilon^{2,n+1}(s,d_lt) = d_l\varepsilon^{2,n}(s,t)$ and we see that, for $0 < l \leq k'$,

$$f_{\mu_0(\sigma^i)\ldots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\ldots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)}(z,\varepsilon^{2,n+1}(s,d_lt))$$

$$= f_{\mu_0(\sigma^i)\dots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}(z, d_l\varepsilon^{2,n}(s,t)) \quad (100)$$
$$= f_{d^l\mu_0(\sigma^i)\dots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}(z, \varepsilon^{2,n}(s,t)).$$

Therefore,

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,d_lt)$$
$$= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{l-1}(\sigma^i)\mu_{l+1}(\sigma^i)\dots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}$$
$$(z,\varepsilon^{2,n}(s,t)), g(z)\big). \quad (101)$$

Now put $d^l\boldsymbol{\mu} = \boldsymbol{\nu} = (\nu_0,\dots,\nu_l,\dots,\nu_n)$ and note that $(\nu_0,\dots,\nu_{l-1}) = (\mu_0,\dots,\mu_{l-1})$ and $(\nu_l,\dots,\nu_n) = (\mu_{l+1},\dots,\mu_{n+1})$. Since $(s,t)\in\Delta^{2,n}_{k'k''}$, we see that

$$\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j\sigma^k}(z,s,t) = \phi_{\nu_0}\big(f_{\nu_0(\sigma^i)\dots\nu_{l-1}(\sigma^i)\nu_l(\sigma^i)\dots\nu_{k'}(\sigma^i)\nu_{k'}(\sigma^j)\dots\nu_{k''}(\sigma^j)\nu_{k''}(\sigma^k)\dots\nu_n(\sigma^k)}$$
$$(z,\varepsilon^{2,n}(s,t)), g(z)\big) \quad (102)$$
$$= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{l-1}(\sigma^i)\mu_{l+1}(\sigma^i)\dots\mu_{k'+1}(\sigma^i)\mu_{k'+1}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}$$
$$(z,\varepsilon^{2,n}(s,t)), g(z)\big).$$

(101) and (102) imply the desired boundary condition. For $l = 0$ a slightly different argument, like the one used in 6.3.1, is required.

We now consider the case when $k' < l \le k''$. By (84), $(s,d_lt)\in\Delta^{2,n+1}_{k',k''+1}$ and thus,

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,d_lt) = \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}$$
$$(z,\varepsilon^{2,n+1}(s,d_lt)), g(z)\big). \quad (103)$$

By (88), $\varepsilon^{2,n+1}(s,d_lt) = d_{l+1}\varepsilon^{2,n}(s,t)$ and we see that,

$$f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}(z,\varepsilon^{2,n+1}(s,d_lt))$$
$$= f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}(z, d_{l+1}\varepsilon^{2,n}(s,t)) \quad (104)$$
$$= f_{d^{l+1}(\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1})(\sigma^k)}(z,\varepsilon^{2,n}(s,t)).$$

Therefore,

$$\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,d_lt)$$
$$= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{l-1}(\sigma^j)\mu_{l+1}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}$$
$$(z,\varepsilon^{2,n}(s,t)), g(z)\big). \quad (105)$$

Since $(s,t)\in\Delta^{2,n}_{k'k''}$, we see that

$$\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j\sigma^k}(z,s,t) = \phi_{\nu_0}\big(f_{\nu_0(\sigma^i)\dots\nu_{k'}(\sigma^i)\nu_{k'}(\sigma^j)\dots\nu_{l-1}(\sigma^j)\nu_l(\sigma^j)\dots\nu_{k''}(\sigma^j)\nu_{k''}(\sigma^k)\dots\nu_n(\sigma^k)}$$
$$(z,\varepsilon^{2,n}(s,t)), g(z)\big) \quad (106)$$
$$= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{l-1}(\sigma^j)\mu_{l+1}(\sigma^j)\dots\mu_{k''+1}(\sigma^j)\mu_{k''+1}(\sigma^k)\dots\mu_{n+1}(\sigma^k)}$$

$$(z, \varepsilon^{2,n}(s,t)), g(z)\big).$$

(105) and (106) imply the desired boundary condition.

Finally, assume that $k'' < l$. By (84), $(s, d_l t) \in \Delta^{2,n+1}_{k',k''}$ and thus,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z, s, d_l t) = \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)} (z, \varepsilon^{2,n+1}(s, d_l t)), g(z)\big). \tag{107}$$

By (88), $\varepsilon^{2,n+1}(s, d_l t) = d_{l+2}\varepsilon^{2,n}(s,t)$ and we see that

$$\begin{aligned} &f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)}(z, \varepsilon^{2,n+1}(s, d_l t)) \\ &\quad = f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)}(z, d_{l+2}\varepsilon^{2,n}(s,t)) \\ &\quad = f_{d^{l+2}(\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n+1}(\sigma^k))}(z, \varepsilon^{2,n}(s,t)). \end{aligned} \tag{108}$$

Therefore,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z, s, d_l t) = \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{l-1}(\sigma^k)\mu_{l+1}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)} (z, \varepsilon^{2,n}(s,t)), g(z)\big). \tag{109}$$

Since $(s,t) \in \Delta^{2,n}_{k'k''}$, we see that

$$\begin{aligned} \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\nu}}(z, s, t) &= \phi_{\nu_0}\big(f_{\nu_0(\sigma^i)\ldots\nu_{k'}(\sigma^i)\nu_{k'}(\sigma^j)\ldots\nu_{k''}(\sigma^j)\nu_{k''}(\sigma^k)\ldots\nu_{l-1}(\sigma^k)\nu_l(\sigma^k)\ldots\nu_n(\sigma^k)} (z, \varepsilon^{2,n}(s,t)), g(z)\big) &(110)\\ &= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots.\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\mu_{l-1}(\sigma^k)\mu_{l+1}(\sigma^k)\ldots\mu_{n+1}(\sigma^k)} (z, \varepsilon^{2,n}(s,t)), g(z)\big). \end{aligned}$$

(109) and (110) imply the desired boundary condition.

7.3.2. *Verification of the degeneracy condition.* Let $(s,t) \in \Delta^{2,n}_{k'k''}$. In determining $\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z, s, s_l t)$ we distinguish three cases, when $k'' \le l$, when $k' \le l < k''$ and when $l < k'$. In the first case, by (85), $(s, s_l t) \in \Delta^{2,n-1}_{k',k''}$ and thus,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z, s, s_l t) = \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n-1}(\sigma^k)} (z, \varepsilon^{2,n-1}(s, s_l t)), g(z)\big). \tag{111}$$

By (89), $\varepsilon^{2,n-1}(s, s_l t) = s_{l+2}\varepsilon^{2,n}(s,t)$ and we see that

$$\begin{aligned} &f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n-1}(\sigma^k)}(z, \varepsilon^{2,n-1}(s, s_l t)) \\ &\quad = f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n-1}(\sigma^k)}(z, s_{l+2}\varepsilon^{2,n}(s,t)) \\ &\quad = f_{s^{l+2}(\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_{n-1}(\sigma^k))}(z, \varepsilon^{2,n}(s,t)). \end{aligned} \tag{112}$$

Therefore,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\boldsymbol{\mu}}(z, s, s_l t) = \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\ldots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\ldots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\ldots\mu_l(\sigma^k)\mu_l(\sigma^k)\ldots\mu_{n-1}(\sigma^k)}$$

$$(z, \varepsilon^{2,n}(s,t)), g(z)\big). \tag{113}$$

Now put $s^l \boldsymbol{\mu} = \boldsymbol{\nu} = (\nu_0, \dots, \nu_l, \dots, \nu_n)$ and note that $(\nu_0, \dots, \nu_l) = (\mu_0, \dots, \mu_l)$ and $(\nu_{l+1}, \dots, \nu_n) = (\mu_l, \dots, \mu_{n-1})$. Since $(s,t) \in \Delta^{2,n}_{k'k''}$, we see that

$$\begin{aligned}\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j\sigma^k}(z,s,t) &= \phi_{\nu_0}\big(f_{\nu_0(\sigma^i)\dots\nu_{k'}(\sigma^i)\nu_{k'}(\sigma^j)\dots\nu_{k''}(\sigma^j)\nu_{k''}(\sigma^k)\dots\nu_l(\sigma^k)\nu_{l+1}(\sigma^k)\dots\nu_n(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n}(s,t)), g(z)\big) \\ &= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''}(\sigma^j)\mu_{k''}(\sigma^k)\dots\mu_l(\sigma^k)\mu_l(\sigma^k)\dots\mu_{n-1}(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n}(s,t)), g(z)\big).\end{aligned} \tag{114}$$

(113) and (114) imply the desired degeneracy condition.

Now assume that $k' \le l < k''$. By (85), $(s, s_l t) \in \Delta^{2,n-1}_{k',k''-1}$ and thus,

$$\begin{aligned}\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,s_l t) &= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n-1}(s,s_l t)), g(z)\big).\end{aligned} \tag{115}$$

By (89), $\varepsilon^{2,n-1}(s, s_l t) = s_{l+1}\varepsilon^{2,n}(s,t)$ and we see that

$$\begin{aligned}& f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)}\big(z, \varepsilon^{2,n-1}(s,s_l t)\big) \\ &\quad = f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)}\big(z, s_{l+1}\varepsilon^{2,n}(s,t)\big) \\ &\quad = f_{s^{l+1}(\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k))}\big(z, \varepsilon^{2,n}(s,t)\big).\end{aligned} \tag{116}$$

Therefore,

$$\begin{aligned}&\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,s_l t) \\ &\quad = \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_l(\sigma^j)\mu_l(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n}(s,t)), g(z)\big).\end{aligned} \tag{117}$$

Since $(s,t) \in \Delta^{2,n}_{k'k''}$, we see that

$$\begin{aligned}\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j\sigma^k}(z,s,t) &= \phi_{\nu_0}\big(f_{\nu_0(\sigma^i)\dots\nu_{k'}(\sigma^i)\nu_{k'}(\sigma^j)\dots\nu_l(\sigma^k)\nu_{l+1}(\sigma^k)\dots\nu_{k''}(\sigma^j)\nu_{k''}(\sigma^k)\dots\nu_n(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n}(s,t)), g(z)\big) \\ &= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'}(\sigma^i)\mu_{k'}(\sigma^j)\dots\mu_l(\sigma^j)\mu_l(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n}(s,t)), g(z)\big).\end{aligned} \tag{118}$$

(117) and (118) imply the desired degeneracy condition.

Finally, assume that $l < k'$. By (85), $(s, s_l t) \in \Delta^{2,n-1}_{k'-1,k''-1}$ and thus,

$$\begin{aligned}\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,s_l t) &= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_{k'-1}(\sigma^i)\mu_{k'-1}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)} \\ &\qquad (z, \varepsilon^{2,n-1}(s,s_l t)), g(z)\big).\end{aligned} \tag{119}$$

By (89), $\varepsilon^{2,n-1}(s,s_lt)=s_l\varepsilon^{2,n}(s,t)$ and we see that

$$\begin{aligned} &f_{\mu_0(\sigma^i)\dots\mu_{k'-1}(\sigma^i)\mu_{k'-1}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)}(z,\varepsilon^{2,n-1}(s,s_lt))\\ &\quad= f_{\mu_0(\sigma^i)\dots\mu_{k'-1}(\sigma^i)\mu_{k'-1}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)}(z,s_l\varepsilon^{2,n}(s,t))\\ &\quad= f_{s^l(\mu_0(\sigma^i)\dots\mu_{k'-1}(\sigma^i)\mu_{k'-1}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k))}(z,\varepsilon^{2,n}(s,t)). \end{aligned} \tag{120}$$

Therefore,

$$\begin{aligned} &\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^j\sigma^k}(z,s,s_lt)\\ &= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_l(\sigma^i)\mu_l(\sigma^i)\dots\mu_{k'-1}(\sigma^i)\mu_{k'-1}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)}\\ &\qquad (z,\varepsilon^{2,n}(s,t)),g(z)\big). \end{aligned} \tag{121}$$

Since $(s,t)\in\Delta^{2,n}_{k'k''}$, we see that

$$\begin{aligned} &\overline{h}_{\boldsymbol{\nu}}^{\sigma^i\sigma^j\sigma^k}(z,s,t)\\ &\quad= \phi_{\nu_0}\big(f_{\nu_0(\sigma^i)\dots\nu_l(\sigma^i)\nu_{l+1}(\sigma^i)\dots\nu_{k'}(\sigma^i)\nu_{k'}(\sigma^j)\dots\nu_{k''}(\sigma^j)\nu_{k''}(\sigma^k)\dots\nu_n(\sigma^k)}\\ &\qquad\quad (z,\varepsilon^{2,n}(s,t)),g(z)\big)\\ &\quad= \phi_{\mu_0}\big(f_{\mu_0(\sigma^i)\dots\mu_l(\sigma^i)\mu_l(\sigma^i)\dots\mu_{k'-1}(\sigma^i)\mu_{k'-1}(\sigma^j)\dots\mu_{k''-1}(\sigma^j)\mu_{k''-1}(\sigma^k)\dots\mu_{n-1}(\sigma^k)}\\ &\qquad\quad (z,\varepsilon^{2,n}(s,t)),g(z)\big). \end{aligned} \tag{122}$$

(121) and (122) imply the desired degeneracy condition.

7.3.3. *Verifying the special condition.* If $z\in Z^{\sigma^i}$ and $(s,t)\in\Delta^{2,n}_{k'k''}$, then (90) shows that

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\overline{\lambda_0}\dots\overline{\lambda_n}}(z,s,t)=\phi_{\overline{\lambda_0}}(f_{\lambda_0\dots\lambda_{k'}\lambda_{k'}\dots\lambda_{k''}\lambda_{k''}\dots\lambda_n}(z,\varepsilon^{2,n}(s,t)),g(z)). \tag{123}$$

Since $(\lambda_0,\dots,\lambda_{k'},\lambda_{k'},\dots,\lambda_{k''},\lambda_{k''}\dots,\lambda_n)=s^{k''+1}s^{k'}(\lambda_0,\dots,\lambda_k,\dots,\lambda_n)$, one concludes by (19) that

$$\begin{aligned} \overline{h}^{\sigma^i\sigma^j\sigma^k}_{\overline{\lambda_0}\dots\overline{\lambda_n}}(z,s,t)&=\phi_{\overline{\lambda_0}}(f_{s^{k''+1}s^{k'}(\lambda_0\dots\lambda_n)}(z,\,\varepsilon^{2,n}(s,t)),g(z))\\ &=\phi_{\overline{\lambda_0}}(f_{\lambda_0\dots\lambda_n}(z,s_{k'}s_{k''+1}\varepsilon^{2,n}(s,t)),g(z)). \end{aligned} \tag{124}$$

Since the restriction of $\phi_{\overline{\lambda_0}}$ to $X_{\lambda_0}\times\sigma^i$ is the inclusion mapping $X_{\lambda_0}\times\sigma^i\to X_{\lambda_0}\times P$, in formula (85) one can erase $\phi_{\overline{\lambda_0}}$. Consequently, to prove (28) it suffices to show that for $(s,t)\in\Delta^{2,n}_{k'k''}$,

$$s_{k'}s_{k''+1}\varepsilon^{2,n}(s,t)=t, \tag{125}$$

i.e., $s_{k'}s_{k''+1}\varepsilon^{2,n}|\Delta^{2,n}_{k'k''}$ coincides with the corresponding restriction of the second projection $\Delta^2\times\Delta^n\to\Delta^n$ to $\Delta^{2,n}_{k'k''}$. Since both mappings are simplicial, the assertion follows from the fact that $s_{k'}s_{k''+1}\varepsilon^{2,n}$ maps the vertices $e_{00},\dots,e_{0k'},e_{1k'},\dots,e_{1k''}$, $e_{2k''},\dots,e_{2n}$ of $\Delta^{2,n}_{k'k''}$ to $e_0,\dots,e_{k'},e_{k'},\dots,e_{k''},e_{k''},\dots,e_n$, respectively, and the second projection does the same.

7.3.4. *Verifying the additional condition.* If $(s,t)\in\Delta^1\times\Delta^n$, there is a v, $0\le v\le n$, such that $(s,t)\in\Delta_v^{1,n}=[e_{00},\dots,e_{0v},e_{1v},\dots,e_{1n}]$. It follows that $(d_0\times 1)(s,t)\in[e_{10},\dots,e_{1v},e_{2v},\dots,e_{2n}]\subseteq[e_{00},,e_{10},\dots,e_{1v},e_{2v},\dots,e_{2n}]=\Delta_{0v}^{2,n}$. Consequently,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\mu_0\dots\mu_n}(z,d_0s,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{2,n}(d_0s,t)),g(z)). \tag{126}$$

A straightforward verification shows that

$$\varepsilon^{2,n}(d_0\times 1)=d_0\varepsilon^{1,n}. \tag{127}$$

Therefore,

$$\begin{aligned}
&f_{\mu_0(\sigma^i)\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{2,n}(d_0s,t))\\
&\quad=f_{\mu_0(\sigma^i)\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,d_0\varepsilon^{1,n}(s,t))\\
&\quad=p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}f_{d^0(\mu_0(\sigma^i)\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k))}(z,\varepsilon^{1,n}(s,t))\\
&\quad=p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}f_{\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{1,n}(s,t)).
\end{aligned} \tag{128}$$

Putting $x=f_{\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{1,n}(s,t))$, we see that $x\in X_{\mu_0(\sigma^j)}$. Moreover, $z\in Z^{\sigma^i}$ implies $g(z)\in\sigma^i\le\sigma^j$ and thus, the points $(x,g(z))\in X_{\mu_0(\sigma^j)}\times\sigma^j\subseteq\tilde{Y}_{\mu_0}$ and $(p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}(x),g(z))\in X_{\mu_0(\sigma^i)}\times\sigma^i\subseteq\tilde{Y}_{\mu_0}$ are $\sim_{\mu_0}$-equivalent. Consequently, $\phi_{\mu_0}(p_{\mu_0(\sigma^i)\mu_0(\sigma^j)}(x),g(z))=\phi_{\mu_0}(x,g(z))$. It follows that

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\mu_0\dots\mu_n}(z,d_0s,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^j)\dots\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{1,n}(s,t)),g(z)). \tag{129}$$

However, since $z\in Z^{\sigma^i}\subseteq Z^{\sigma^j}$, $(s,t)\in\Delta_v^{1,n}$, the right-hand side of (129) equals $\overline{h}^{\sigma^j\sigma^k}_{\mu_0\dots\mu_n}(z,s,t)$ and we obtained the desired relation (31), for $l=0$.

Now assume that $l=1$. Since $(s,t)\in\Delta_v^{1,n}=[e_{00},\dots,e_{0v},e_{1v},\dots,e_{1n}]$, one sees that $(d_1\times 1)(s,t)\in[e_{00},\dots,e_{0v},e_{2v},\dots,e_{2n}]\subseteq\Delta_{vv}^{2,n}$. Consequently,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\mu_0\dots\mu_n}(z,d_1s,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\dots\mu_v(\sigma^i)\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{2,n}(d_1s,t)),g(z)). \tag{130}$$

A straightforward verification shows that

$$\varepsilon^{2,n}(d_1\times 1)=d_{v+1}\varepsilon^{1,n}. \tag{131}$$

Therefore,

$$\begin{aligned}
&f_{\mu_0(\sigma^i)\dots\mu_v(\sigma^i)\mu_v(\sigma^j)\mu_(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{2,n}(d_1s,t))\\
&\quad=f_{\mu_0(\sigma^i)\dots\mu_v(\sigma^i)\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,d_{v+1}\varepsilon^{1,n}(s,t))\\
&\quad=f_{d^{v+1}(\mu_0(\sigma^i)\dots\mu_v(\sigma^i)\mu_v(\sigma^j)\mu_v(\sigma^k)\dots\mu_n(\sigma^k))}(z,\varepsilon^{1,n}(s,t))\\
&\quad=f_{\mu_0(\sigma^i)\dots\mu_v(\sigma^i)\mu_v(\sigma^k)\dots\mu_n(\sigma^k)}(z,\varepsilon^{1,n}(s,t)).
\end{aligned} \tag{132}$$

It follows that

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\mu_0\ldots\mu_n}(z,d_1s,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^k)\ldots\mu_n(\sigma^k)}(z,\varepsilon^{1,n}(s,t)),g(z)). \tag{133}$$

However, since $z\in Z^{\sigma^i}$ and $(s,t)\in\Delta^{1,n}_v$, the right-hand side of (133) equals $\overline{h}^{\sigma^j\sigma^k}_{\mu_0\ldots\mu_n}(z,s,t)$ and we obtained the desired relation (31), for $l=1$. Now assume that $l=2$. Since $(s,t)\in\Delta^{1,n}_v=[e_{00},\ldots,e_{0v},e_{1v},\ldots,e_{1n}]$, one sees that $(d_2\times 1)(s,t)\in[e_{00},\ldots,e_{0v},e_{1v},\ldots,e_{1n}]\subseteq\Delta^{2,n}_{vn}$. Consequently,

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\mu_0\ldots\mu_n}(z,d_2s,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^j)\ldots\mu_n(\sigma^j)\mu_n(\sigma^k)}(z,\varepsilon^{2,n}(d_2s,t)),g(z)). \tag{134}$$

A straightforward verification shows that

$$\varepsilon^{2,n}(d_2\times 1)=d_{n+2}\varepsilon^{1,n}. \tag{135}$$

Therefore,

$$\begin{aligned}
&f_{\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^j)\ldots\mu_n(\sigma^j)\mu_n(\sigma^k)}(z,\varepsilon^{2,n}(d_2s,t))\\
&\quad=f_{\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^j)\ldots\mu_n(\sigma^j)\mu_n(\sigma^k)}(z,d_{n+2}\varepsilon^{1,n}(s,t))\\
&\quad=f_{d^{n+2}(\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^j)\ldots\mu_n(\sigma^j)\mu_n(\sigma^k))}(z,\varepsilon^{1,n}(s,t))\\
&\quad=f_{\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^j)\ldots\mu_n(\sigma^j)}(z,\varepsilon^{1,n}(s,t)).
\end{aligned} \tag{136}$$

It follows that

$$\overline{h}^{\sigma^i\sigma^j\sigma^k}_{\mu_0\ldots\mu_n}(z,d_2s,t)=\phi_{\mu_0}(f_{\mu_0(\sigma^i)\ldots\mu_v(\sigma^i)\mu_v(\sigma^j)\ldots\mu_n(\sigma^j)}(z,\varepsilon^{1,n}(s,t)),g(z)). \tag{137}$$

However, since $z\in Z^{\sigma^i}$ and $(s,t)\in\Delta^{1,n}_v$, the right-hand side of (137) equals $\overline{h}^{\sigma^j\sigma^k}_{\mu_0\ldots\mu_n}(z,s,t)$ and we obtained the desired relation (31), for $l=2$. □

8. Coherent homotopy extension properties

8.1. The standard homotopy extension property (HEP) was stated in Subsection 2.2. In this subsection we will first define a coherent version of (HEP) called the *coherent homotopy extension property*, abbreviated (CHEP).

Definition 1. *A pair of spaces (A,B), where $B\subseteq A$ is a closed subset of A, is said to have the coherent homotopy extension property* (CHEP) *with respect to an inverse system of spaces $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$, provided the following holds. For any coherent mapping $\boldsymbol{k}=(k_{\boldsymbol{\mu}})\colon((A\times e_1)\cup(B\times[e_0,e_1]))\to\boldsymbol{Y}$, there exists a coherent mapping $\boldsymbol{h}=(h_{\boldsymbol{\mu}})\colon A\times[e_0,e_1])\to\boldsymbol{Y}$, which extends $\boldsymbol{k}$, i.e., $h_{\boldsymbol{\mu}}$ extends $k_{\boldsymbol{\mu}}$, for every multiindex $\boldsymbol{\mu}$ in M.*

Lemma 10. *Let $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$ be an inverse system consisting of spaces Y_μ, which are* ANE*s for metrizable spaces. Then every metrizable pair (A,B), B closed in A, has the* (CHEP) *with respect to $\boldsymbol{Y}$.*

Proof. By assumption, for multiindices $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ in M, we have mappings $k_{\boldsymbol{\mu}} \colon ((A \times e_1) \cup (B \times [e_0, e_1])) \times \Delta^n \to Y_{\mu_0}$ such that coherence conditions (10) and (11) hold. We must extend mappings $k_{\boldsymbol{\mu}}$ to mappings $h_{\boldsymbol{\mu}} \colon (A \times [e_0, e_1] \times \Delta^n) \to Y_{\mu_0}$ in such a way that the coherence conditions continue to hold and therefore, $\boldsymbol{h} = (h_{\boldsymbol{\mu}})$ is a coherent mapping. We will define the desired mappings $h_{\boldsymbol{\mu}}$ by induction on the length $n = |\boldsymbol{\mu}|$.

If $n = 0$, then $\boldsymbol{\mu} = (\mu_0)$, $\Delta^n = [e_0]$ and $k_{\mu_0} \colon ((A \times e_1) \cup (B \times [e_0, e_1])) \times e_0 \to Y_{\mu_0}$ is a mapping. Since Y_{μ_0} is an ANE for metrizable spaces, A is metrizable and $B \subseteq A$ is closed, Lemma 2 shows that the pair (A, B) has property (HEP) with respect to Y_{μ_0}. Consequently, there exists an extension $h_{\mu_0} \colon (A \times [e_0, e_1]) \times e_0 \to Y_{\mu_0}$ of k_{μ_0}. In this case the coherence conditions are empty.

Now assume that $n = 1$, i.e., $\boldsymbol{\mu} = (\mu_0, \mu_1)$, $\Delta^1 = [e_0, e_1]$. Since $e_0 = d_1 e_0$ and $e_1 = d_0 e_0$, $k_{\mu_0\mu_1} \colon (A \times e_1 \times \Delta^1) \cup (B \times [e_0, e_1] \times \Delta^1) \to Y_{\mu_0}$ is a mapping such that

$$k_{\mu_0\mu_1}(a, e_1, e_0) = k_{\mu_0}(a, e_1, e_0), \quad a \in A, \tag{138}$$

$$k_{\mu_0\mu_1}(b, s, e_0) = k_{\mu_0}(b, s, e_0), \quad b \in B,\ s \in [e_0, e_1], \tag{139}$$

$$k_{\mu_0\mu_1}(a, e_1, e_1) = q_{\mu_0\mu_1} k_{\mu_1}(a, e_1, e_0), \quad a \in A, \tag{140}$$

$$k_{\mu_0\mu_1}(b, s, e_1) = q_{\mu_0\mu_1} k_{\mu_1}(b, s, e_0), \quad b \in B,\ s \in [e_0, e_1]. \tag{141}$$

If $\boldsymbol{\mu} = (\mu_0, \mu_1)$ is nondegenerate, we define $h_{\mu_0\mu_1} \colon (A \times e_1 \times \Delta^1) \cup (A \times \Delta^1 \times \{e_0, e_1\}) \cup (B \times [e_0, e_1] \times \Delta^1) \to Y_{\mu_0}$ by the following formulae.

$$h_{\mu_0\mu_1}(a, e_1, t) = k_{\mu_0\mu_1}(a, e_1, t), \quad a \in A, \tag{142}$$

$$h_{\mu_0\mu_1}(a, s, e_0) = h_{\mu_0}(a, s, e_0), \quad a \in A,\ s \in [e_0, e_1], \tag{143}$$

$$h_{\mu_0\mu_1}(a, s, e_1) = q_{\mu_0\mu_1} h_{\mu_1}(a, s, e_0), \quad a \in A,\ s \in [e_0, e_1], \tag{144}$$

$$h_{\mu_0\mu_1}(b, s, t) = k_{\mu_0\mu_1}(b, s, t), \quad b \in B,\ s \in [e_0, e_1]. \tag{145}$$

(142) is compatible with (143), because $k_{\mu_0\mu_1}(a, e_1, e_0) = k_{\mu_1}(a, e_1, e_0)$ and also $h_{\mu_0}(a, e_1, e_0) = k_{\mu_0}(a, e_1, e_0)$. (142) is compatible with (144), because $k_{\mu_0\mu_1}(a, e_1, e_1) = q_{\mu_0\mu_1} k_{\mu_1}(a, e_1, e_0) = q_{\mu_0\mu_1} h_{\mu_1}(a, e_1, e_0)$. (142) and (143) are compatible with (145), because, for $a = b$ and $s = e_1$, all three expressions assume the value $k_{\mu_0\mu_1}(b, e_1, t)$. Compatibility of (144) and (145) follows from $k_{\mu_0\mu_1}(b, s, e_1) = q_{\mu_0\mu_1} k_{\mu_1}(b, s, e_0) = q_{\mu_0\mu_1} h_{\mu_1}(b, s, e_0)$. Since Y_{μ_0} is an ANE for metrizable spaces and $(A \times \Delta^1, A \times \{e_0, e_1\} \cup B \times \Delta^1)$ is a closed pair of metrizable spaces, one can apply Lemma 2 and conclude that the pair has property (HEP). Consequently, the mapping $k_{\mu_0\mu_1}$ admits an extension $h_{\mu_0\mu_1} \colon A \times \Delta^1 \times \Delta^1 \to Y_{\mu_0}$ such that formulae (142)–(145) continue to hold. Formulae (143) and (144) show that the required boundary conditions are fulfilled.

If $\boldsymbol{\mu}$ is degenerate, i.e., $\boldsymbol{\mu} = (\mu_0, \mu_0)$, we put

$$h_{\mu_0\mu_0}(a, s, t) = h_{\mu_0}(a, s, e_0), \quad a \in A,\ s \in [e_0, e_1],\ t \in [e_0, e_1]. \tag{146}$$

Note that $h_{\mu_0\mu_0}(a, s, d_0 e_0) = h_{\mu_0}(a, s, e_0) = q_{\mu_0\mu_0} h_{d^0(\mu_0, \mu_0)}(a, s, e_0)$ and also $h_{\mu_0\mu_0}(a, s, d_1 e_0) = h_{\mu_0}(a, s, e_0) = h_{d^1(\mu_0, \mu_0)}(a, s, e_0)$, which shows that the required boundary conditions are fulfilled. (146) shows that the corresponding degeneracy conditions are fulfilled, because $s_j(t) = e_0$, for $t \in [e_0, e_1]$ and $j \in \{0, 1\}$. To verify that $h_{\mu_0\mu_0}$ extends $k_{\mu_0\mu_0}$ note that $h_{\mu_0\mu_0}(a,e_1,t) = h_{\mu_0}(a,e_1,e_0) = k_{\mu_0}(a,e_1,e_0) = k_{\mu_0\mu_0}(a,e_1,t)$, because $e_0 = s_1(t)$. Similarly, $h_{\mu_0\mu_0}(a,s,e_j) = h_{\mu_0}(a,s,e_0) = k_{\mu_0}(a,s,e_0) = k_{\mu_0\mu_0}(a,s,e_j)$, $j \in \{0,1\}$. Finally, $h_{\mu_0\mu_0}(a,s,e_j) = h_{\mu_0}(b,s,e_0) = k_{\mu_0}(b,s,e_0) = k_{\mu_0\mu_0}(b,s,e_j)$, because $s_j(t) = e_0$.

Now assume that $n \geq 2$ and that we have already defined mappings $h_{\mu_0\dots\mu_l} : A \times \Delta^1 \times \Delta^l \to Y_{\mu_0}$, $0 \leq l \leq n-1$, which extend $k_{\mu_0\dots\mu_l} : (A \times e_1 \times \Delta^l) \cup (B \times [e_0,e_1] \times \Delta^l) \to Y_{\mu_0}$ and satisfy the coherence conditions applicable at this stage of the induction process. We will first define mappings $h_{\boldsymbol{\mu}} = h_{\mu_0\dots\mu_n} : A \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$ in the case when $\boldsymbol{\mu}$ is nondegenerate.

Put

$$h_{\boldsymbol{\mu}}(a,s,d_j t) = \begin{cases} q_{\mu_0\mu_1} h_{d^0\boldsymbol{\mu}}(a,s,t), & j = 0, \\ h_{d^j\boldsymbol{\mu}}(a,s,t), & 1 \leq j \leq n, \end{cases} \tag{147}$$

where $a \in A$, $s \in \Delta^1$ and $t \in \Delta^{n-1}$. Since the length $|d^j\boldsymbol{\mu}| = n-1 < n$, the right-hand side of (147) is defined. Note that (147) corresponds to formulae (143) and (144), for $n = 1$. Let us show that the expressions on the right-hand side of (147) are compatible and therefore, define a mapping $h_{\boldsymbol{\mu}} : A \times \Delta^1 \times \partial\Delta^n \to Y_{\mu_0}$, where $\partial\Delta^n$ denotes the boundary of Δ^n. Since $\partial\Delta^n = \cup_{j=0}^{j=n} d_j(\Delta^{n-1})$, every point of $\partial\Delta^n$ is of the form $d_j t$, for some $0 \leq j \leq n$ and some $t \in \Delta^{n-1}$. It can happen that $d_j t = d_k t'$, for some $t, t' \in \Delta^{n-1}$ and $j \neq k$, say $j < k$. We must show that the values obtained by (147) using t and using t' coincide.

Let us first show that there exists a point $t^* \in \Delta^{n-2}$ such that $t = d_{k-1}t^*$ and $t' = d_j t^*$. Indeed, the barycentric coordinate α_{k-1} of t must be 0, because $j < k$ implies that α_{k-1} is the k-th barycentric coordinate of $d_j t$ and the latter is 0, because $d_j t = d_k t' \in d_k(\Delta^{n-1})$. Consequently, $t \in d_{k-1}(\Delta^{n-2})$ and there exists a point $t^* \in \Delta^{n-2}$ such that $t = d_{k-1}t^*$. Similarly, the barycentric coordinate α'_j of t' must be 0, because $j < k$ implies that α'_j is the j-th barycentric coordinate of $d_k t'$ and the latter must be 0, because $d_k t' = d_j t \in d_j(\Delta^{n-1})$. Consequently, $t' \in d_j(\Delta^{n-2})$ and there exists a point $t'^* \in \Delta^{n-2}$ such that $t' = d_j t'^*$. Now note that $d_k d_j = d_j d_{k-1} : \Delta^{n-2} \to \Delta^n$ (see the analogue of (1.2.16) in [15]). Therefore, $d_j d_{k-1} t^* = d_j t = d_k t' = d_k d_j t'^* = d_j d_{k-1} t'^*$, Since $d_j d_{k-1} : \Delta^{n-2} \to \Delta^n$ is an injection, it follows that $t'^* = t^*$ and thus, also $t' = d_j t^*$.

If $j = 0$, then $d_0 t = d_0 d_{k-1} t^*$. Therefore, (147) shows that $h_{\boldsymbol{\mu}}(a,s,d_0 t) = q_{\mu_0\mu_1} h_{d^0\boldsymbol{\mu}}(a,s,d_{k-1}t^*)$. If $k > 1$, this equals $q_{\mu_0\mu_1} h_{d^{k-1}d^0\boldsymbol{\mu}}(a,s,t^*)$. On the other hand, $d_k t' = d_k d_0 t^*$ and therefore, by (147), $h_{\boldsymbol{\mu}}(a,s,d_k t') = h_{d^k\boldsymbol{\mu}}(a,s,d_0 t^*) = q_{\mu_0\mu_1} h_{d^0 d^k\boldsymbol{\mu}}(a,s,t^*)$. Since $d^{k-1}d^0 = d^0 d^k$ (see (1.2.19) in [15]), we conclude that indeed, $h_{\boldsymbol{\mu}}(a,s,d_0 t) = h_{\boldsymbol{\mu}}(a,s,d_k t')$. The same conclusion holds if $j = 0$ and $k = 1$. Then $d_0 t = d_0 d_0 t^*$, $h_{\boldsymbol{\mu}}(a,s,d_0 t) = q_{\mu_0\mu_1} h_{d^0\boldsymbol{\mu}}(a,s,d_0 t^*) = q_{\mu_0\mu_1} q_{\mu_1\mu_2} h_{d^0 d^0\boldsymbol{\mu}}(a,s,t^*)$. On the other hand, $d_1 t' = d_1 d_0 t^*$ and we see that $h_{\boldsymbol{\mu}}(a,s,d_1 t') = h_{d^1\boldsymbol{\mu}}(a,s,d_0 t^*) = q_{\mu_0\mu_2} h_{d^0 d^1\boldsymbol{\mu}}(a,s,t^*)$. Since $d^0 d^0 = d^0 d^1$, we see that again $h_{\boldsymbol{\mu}}(a,s,d_j t) = h_{\boldsymbol{\mu}}(a,s,d_1 t')$.

If $j > 0$, the verification is simpler. Indeed, since $d_j t = d_j d_{k-1} t^*$, we see that $h_{\boldsymbol{\mu}}(a,s,d_j t) = h_{d^j\boldsymbol{\mu}}(a,s,d_{k-1}t^*) = h_{d^{k-1}d^j\boldsymbol{\mu}}(a,s,t^*)$. On the other hand, $d_k t' = d_k d_j t^*$ yields $h_{\boldsymbol{\mu}}(a,s,d_k t') = h_{d^k\boldsymbol{\mu}}(a,s,d_j t^*) = h_{d^j d^k\boldsymbol{\mu}}(a,s,t^*)$, which coincides with $h_{\boldsymbol{\mu}}(a,s,d_j t)$, because $d^{k-1}d^j = d^j d^k$ (see (1.2.19) in [15]).

In analogy with (142) and (145), we define $h_{\boldsymbol{\mu}}$ on $(A\times e_1\times\Delta^n)\cup(B\times\Delta^1\times\Delta^n)$ by putting

$$h_{\boldsymbol{\mu}}(a,e_1,t)=k_{\boldsymbol{\mu}}(a,e_1,t),\quad a\in A, \tag{148}$$

$$h_{\boldsymbol{\mu}}(b,s,t)=k_{\boldsymbol{\mu}}(b,s,t),\quad b\in B. \tag{149}$$

Let us verify that formulae (148) and (149) are compatible with (147) and therefore, yield a well-defined mapping $h_{\boldsymbol{\mu}}\colon (A\times e_1\times\Delta^n)\cup(B\times\Delta^1\times\Delta^n)\cup(A\times\Delta^1\times\partial\Delta^n)\to Y_{\mu_0}$. By (148), $h_{\boldsymbol{\mu}}(a,e_1,d_0t)=k_{\boldsymbol{\mu}}(a,e_1,d_0t)=q_{\mu_0\mu_1}k_{d^0\boldsymbol{\mu}}(a,e_1,t)$ and by (146), $h_{\boldsymbol{\mu}}(a,e_1,d_0t)=q_{\mu_0\mu_1}h_{d^0\boldsymbol{\mu}}(a,e_1,t)$. Since the length $|d^0\boldsymbol{\mu}|=n-1$, (148) for $n-1$ shows that $h_{d^0\boldsymbol{\mu}}(a,e_1,t)=k_{d^0\boldsymbol{\mu}}(a,e_1,t)$ and thus, the right-hand sides of (147) and (148) assume the same value at the point (a,e_1,t). A similar and simpler argument proves the same fact if $j>0$. To show that (147) and (149) are compatible, note that, by (147), $h_{\boldsymbol{\mu}}(b,s,d_0t)=q_{\mu_0\mu_1}h_{d^0\boldsymbol{\mu}}(b,s,t)$ and by (149), $h_{\boldsymbol{\mu}}(b,s,d_0t)=k_{\boldsymbol{\mu}}(b,s,d_0t)=q_{\mu_0\mu_1}k_{d^0\boldsymbol{\mu}}(b,s,t)$. Since the length $|d^0\boldsymbol{\mu}|=n-1$, (149) for $n-1$ shows that $h_{d^0\boldsymbol{\mu}}(b,s,t)=k_{d^0\boldsymbol{\mu}}(b,s,t)$ and thus, the right-hand sides of (147) and (149) assume the same value at the point (b,s,t). A similar and simpler argument proves the same fact if $j>0$.

Applying (HEP) to the metrizable pair $(A\times\Delta^n, A\times\partial\Delta^n\cup B\times\Delta^n)$, we obtain a further extension $h_{\boldsymbol{\mu}}\colon A\times\Delta^1\times\Delta^n\to Y_{\mu_0}$ of $h_{\boldsymbol{\mu}}$, which extends $k_{\boldsymbol{\mu}}$ and satisfies the boundary conditions (147). Since $\boldsymbol{\mu}$ was assumed nondegenerate, the degeneracy condition $h_{\boldsymbol{\mu}}(a,s_jt)=h_{s^j\boldsymbol{\mu}}(a,t)$ does not apply at this stage of the induction process, because $|s^j\boldsymbol{\mu}|=n+1>n$.

In order to define $h_{\boldsymbol{\mu}}$ for degenerate $\boldsymbol{\mu}$, we follow the procedure used in the proof of Lemma 1.13 in [15]. Clearly, there are k uniquely determined integers $0<u_0<\dots<u_{k-1}<n$ such that $\mu_0=\dots=\mu_{u_0-1}<\mu_{u_0}=\dots=\mu_{u_1-1}<\mu_{u_1}\cdot\dots\cdot\mu_{u_{k-1}-1}<\mu_{u_{k-1}}=\dots=\mu_n$, where $<$ stands for $\le$ and $\ne$. Define an increasing function $u\colon\{0,1,\dots,n\}\to\{0,1,\dots,k\}$, by putting $u(j)=0$, if $0\le j<u_0$, putting $u(j)=i$, if $u_{i-1}\le j<u_i$, where $1\le i\le k-1$, and putting $u(j)=k$, if $u_{k-1}\le j\le n$. The function u induces a simplicial mapping $u_*\colon\Delta^n\to\Delta^k$, defined by putting $u_*(e_i)=e_{u_i}$, $0\le i\le n$. Define $\boldsymbol{\nu}=(\nu_0,\dots,\nu_k)$, by putting $\nu_0=\mu_0$ and $\nu_i=\mu_{u_{i-1}}$, for $1\le i\le k$. Note that $\nu_k=\mu_{u_{k-1}}=\mu_n$. Also note that $\boldsymbol{\nu}$ is a nondegenerate multiindex of length k.

We now define $h_{\boldsymbol{\mu}}$ by the formula

$$h_{\boldsymbol{\mu}}(a,s,t)=h_{\boldsymbol{\nu}}(a,s,u_*t),\quad a\in A,\ s\in\Delta^1,\ t\in\Delta^n. \tag{150}$$

Since $\boldsymbol{\nu}$ is a nondegenerate multiindex of length $k<n$ and $u_*t\in\Delta^k$, we see that the right-hand side of (150) is well defined.

In the proof of Lemma 1.13 in [15] one finds a proof of the fact that $h_{\boldsymbol{\mu}}$, defined by (150), satisfies the boundary conditions. There it is also proved that the degeneracy condition $h_{\boldsymbol{\mu}}(a,s,s_jt)=h_{s^j\boldsymbol{\mu}}(a,s,t)$ is fulfilled. There is one more degeneracy condition, which at this stage of the induction process makes sense and must be verified. It is the condition $h_{s^j\boldsymbol{\mu}}(a,s,t)=h_{\boldsymbol{\mu}}(a,s,s_jt)$, where $|\boldsymbol{\mu}|=n-1$ and $t\in\Delta^{n-1}$. Indeed, if the role of $\boldsymbol{\mu}$ is played by $s^j\boldsymbol{\mu}$, then $\boldsymbol{\mu}$ assumes the role of $\boldsymbol{\nu}$ and s_j assumes the role of u_*. Consequently, formula (150) assumes the desired form $h_{s^j\boldsymbol{\mu}}(a,s,t)=h_{\boldsymbol{\mu}}(a,s,s_jt)$.

It remains to prove that also in the case of degenerate $\boldsymbol{\mu}$, $h_{\boldsymbol{\mu}}$ extends $k_{\boldsymbol{\mu}}$. Indeed, for $a \in A$, $s \in \Delta^1$ and $t \in \Delta^n$, one has $h_{\boldsymbol{\mu}}(a,s,t) = h_{\boldsymbol{\nu}}(a,s,u_*t)$. Since $|\boldsymbol{\nu}| < n$, we already know that $h_{\boldsymbol{\nu}}$ extends $k_{\boldsymbol{\nu}}$ and thus, $h_{\boldsymbol{\nu}}(a,s,u_*t) = k_{\boldsymbol{\nu}}(a,s,u_*t)$. However, $k_{\boldsymbol{\mu}}(a,s,t) = k_{\boldsymbol{\nu}}(a,s,u_*t)$, because $\boldsymbol{\mu} = u^*(\boldsymbol{\nu})$ and the degeneracy conditions for $\boldsymbol{\nu}$ imply $k_{\boldsymbol{\nu}}(a,s,u_*t) = k_{u^*\boldsymbol{\nu}}(a,s,t) = k_{\boldsymbol{\mu}}(a,s,t)$. Hence, $h_{\boldsymbol{\mu}}(a,s,t) = k_{\boldsymbol{\mu}}(a,s,t)$. □

8.2. In this subsection, we will prove two coherent homotopy extension lemmas, needed to perform constructions (C1)–(C4). To state the first lemma, we introduce some terminology. Recall that $\boldsymbol{q} = (q_{\boldsymbol{\mu}})\colon X \times P \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is the standard resolution of $X \times P$, $\boldsymbol{f} = (f_\lambda)\colon Z \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is a coherent mapping and $g\colon Z \to P$ is a mapping. If A is a closed subset of Z, we say that a coherent mapping $\boldsymbol{h} = (h_{\boldsymbol{\mu}})\colon A \to \boldsymbol{Y}$ has the special property provided (22) holds. We say that a coherent mapping $\boldsymbol{h} = (h_{\boldsymbol{\mu}})\colon A \times \Delta^1 \to \boldsymbol{Y}$ has the special property provided

$$h_{\overline{\lambda_0}\ldots\overline{\lambda_n}}(a,s,t) = (f_{\lambda_0\ldots\lambda_n}(a,t), g(a)), \quad a \in A,\ s \in \Delta^1,\ t \in \Delta^n. \tag{151}$$

Similarly, we say that $\boldsymbol{h} = (h_{\boldsymbol{\mu}})\colon A \times \Delta^2 \to \boldsymbol{Y}$ has the special property provided (151) holds, for $s \in \Delta^2$.

Lemma 11. *Let the space Z be metrizable and let $B \subseteq A$ be closed subsets of Z. If $\boldsymbol{k} = (k_{\boldsymbol{\mu}})\colon ((A \times e_1) \cup (B \times [e_0,e_1])) \to \boldsymbol{Y}$ is a coherent mapping, which has the special property, then there exists a coherent mapping $\boldsymbol{h} = (h_{\boldsymbol{\mu}})\colon (A \times [e_0,e_1]) \to \boldsymbol{Y}$, which extends $\boldsymbol{k}$ and also has the special property.*

Proof. The assumption that $\boldsymbol{k}$ has the special property means that both coherent mappings $\boldsymbol{k}|(A \times e_1)$ and $\boldsymbol{k}|(B \times [e_0,e_1])$ have that property. If $n = 0$ and $\boldsymbol{\mu} = (\overline{\lambda_0})$ is special, we put $h_{\overline{\lambda_0}}(a,s,e_0) = (f_{\lambda_0}(a,t), g(a))$. The mapping $h_{\overline{\lambda_0}}$ extends $k_{\overline{\lambda_0}}$, because the latter mapping satisfies the special condition. If $n = 1$, i.e., $\boldsymbol{\mu} = (\mu_0, \mu_1)$, one first extends $k_{\boldsymbol{\mu}}$ to $h_{\boldsymbol{\mu}}$ in the cases when $\boldsymbol{\mu}$ is special, by putting $h_{\overline{\lambda_0\lambda_1}}(a,s,t) = (f_{\lambda_0\lambda_1}(a,t), g(a))$. Then one proceeds to the cases when $\boldsymbol{\mu}$ is not special, following the proof of Lemma 10.

Now assume that $n \ge 2$ and that we have already defined mappings $h_{\mu_0\ldots\mu_l}\colon A \times \Delta^1 \times \Delta^l \to Y_{\mu_0}$, $0 \le l \le n-1$, which extend $k_{\mu_0\ldots\mu_l}\colon (A \times e_1 \times \Delta^l) \cup (B \times [e_0,e_1] \times \Delta^l) \to Y_{\mu_0}$ and satisfy the coherence and the special condition in situations when they are applicable at this stage of the induction process. We first define $h_{\boldsymbol{\mu}} = h_{\mu_0\ldots\mu_n}\colon A \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$, for $\boldsymbol{\mu}$ nondegenerate.

If $\boldsymbol{\mu}$ is also special, i.e., $\boldsymbol{\mu} = \overline{\boldsymbol{\lambda}} = (\overline{\lambda_0}, \ldots, \overline{\lambda_n})$, then we put $h_{\overline{\lambda_0}\ldots\overline{\lambda_n}}(a,s,t) = (f_{\lambda_0\ldots\lambda_n}(a,t), g(a))$. This insures the validity of special condition (151). Note that we do obtain an extension of $k_{\overline{\lambda_0}\ldots\overline{\lambda_n}}$ because the latter mapping satisfies the special condition. Also $h_{\overline{\lambda_0}\ldots\overline{\lambda_n}}(a,s,d_jt) = (f_{\lambda_0\ldots\lambda_n}(a,d_jt), g(a)) = (f_{d^j(\lambda_0\ldots\lambda_n)}(a,t), g(a))$, for $j > 0$, because $\boldsymbol{f}$ satisfies the boundary condition. Since $d^j(\overline{\lambda_0}, \ldots, \overline{\lambda_n})$ is a special nondegenerate multiindex of length $n-1$, the induction hypothesis implies that $h_{d^j(\overline{\lambda_0}\ldots\overline{\lambda_n})}(a,s,t) = (f_{d^j(\lambda_0\ldots\lambda_n)}(a,t), g(a))$ and thus, $h_{\overline{\lambda_0}\ldots\overline{\lambda_n}}(a,s,d_jt) = h_{d^j(\overline{\lambda_0}\ldots\overline{\lambda_n})}(a,s,t)$, as required by the boundary condition. The case when $j = 0$ is established by a similar argument. Since $s^j(\overline{\lambda_0}\ldots\overline{\lambda_{n-1}})$ is degenerate and $s^j(\overline{\lambda_0}\ldots\overline{\lambda_n})$ is of length $n+1$, at this stage there are no degenerate conditions to be verified.

If $\boldsymbol{\mu}$ is nondegenerate, but not special, we proceed as in the first part of the proof of Lemma 10. This is possible, because by Lemma 1, $\boldsymbol{Y}$ consists of ANEs for metrizable spaces. We obtain mappings $h_{\boldsymbol{\mu}}$, which satisfy the coherence conditions and extend $k_{\boldsymbol{\mu}}$.

In the case when $\boldsymbol{\mu}$ is degenerate, we proceed as in the second part of the proof of Lemma 10, i.e., we define $h_{\boldsymbol{\mu}}$, by formula (149). As in that proof, the obtained mappings $h_{\boldsymbol{\mu}}$ satisfy the coherence conditions and extend $k_{\boldsymbol{\mu}}$. It only remains to show that in the case of degenerate special multiindices $\boldsymbol{\mu} = \overline{\boldsymbol{\lambda}}$, the special condition remains valid. Indeed, by definition, $h_{\overline{\lambda_0}\dots\overline{\lambda_n}}(a,s,t) = h_{\boldsymbol{\nu}}(a,s,u_*t)$, where $u\colon \{0,\dots,n\} \to \{0,\dots,k\}$, u^* and $\boldsymbol{\nu}$ are defined as in the proof of Lemma 10. Since $\boldsymbol{\nu}$ is nondegenerate and of the form $\boldsymbol{\nu} = (\overline{\lambda_{i_0}},\dots,\overline{\lambda_{i_k}})$, where the length $k = |\boldsymbol{\nu}| < n$, the induction hypothesis implies that $h_{\boldsymbol{\nu}}$ has the special property and thus, $h_{\boldsymbol{\nu}}(a,s,u_*t) = h_{\overline{\lambda_{i_0}}\dots\overline{\lambda_{i_k}}}(a,s,u_*t) = (f_{\lambda_{i_0}\dots\lambda_{i_k}}(a,u_*t), g(a))$. By the degeneracy property of $\boldsymbol{f}$, $f_{\lambda_{i_0}\dots\lambda_{i_k}}(a,u_*t) = f_{\lambda_0\dots\lambda_n}(a,t)$ and thus, $h_{\overline{\lambda_0}\dots\overline{\lambda_n}}(a,s,t) = (f_{\lambda_0\dots\lambda_n}(a,t), g(a))$, as required by the special property (151) for $h_{\overline{\boldsymbol{\lambda}}}$. □

Lemma 12. *Let the space Z be metrizable and let $Z^0 \subseteq Z^1$ be closed subsets of Z. Let $\boldsymbol{h}^{01} = (h^{01}_{\boldsymbol{\mu}})\colon Z^1 \times [e_0,e_1] \to \boldsymbol{Y}$, $\boldsymbol{h}^{12} = (h^{12}_{\boldsymbol{\mu}})\colon Z^1 \times [e_1,e_2] \to \boldsymbol{Y}$ and $\boldsymbol{h}^{012} = (h^{012}_{\boldsymbol{\mu}})\colon Z^0 \times [e_0,e_1,e_2] \to \boldsymbol{Y}$ be coherent mappings, satisfying the special condition, and let the following conditions be fulfilled.*

$$h^{01}_{\boldsymbol{\mu}}(z,e_1,t) = h^{12}_{\boldsymbol{\mu}}(z,e_1,t), \quad z \in Z^1,\ t \in \Delta^n, \tag{152}$$

$$h^{012}_{\boldsymbol{\mu}}(z,s,t) = h^{01}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^0,\ s \in [e_0,e_1],\ t \in \Delta^n, \tag{153}$$

$$h^{012}_{\boldsymbol{\mu}}(z,s,t) = h^{12}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^0,\ s \in [e_1,e_2],\ t \in \Delta^n. \tag{154}$$

Then there exists a coherent mapping $\boldsymbol{h} = (\boldsymbol{h}_{\boldsymbol{\mu}})\colon Z^1 \times [e_0,e_1,e_2] \to Y$, which extends the coherent mappings $\boldsymbol{h}^{01}, \boldsymbol{h}^{12}$ and $\boldsymbol{h}^{012}$ and satisfies the special condition.

Proof. Recall that the standard triangulation $\Delta^{1,1}$ of the Cartesian product $\Delta^1 \times \Delta^1$ consists of 2-simplices $\Delta^{1,1}_0 = [e_{00},e_{10},e_{11}]$, $\Delta^{1,1}_1 = [e_{00},e_{01},e_{11}]$ and their faces. Also recall the simplicial mapping $\varepsilon^{1,1}\colon \Delta^{1,1} \to \Delta^2$, defined in 6.1, by $\varepsilon^{1,1}(e_{0j}) = e_j$ and $\varepsilon^{1,1}(e_{1j}) = e_{j+1}$, for $0 \le j \le 1$. Define mappings $\widehat{h}^{01}_{\boldsymbol{\mu}}\colon Z^1 \times [e_{00},e_{10}] \times \Delta^n \to Y_{\mu_0}$, $\widehat{h}^{12}_{\boldsymbol{\mu}}\colon Z^1 \times [e_{10},e_{11}] \times \Delta^n \to Y_{\mu_0}$ and $\widehat{h}^{012}_{\boldsymbol{\mu}}\colon Z^0 \times (\Delta^1 \times \Delta^1) \times \Delta^n \to Y_{\mu_0}$, by putting $\widehat{h}^{01}_{\boldsymbol{\mu}}(z,s,t) = h^{01}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),t)$, $\widehat{h}^{12}_{\boldsymbol{\mu}}(z,s,t) = h^{12}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),t)$ and $\widehat{h}^{012}_{\boldsymbol{\mu}} = h^{012}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),t)$.

It is readily seen that $\widehat{\boldsymbol{h}}^{01} = (\widehat{h}^{01}_{\boldsymbol{\mu}})\colon Z^1 \times [e_{00},e_{10}] \to \boldsymbol{Y}$, $\widehat{\boldsymbol{h}}^{12} = (\widehat{h}^{12}_{\boldsymbol{\mu}})\colon Z^1 \times [e_{10},e_{11}] \to \boldsymbol{Y}$ and $\widehat{\boldsymbol{h}}^{012} = (\widehat{h}^{012}_{\boldsymbol{\mu}})\colon Z^0 \times (\Delta^1 \times \Delta^1) \to \boldsymbol{Y}$ are coherent mappings, satisfying the special condition. Indeed, if $j > 0$, then $\widehat{h}^{01}_{\boldsymbol{\mu}}(z,s,d_jt) = h^{01}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),d_jt) = h^{01}_{d^j\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),t) = \widehat{h}^{01}_{d^j\boldsymbol{\mu}}(z,s,t)$, because $\boldsymbol{h}^{01}$ satisfies the boundary conditions. A similar argument proves the assertion in the case $j = 0$. Furthermore, since $\boldsymbol{h}^{01}$ satisfies the degeneracy condition, $\widehat{h}^{01}_{\boldsymbol{\mu}}(z,s,s_jt) = h^{01}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),s_jt) = h^{01}_{s^j\boldsymbol{\mu}}(z,\varepsilon^{1,1}(s),t) = \widehat{h}^{01}_{s^j\boldsymbol{\mu}}(z,s,t)$. Also note that $\widehat{\boldsymbol{h}}^{01}_{\overline{\lambda_0}\dots\overline{\lambda_n}}(z,s,t) = h^{01}_{\overline{\lambda_0}\dots\overline{\lambda_n}}(z,\varepsilon^{1,1}(s),t) = (f_{\lambda_0\dots\lambda_n}(z,t), g(z))$, because $\boldsymbol{h}^{01}$ has the special property. Analogous arguments show that $\widehat{\boldsymbol{h}}^{12}$ and $\widehat{\boldsymbol{h}}^{012}$ are coherent mappings, satisfying the special condition.

Formulae (152), (153) and (154) show that mappings $\widehat{h}^{01}_{\boldsymbol{\mu}}, \widehat{h}^{12}_{\boldsymbol{\mu}}$ and $\widehat{h}^{012}_{\boldsymbol{\mu}}$ are compatible, i.e, $\widehat{h}^{01}_{\boldsymbol{\mu}}(z,e_{10},t) = \widehat{h}^{12}_{\boldsymbol{\mu}}(z,e_{10},t)$, for $z \in Z^1$, $\widehat{h}^{012}_{\boldsymbol{\mu}}(z,s,t) = \widehat{h}^{01}_{\boldsymbol{\mu}}(z,s,t)$, for

$z \in Z^0$, $s \in [e_{00}, e_{10}]$ and $\widehat{h}^{012}_{\boldsymbol{\mu}}(z,s,t) = \widehat{h}^{12}_{\boldsymbol{\mu}}(z,s,t)$, for $z \in Z^0$, $s \in [e_{10}, e_{11}]$. Consequently, these mappings determine a mapping $k_{\boldsymbol{\mu}} \colon (A \times \times e_0) \cup (B \times [e_0, e_1]) \to Y_{\mu_0}$, where $A = Z^1 \times [e_0, e_1] \times \Delta^n$ and $B = (Z^1 \times e_1 \times \Delta^n) \cup (Z^0 \times [e_0, e_1] \times \Delta^n)$. Since $\widehat{\boldsymbol{h}}^{01}, \widehat{\boldsymbol{h}}^{12}$ and $\widehat{\boldsymbol{h}}^{012}$ are coherent mappings, it follows that $\boldsymbol{k} = (k_{\boldsymbol{\mu}}) \colon (A \times e_0) \cup (B \times [e_0, e_1]) \to \boldsymbol{Y}$ is also a coherent mapping. We will now extend $\boldsymbol{k}$ to a coherent mapping $\widehat{\boldsymbol{h}} = (\widehat{h}_{\boldsymbol{\mu}}) \colon Z^1 \times [e_0, e_1] \to \boldsymbol{Y}$, which satisfies the special condition, proceeding as in the proof of Lemma 11.

Assume that we have already defined mappings $\widehat{h}_{\mu_0 \dots \mu_l} \colon A \times \Delta^1 \times \Delta^l \to Y_{\mu_0}$, $0 \le l \le n-1$, which extend $k_{\mu_0 \dots \mu_l} \colon (A \times e_1 \times \Delta^l) \cup (B \times [e_0, e_1] \times \Delta^l) \to Y_{\mu_0}$ and satisfy the coherence and the special condition in situations when they are applicable at this stage of the induction process. We first define $\widehat{h}_{\boldsymbol{\mu}} = \widehat{h}_{\mu_0 \dots \mu_n} \colon A \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$, for nondegenerate $\boldsymbol{\mu}$.

If $\boldsymbol{\mu}$ is also special, i.e., $\boldsymbol{\mu} = \overline{\boldsymbol{\lambda}} = (\overline{\lambda_0}, \dots, \overline{\lambda_n})$, then we put $\widehat{h}_{\overline{\lambda_0} \dots \overline{\lambda_n}}(a,s,t) = (f_{\lambda_0 \dots \lambda_n}(a,t), g(a))$ and thus, insure the validity of the special condition. Note that, $\widehat{h}_{\overline{\lambda_0} \dots \overline{\lambda_n}}(a,s,d_j t) = (f_{\lambda_0 \dots \lambda_n}(a, d_j t), g(a)) = (f_{d^j(\lambda_0 \dots \lambda_n)}(a,t), g(a))$, for $j > 0$, because $\boldsymbol{f}$ satisfies the boundary condition. Since $d^j(\overline{\lambda_0}, \dots, \overline{\lambda_n})$ is a special nondegenerate multiindex of length $n-1$, the induction hypothesis implies that $\widehat{h}_{d^j(\overline{\lambda_0} \dots \overline{\lambda_n})}(a,s,t) = (f_{d^j(\lambda_0 \dots \lambda_n)}(a,t), g(a))$, hence $\widehat{h}_{\overline{\lambda_0} \dots \overline{\lambda_n}}(a,s,d_j t) = \widehat{h}_{d^j(\overline{\lambda_0} \dots \overline{\lambda_n})}(a,s,t)$, as required by the boundary condition. The case $j = 0$ is established by a similar argument. Since $s^j(\overline{\lambda_0} \dots \overline{\lambda_{n-1}})$ is degenerate and $s^j(\overline{\lambda_0} \dots \overline{\lambda_n})$ is of length $n+1$, at this stage there are no degenerate conditions to be verified.

If $\boldsymbol{\mu}$ is nondegenerate and not special, we proceed as in the first part of the proof of Lemma 10. This is possible, because by Lemma 1, $\boldsymbol{Y}$ consists of ANEs for metrizable spaces. We obtain mappings $\widehat{h}_{\boldsymbol{\mu}}$, which satisfy the coherence conditions and extend $k_{\boldsymbol{\mu}}$.

In the case when $\boldsymbol{\mu}$ is degenerate, we proceed as in the second part of the proof of Lemma 10, i.e., we define $\widehat{h}_{\boldsymbol{\mu}}$, by formula (150) (with h replaced by $\widehat{h}$). Following that proof, we obtain mappings $\widehat{h}_{\boldsymbol{\mu}}$, which satisfy the coherence condition and extend $k_{\boldsymbol{\mu}}$. It only remains to show that in the case of degenerate special multiindices $\boldsymbol{\mu} = \overline{\boldsymbol{\lambda}}$, the special condition remains valid. Indeed, by definition, $\widehat{h}_{\overline{\lambda_0} \dots \overline{\lambda_n}}(a,s,t) = \widehat{h}_{\boldsymbol{\nu}}(a,s,u_* t)$, where $u \colon \{0, \dots, n\} \to \{0, \dots, k\}$, u^* and $\boldsymbol{\nu}$ are defined as in the proof of Lemma 10. Since $\boldsymbol{\nu}$ is nondegenerate and of the form $\boldsymbol{\nu} = (\overline{\lambda_{i_0}}, \dots, \overline{\lambda_{i_k}})$, where the length $k = |\boldsymbol{\nu}| < n$, the induction hypothesis implies that $\widehat{h}_{\boldsymbol{\nu}}$ has the special property and thus, $\widehat{h}_{\boldsymbol{\nu}}(a,s,u_* t) = \widehat{h}_{\overline{\lambda_{i_0}} \dots \overline{\lambda_{i_k}}}(a,s,u_* t) = (f_{\lambda_{i_0} \dots \lambda_{i_k}}(a, u_* t), g(a))$. By the degeneracy property of $\boldsymbol{f}$, $f_{\lambda_{i_0} \dots \lambda_{i_k}}(a, u_* t) = f_{\lambda_0 \dots \lambda_n}(a,t)$ and thus, $\widehat{h}_{\overline{\lambda_0} \dots \overline{\lambda_n}}(a,s,t) = (f_{\lambda_0 \dots \lambda_n}(a,t), g(a))$, as required by the special property (151) for $\widehat{h}_{\overline{\boldsymbol{\lambda}}}$.

Now note that the restriction $\overline{\varepsilon^{1,1}}$ of $\varepsilon^{1,1}$ to $[e_{00}, e_{10}, e_{11}]$ is a homeomorphism $\overline{\varepsilon^{1,1}} \colon [e_{00}, e_{10}, e_{11}] \to [e_0, e_1, e_2]$. Therefore, $(1_{Z^1} \times \overline{\varepsilon^{1,1}}) \colon Z^1 \times [e_{00}, e_{10}, e_{11}] \to Z^1 \times [e_0, e_1, e_2]$ is also a homeomorphism. Denote by η its inverse and define mappings $h_{\boldsymbol{\mu}} \colon Z^1 \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$ by putting $h_{\boldsymbol{\mu}}(z,s,t) = \widehat{h}_{\boldsymbol{\mu}}(z, \eta(s), t)$. Since $\widehat{\boldsymbol{h}}$ has the boundary property, we see that $h_{\boldsymbol{\mu}}(z,s,d_0 t) = \widehat{h}_{\boldsymbol{\mu}}(z, \eta(s), d_0 t) = q_{\mu_0 \mu_1} \widehat{h}_{d^0 \boldsymbol{\mu}}(z, \eta(s), t) = q_{\mu_0 \mu_1} h_{d^0 \boldsymbol{\mu}}(z,s,t)$. Similarly, for $j > 0$, $h_{\boldsymbol{\mu}}(z,s,d_j t) = h_{d^j \boldsymbol{\mu}}(z,s,t)$. Furthermore, $h_{\boldsymbol{\mu}}(z,s,s_j t) = \widehat{h}_{\boldsymbol{\mu}}(z, \eta(s), s_j t) = \widehat{h}_{s^j \boldsymbol{\mu}}(z, \eta(s), t) = h_{s^j \boldsymbol{\mu}}(z,s,t)$. Consequently, the mappings $h_{\boldsymbol{\mu}} \colon Z^1 \times \Delta^2 \times \Delta^n$ form a coherent mapping $\boldsymbol{h} = (h_{\boldsymbol{\mu}}) \colon Z^1 \times \Delta^2 \to \boldsymbol{Y}$. Since

$\widehat{\boldsymbol{h}}$ has the special property, so does $\boldsymbol{h}$, because $h_{\overline{\lambda_0}\ldots\overline{\lambda_n}}(z,s,t) = \widehat{h}_{\overline{\lambda_0}\ldots\overline{\lambda_n}}(z,\eta(s),t) = (f_{\lambda_0\ldots\lambda_n}(z,t), g(z))$.

Finally, the fact that $\widehat{h}_{\boldsymbol{\mu}}$ is an extension of $\widehat{h}^{01}_{\boldsymbol{\mu}}$, $\widehat{h}^{12}_{\boldsymbol{\mu}}$ and $\widehat{h}^{012}_{\boldsymbol{\mu}}$ implies that $h_{\boldsymbol{\mu}}$ is an extension of $h^{01}_{\boldsymbol{\mu}}, h^{12}_{\boldsymbol{\mu}}$ and $h^{012}_{\boldsymbol{\mu}}$ and thus, $\boldsymbol{h}$ is an extension of $\boldsymbol{h}^{01}$, $\boldsymbol{h}^{12}$ and h^{012}. Indeed, for $(z,s,t) \in Z^1 \times [e_0,e_1] \times \Delta^n$, we have $(z,\eta(s),t) \in Z^1 \times [e_{00},e_{10}] \times \Delta^n$, $h_{\boldsymbol{\mu}}(z,s,t) = \widehat{h}_{\boldsymbol{\mu}}(z,\eta(s),t) = \widehat{h}^{01}_{\boldsymbol{\mu}}(z,\eta(s),t) = h^{01}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}\eta(s),t) = h^{01}_{\boldsymbol{\mu}}(z,s,t)$. For $(z,s,t) \in Z^1 \times [e_1,e_2] \times \Delta^n$, we have $(z,\eta(s),t) \in Z^1 \times [e_{10},e_{11}] \times \Delta^n$ and $h_{\boldsymbol{\mu}}(z,s,t) = \widehat{h}_{\boldsymbol{\mu}}(z,\eta(s),t) = \widehat{h}^{12}_{\boldsymbol{\mu}}(z,\eta(s),t) = h^{12}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}\eta(s),t) = h^{12}_{\boldsymbol{\mu}}(z,s,t)$. Finally, for $(z,s,t) \in Z^1 \times [e_1,e_1,e_2] \times \Delta^n$, we have $(z,\eta(s),t) \in Z^1 \times [e_{00},e_{10},e_{11}] \times \Delta^n$ and $h_{\boldsymbol{\mu}}(z,s,t) = \widehat{h}_{\boldsymbol{\mu}}(z,\eta(s),t) = \widehat{h}^{012}_{\boldsymbol{\mu}}(z,\eta(s),t) = h^{012}_{\boldsymbol{\mu}}(z,\varepsilon^{1,1}\eta(s),t) = h^{012}_{\boldsymbol{\mu}}(z,s,t)$. □

9. The constructions (C1)–(C4)

In this section we describe the four constructions (C1)–(C4), which yield coherent mappings satisfying special and appropriate additional conditions. This will complete the proof of Theorem 3.

9.1. *Construction* (C1). This construction is based on Lemma 12. Consider the pair of metric spaces (Z_1, Z_0), where $Z_1 = Z^{\sigma^i} \times \Delta^n$ and $Z_0 = Z^{\sigma^{i-1}} \times \Delta^n$. Define the mappings $h^{01}\colon Z_1 \times [e_0,e_1] \to Y_{\mu_0}$, $h^{12}\colon Z_1 \times [e_1,e_2] \to Y_{\mu_0}$ and $h^{012}\colon Z_0 \times [e_0,e_1,e_2] \to Y_{\mu_0}$, by putting

$$h^{01}(z,d_2s,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^i},\ s \in \Delta^1, \tag{155}$$

$$h^{12}(z,d_0s,t) = \overline{h}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^i},\ s \in \Delta^1, \tag{156}$$

$$h^{012}(z,s,t) = H^{\sigma^{i-1}\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^{i-1}},\ s \in \Delta^2. \tag{157}$$

Let us verify conditions (152), (153) and (154). Since $d_2e_1 = e_1 = d_0e_0$, we see that, for $z \in Z^{\sigma^i}$, $h^{01}(z,e_1,t) = h^{01}(z,d_2e_1,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{h}^{\sigma^i}_{\boldsymbol{\mu}}(z,t)$. However, one also has $h^{12}(z,e_1,t) = h^{12}(z,d_0e_0,t) = \overline{h}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,e_0,t) = \overline{h}^{\sigma^i}_{\boldsymbol{\mu}}(z,t)$. Furthermore, for $z \in Z^{\sigma^{i-1}}$, $s \in \Delta^1$, by (46), $h^{012}(z,d_2s,t) = H^{\sigma^{i-1}\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_2s,t) = H^{\sigma^{i-1}\sigma^i}_{\boldsymbol{\mu}}(z,s,t)$ and also $h^{01}(z,d_2s,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = H^{\sigma^{i-1}\sigma^i}_{\boldsymbol{\mu}}(z,s,t)$, because of (35). Finally, by (46), for $z \in Z^{\sigma^{i-1}}$, $s \in \Delta^1$, $h^{012}(z,d_0s,t) = H^{\sigma^{i-1}\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,d_0s,t) = \overline{h}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t)$ and also $h^{12}(z,d_0s,t) = \overline{h}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,s,t)$.

This enables us to apply Lemma 12 and conclude that there exists a coherent mapping $\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+1}}$, which has the special property, and consists of mappings $\overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}\colon Z^{\sigma^i} \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$, which extend mappings h^{01}, h^{12} and h^{012}. Consequently, additional conditions (41), for $j = 0,2$, and (42) are satisfied. For $j = 1$, (41) holds by 4.5.2.

9.2. *Construction* (C2). This construction is also based on Lemma 12. Let us first show that

$$H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = \overline{h}^{\sigma^0\sigma^i}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^0}. \tag{158}$$

Indeed, by (35), $H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = H^{\sigma^{i-1}\sigma^i}_{\boldsymbol{\mu}}(z,s,t)$, for $z \in Z^{\sigma^{i-1}}$. Moreover, by (40) for $i-1$, $H^{\sigma^{i-1}\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = H^{\sigma^{i-2}\sigma^i}_{\boldsymbol{\mu}}(z,s,t)$, for $z \in Z^{\sigma^{i-2}}$ and by (45) for $i-2$, $H^{\sigma^{i-2}\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = \overline{h}^{\sigma^0\sigma^i}_{\boldsymbol{\mu}}$, for $z \in Z^{\sigma^0}$. Since $Z^{\sigma^0} \subseteq Z^{\sigma^{i-2}} \subseteq Z^{\sigma^{i-1}}$, (158) follows.

Now consider the pair of metric spaces (Z_1, Z_0), where $Z_1 = Z^{\sigma^i} \times \Delta^n$ and $Z_0 = Z^{\sigma^0} \times \Delta^n$. Define the mappings $h^{01}\colon Z_1 \times [e_0,e_1] \to Y_{\mu_0}$, $h^{12}\colon Z_1 \times [e_1,e_2] \to Y_{\mu_0}$ and $h^{012}\colon Z_0 \times [e_0,e_1,e_2] \to Y_{\mu_0}$, by putting

$$h^{01}(z,d_2s,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^i},\ s \in \Delta^1, \tag{159}$$

$$h^{12}(z,d_0s,t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^i},\ s \in \Delta^1 \tag{160}$$

$$h^{012}(z,s,t) = \overline{h}^{\sigma^0\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t), \quad z \in Z^{\sigma^0},\ s \in \Delta^2. \tag{161}$$

Let us verify conditions (152), (153) and (154). Since $d_2e_1 = e_1 = d_0e_0$, we see that, for $z \in Z^{\sigma^i}$, (34) implies $h^{01}(z,e_1,t) = h^{01}(z,d_2e_1,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,e_1,t) = \overline{h}^{\sigma^i}_{\boldsymbol{\mu}}(z,t)$. Also $h^{12}(z,e_1,t) = h^{12}(z,d_0e_0,t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,e_0,t) = \overline{h}^{\sigma^i}_{\boldsymbol{\mu}}(z,t)$, which establishes (152). Furthermore, by (158), for $z \in Z^{\sigma^0}$, $h^{01}(z,d_2s,t) = H^{\sigma^i}_{\boldsymbol{\mu}}(z,s,t) = \overline{h}^{\sigma^0\sigma^i}_{\boldsymbol{\mu}}(z,s,t)$ and by (31), $h^{012}(z,d_2s,t) = \overline{h}^{\sigma^0\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_2s,t) = \overline{h}^{\sigma^0\sigma^i}_{\boldsymbol{\mu}}(z,s,t)$, which establishes (153). Finally, for $z \in Z^{\sigma^0}$, $h^{012}(z,d_0s,t) = \overline{h}^{\sigma^0\sigma^i\sigma^{i+2}}(z,d_0s,t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t)$ and also $h^{12}(z,d_0s,t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t)$, which establishes (154).

This enables us to apply Lemma 12 and conclude that there exists a coherent mapping $\overline{\boldsymbol{H}}^{\sigma^i\sigma^{i+2}}$, which has the special property, and consists of mappings $\overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}\colon Z^{\sigma^i} \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$, which extend the mappings h^{01}, h^{12} and h^{012}. Consequently, the additional conditions (36), for $j = 0, 2$, and (37) are satisfied. For $j = 1$, (36) holds by 4.5.3.

9.3. *Construction* (C3). Denote by b the barycenter of the standard 2-simplex $\Delta^2 = [e_0,e_1,e_2]$. Consider three 2-simplices $\Delta^2_0 = [b,e_1,e_2]$, $\Delta^2_1 = [b,e_0,e_2]$ and $\Delta^2_2 = [b,e_0,e_1]$. Clearly, these 2-simplices and their faces form a triangulation of Δ^2. Consider the simplicial mappings $\alpha_k\colon \Delta^2_k \to \Delta^2$, $k = 0,1,2$, where α_0 maps b,e_1,e_2; α_1 maps e_0,b,e_2 and α_2 maps e_0,b,e_1 to e_0,e_1,e_2, respectively.

We define the mapping $H^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}}\colon Z^{\sigma^i} \times \Delta^2 \times \Delta^n \to Y_{\mu_0}$ by putting

$$H^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s,t) = \begin{cases} \overline{h}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}}(z,\alpha_0(s),t), & s \in \Delta^2_0, \\ \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,\alpha_1(s),t), & s \in \Delta^2_1, \\ \overline{H}^{\sigma^i\sigma^{i+1}}_{\boldsymbol{\mu}}(z,\alpha_2(s),t), & s \in \Delta^2_2. \end{cases} \tag{162}$$

Let us first verify that the mapping $H^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}}$ is well defined by (162). If $s \in \Delta^2_0 \cap \Delta^2_1 = [b,e_2]$, then s is of the form $s = (1-u)b + ue_2$, where $0 \le u \le 1$. Consider the point $s' = (1-u)e_0 + ue_1 \in \Delta^1$ and note that $\alpha_0(s) = (1-u)e_0 + ue_2 = d_1s'$ and $\alpha_1(s) = (1-u)e_1 + ue_2 = d_0s'$. Therefore, by (31), $\overline{h}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}}(z,\alpha_0(s),t) = \overline{h}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_1s',t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s',t)$. However, by (41), we also have $\overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,\alpha_1(s),t) = \overline{H}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,d_0s',t) = \overline{h}^{\sigma^i\sigma^{i+2}}_{\boldsymbol{\mu}}(z,s',t)$.

If $s \in \Delta_0^2 \cap \Delta_2^2 = [b, e_1]$, then s is of the form $s = (1-u)b + ue_1$, where $0 \le u \le 1$. Consider the point $s' = (1-u)e_0 + ue_1 \in \Delta^1$ and note that $\alpha_0(s) = (1-u)e_0 + ue_1 = d_2 s'$ and $\alpha_2(s) = (1-u)e_1 + ue_2 = d_0 s'$. Therefore, by (31), $\overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, \alpha_0(s), t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_2 s', t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, s', t)$. However, by (36), we also have $\overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, \alpha_2(s), t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, d_0 s', t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, s', t)$.

Finally, consider the case when $s \in \Delta_1^2 \cap \Delta_2^2 = [e_0, b]$. Then s is of the form $s = (1-u)e_0 + ub$, where $0 \le u \le 1$. Let $s' = (1-u)e_0 + ue_1 \in \Delta^1$ and note that $\alpha_1(s) = (1-u)e_0 + ue_1 = d_2 s'$ and $\alpha_2(s) = (1-u)e_0 + ue_1 = d_2 s'$. Therefore, by (41), $\overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z, \alpha_1(s), t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z, d_2 s', t) = H_{\boldsymbol{\mu}}^{\sigma^i}(z, s', t)$. However, by (36), we also have $\overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, \alpha_2(s), t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, d_2 s', t) = H_{\boldsymbol{\mu}}^{\sigma^i}(z, s', t)$.

It remains to verify (46). Indeed, if $s \in \Delta^1$, then $d_0 s \in [e_1, e_2] \subseteq \Delta_0^2$ and therefore, by (162), $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_0 s, t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, \alpha_0 d_0 s, t)$. Since $\alpha_0|[e_1, e_2]$ is the identity mapping, we see that $\alpha_0 d_0 s = d_0 s$. Consequently, by (31), $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_0 s, t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_0 s, t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+1}\sigma^{i+2}}(z, s, t)$. Furthermore, $d_1 s \in [e_0, e_2] \subseteq \Delta_1^2$ and therefore, by (162), $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_1 s, t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z, \alpha_1 d_1 s, t)$. However, $\alpha_1 d_1 s = d_1 s$ and thus, by (41), for $j = 1$, $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_1 s, t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z, d_1 s, t) = H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+2}}(z, s, t)$. Finally, $d_2 s \in [e_0, e_1] \subseteq \Delta_2^2$ and therefore, $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_2 s, t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, \alpha_2 d_2 s, t)$. If $s = (1-u)e_0 + ue_1$, then $d_2 s = (1-u)e_0 + ue_1$ and $\alpha_2 d_2 s = (1-u)e_0 + ue_2 = d_1 s$. Consequently, by (36), for $j = 1$, $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}\sigma^{i+2}}(z, d_2 s, t) = \overline{H}_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, d_1 s, t) = H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}(z, s, t)$. This completes the proof of (46).

9.4. *Construction* (C4).

We will first define $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}$ on $Z^{\partial\sigma^{i+1}} \times \Delta^1 \times \Delta^n$, where by definition, $Z^{\partial\sigma^{i+1}} = g^{-1}(\partial\sigma^{i+1}) \subseteq g^{-1}(\sigma^{i+1}) = Z^{\sigma^{i+1}}$. If $\sigma_0^i, \dots, \sigma_i^i$ are all i-faces of σ^{i+1}, then $\partial\sigma^{i+1} = \sigma_0^i \cup \ldots \cup \sigma_i^i$ and $Z^{\partial\sigma^{i+1}} = Z^{\sigma_0^i} \cup \ldots \cup Z^{\sigma_i^i}$. Therefore, it suffices to define mappings $H_{l\boldsymbol{\mu}}^{\sigma^{i+1}}$ on $Z^{\sigma_l^i} \times \Delta^1 \times \Delta^n$, $l = 0, \dots, i$, such that any two of these mappings coincide at the intersection of their domains. Then, putting

$$H_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, s, t) = H_{l\boldsymbol{\mu}}^{\sigma^{i+1}}(z, s, t), \quad z \in Z^{\sigma_l^i}, \tag{163}$$

we obtain a well-defined mapping $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}$ on $Z^{\partial\sigma^{i+1}} \times \Delta^1 \times \Delta^n$.

We define $H_{l\boldsymbol{\mu}}^{\sigma^{i+1}}$ by the formula

$$H_{l\boldsymbol{\mu}}^{\sigma^{i+1}}(z, s, t) = H_{\boldsymbol{\mu}}^{\sigma_l^i\sigma^{i+1}}(z, s, t), \quad z \in Z^{\sigma_l^i}. \tag{164}$$

If $l \ne l'$, then $\sigma_l^i \cap \sigma_{l'}^i$ is an $(i-1)$-face σ^{i-1} of both i-simplices $\sigma_l^i, \sigma_{l'}^i$. Therefore, by (40), for $z \in Z^{\sigma_l^i} \cap Z^{\sigma_{l'}^i} = Z^{\sigma^{i-1}}$, one has $H_{\boldsymbol{\mu}}^{\sigma_l^i\sigma^{i+1}}(z, s, t) = H_{\boldsymbol{\mu}}^{\sigma^{i-1}\sigma^{i+1}}(z, s, t) = H_{\boldsymbol{\mu}}^{\sigma_{l'}^i\sigma^{i+1}}(z, s, t)$. Note that in the sequence (47), $H_{\boldsymbol{\mu}}^{\sigma^{i-1}\sigma^{i+1}}$ precedes $H_{\boldsymbol{\mu}}^{\sigma^i\sigma^{i+1}}$.

Let us now show that the mapping $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}\colon Z^{\partial\sigma^{i+1}} \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$ extends to $Z^{\sigma^{i+1}} \times e_1 \times \Delta^n$, by putting

$$H_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, e_1, t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, t), \quad z \in Z^{\sigma^{i+1}}. \tag{165}$$

Indeed, if $z \in Z^{\sigma_l^i}$, by (35), $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, e_1, t) = H_{l\boldsymbol{\mu}}^{\sigma^{i+1}}(z, e_1, t) = H_{\boldsymbol{\mu}}^{\sigma_l^i \sigma^{i+1}}(z, e_1, t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, t)$. This enables us to apply (HEP) and obtain a further extension of $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}$ to the desired mapping $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}\colon Z^{\sigma^{i+1}} \times \Delta^1 \times \Delta^n \to Y_{\mu_0}$. Clearly, $H_{\boldsymbol{\mu}}^{\sigma^{i+1}}$ has properties (34) and (35) (for $i+1$), i.e.,

$$H_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, e_1, t) = \overline{h}_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, t), \quad z \in Z^{\sigma^{i+1}}, \tag{166}$$

$$H_{\boldsymbol{\mu}}^{\sigma^{i+1}}(z, s, t) = H_{\boldsymbol{\mu}}^{\sigma^i \sigma^{i+1}}(z, s, t), \quad z \in Z^{\sigma^i}. \tag{167}$$

Note that (33) holds because of 4.5.1.

10. The case when Z is a CW-complex

10.1. An easy consequence of Theorem 1 is the following corollary.

Corollary 1. *Let X be a compact Hausdorff space and let P be a polyhedron. Then the existence condition* (ESS)$_Z$ *holds for every* CW*-complex Z.*

We will first prove the following lemma.

Lemma 13. *Let X be a compact Hausdorff space and let P be a polyhedron. If Z, Z' are spaces such that Z is strong shape dominated by Z', then* (ESS)$_{Z'}$ *implies* (ESS)$_Z$*.*

Proof. Let $F\colon Z \to X$ be a strong shape morphism and let $[g]\colon Z \to P$ be a homotopy class of mappings. We must produce a strong shape morphism $H\colon Z \to X \times P$ such that $\overline{S}[\pi_X]H = F$ and $\overline{S}[\pi_P]H = \overline{S}[g]$. By assumption, there are strong shape morphisms $\Phi\colon Z \to Z'$ and $\Psi\colon Z' \to Z$ such that $\Psi\Phi = 1_Z$. Consider the strong shape morphism $F' = F\Psi\colon Z' \to X$ and note that the strong shape morphism $\overline{S}[g]\Psi\colon Z' \to P$ is of the form $\overline{S}[g']\colon Z' \to P$, where $[g']\colon Z' \to P$ is a homotopy class of mappings. This is so because P is a polyhedron. By (ESS)$_{Z'}$, there is a strong shape morphism $H'\colon Z' \to X \times P$ such that $\overline{S}[\pi_X]H' = F'$ and $\overline{S}[\pi_P]H' = \overline{S}[g']$. Now put $H = H'\Phi\colon Z \to X \times P$. Then, $\overline{S}[\pi_X]H = \overline{S}[\pi_X]H'\Phi = F'\Phi = F\Psi\Phi = F$. Similarly, $\overline{S}[\pi_P]H = \overline{S}[\pi_P]H'\Phi = \overline{S}[g']\Phi = \overline{S}[g]\Psi\Phi = \overline{S}[g]$ and we see that (ESS)$_Z$ holds. □

Proof of Corollary 1. It is well known that every CW-complex has the homotopy type of an ANR for metric spaces. All the more, every CW-complex is strong shape dominated by an ANR. Since ANRs are metrizable spaces, the statement follows from Theorem 1 and Lemma 13. □

Remark 2. *If the standard resolution $\boldsymbol{Y}$ of $X \times P$ consists of spaces Y_μ which are polyhedra or CW-complexes, the assertion of Theorems 1 and 3 can be strengthened by allowing the spaces Z to be stratifiable. The only change in the proof is that, instead of using the fact that the spaces Y_μ are ANEs for metrizable spaces (Lemma 2), one uses the fact that polyhedra and CW-complexes are ANEs for stratifiable spaces (see 2.2). Consequently, as in Lemma 2, every pair (Z, B), which consists of a stratifiable space Z and a closed subset $B \subseteq Z$, has the homotopy extension property with respect to every polyhedron and every CW-complex.*

11. The case when X is a metric compactum

11.1. We will first prove the following lemma.

Lemma 14. *Let $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ be an inverse sequence of metric compacta and let P be a polyhedron. If Z is a topological space, $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ is a coherent mapping and $g\colon Z \to P$ is a mapping, then there exist a metrizable space Z', a mapping $u\colon Z \to Z'$, a coherent mapping $\boldsymbol{f}'\colon Z' \to \boldsymbol{X}$ and a mapping $g'\colon Z' \to P$ such that*

$$\boldsymbol{f}'C(u) \simeq \boldsymbol{f}, \quad g'u \simeq g. \tag{168}$$

Proof. In 4.2 of [15], with every cofinite inverse system of compact Hausdorff spaces $\boldsymbol{X}$ was associated a space $T(\boldsymbol{X})$, called the the cotelescope of $\boldsymbol{X}$, and a coherent mapping $\boldsymbol{\tau}_X\colon T(\boldsymbol{X}) \to \boldsymbol{X}$ such that, whenever $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ is a coherent mapping from a space Z, then there exists a mapping $v\colon Z \to T(\boldsymbol{X})$, unique up to homotopy, such that $\boldsymbol{f} \simeq \boldsymbol{\tau}_X C(v)$ (Lemma 4.17 in [15]). According to its construction, $T(\boldsymbol{X})$ is a subset of the product $\prod_{\boldsymbol{\lambda}} (X_{\lambda_0})^{\Delta^n}$, where $\boldsymbol{\lambda}$ ranges over all multiindices $(\lambda_0, \dots, \lambda_n)$ in Λ. In our case, $\Lambda = \mathbb{N}$ and therefore there are $\aleph_0$ factors in that product. The factors $(X_{i_0})^{\Delta^n}$ are spaces of singular n-simplices in X_{i_0}, endowed with the compact open topology. Since Δ^n is compact and X_{i_0} is metrizable, the mapping space $(X_{i_0})^{\Delta^n}$ is metrizable. Therefore, the whole product, hence also $T(\boldsymbol{X})$, is a metrizable space.

Mappings $v\colon Z \to T(\boldsymbol{X})$ and $g\colon Z \to P$ induce a mapping $w\colon Z \to T(\boldsymbol{X}) \times P$ such that $\pi' w = v$ and $\pi'' w = g$, where π', π'' are canonical projections of $T(\boldsymbol{X}) \times P$. Denote by K a triangulation of P and let P_m be the carrier $|K| = P$, endowed with the metric topology. It is well known that P (with the CW-topology) and P_m have the same homotopy type. Therefore, there exist mappings $k\colon P \to P_m$ and $k'\colon P_m \to P$ such that $kk' \simeq \mathrm{id}$ and $k'k \simeq \mathrm{id}$. Now put $Z' = T(\boldsymbol{X}) \times P_m$, $u = (1_{T(\boldsymbol{X})} \times k)w$ and note that Z' is a metrizable space. Moreover, put $\boldsymbol{f}' = \boldsymbol{\tau}_X C(\pi'(1_{T(\boldsymbol{X})} \times k'))$ and $g' = \pi''(1_{T(\boldsymbol{X})} \times k')$. Note that $[\boldsymbol{f}'] = [\boldsymbol{\tau}_X][C(\pi')][C(1_{T(\boldsymbol{X})} \times k')]$ and $[C(u)] = [C(1_{T(\boldsymbol{X})} \times k)][C(w)]$ and therefore, $[\boldsymbol{f}'C(u)] = [\boldsymbol{f}'][C(u)] = [\boldsymbol{\tau}_X][C(\pi')][C(1_{T(\boldsymbol{X})} \times k')][C(1_{T(\boldsymbol{X})} \times k)][C(w)] = [\boldsymbol{\tau}_X][C(\pi')][C(1_{T(\boldsymbol{X})} \times k'k)][C(w)] = [\boldsymbol{\tau}_X][C(\pi')][C(w)] = [\boldsymbol{\tau}_X][C(\pi' w)] = [\boldsymbol{\tau}_X][C(v)] = [\boldsymbol{\tau}_X C(v)] = [\boldsymbol{f}]$, i.e., $\boldsymbol{f}'C(u) \simeq f$. In this argument we used the property of the operator C that $C(\boldsymbol{kh}) = C(\boldsymbol{k})C(\boldsymbol{h})$ (Lemma 1.17 in [15]) and we used homotopy classes of coherent mappings and the associativity law because conditions of Lemma 3 are fulfilled (for all coherent mappings involved either the domain is rudimentary or the codomain is cofinite). Finally, $g'u = \pi''(1_{T(\boldsymbol{X})} \times k')(1_{T(\boldsymbol{X})} \times k)w = \pi''(1_{T(\boldsymbol{X})} \times k'k)w \simeq \pi'' w = g$. □

11.2. Proof of Theorem 2. Let X be a compact metric space and let P be a polyhedron. There exist an inverse sequence of compact polyhedra $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ and an inverse limit $\boldsymbol{p} = (p_i)\colon X \to \boldsymbol{X}$. Moreover, let K be a triangulation of P. Note that K is countable. By Proposition 2, to prove that for every topological space Z condition $(\mathrm{ESS})_Z$ for X, P holds, it suffices to prove that condition $(\mathrm{ECH})_Z$ for $\boldsymbol{X}$, K holds.

Let $\boldsymbol{f}\colon Z \to \boldsymbol{X}$ be a coherent mapping and $g\colon Z \to P$ a mapping. By Lemma 14, there exists a metrizable space Z', a mapping $u\colon Z \to Z'$, a coherent mapping $\boldsymbol{f}'\colon Z' \to \boldsymbol{X}$ and a mapping $g'\colon Z' \to P$ such that $\boldsymbol{f}'C(u) \simeq \boldsymbol{f}$ and $g'u \simeq g$. By Theorem 3, there exists a coherent mapping $\boldsymbol{h}'\colon Z' \to \boldsymbol{Y}$ such that $C(\boldsymbol{\pi}_X)\boldsymbol{h}' \simeq \boldsymbol{f}'$ and $C(\boldsymbol{\pi}_P)\boldsymbol{h}' \simeq C(g')$. Now define a coherent mapping $\boldsymbol{h}\colon Z \to \boldsymbol{Y}$, by putting $\boldsymbol{h} = \boldsymbol{h}'C(u)$. Clearly, $\boldsymbol{\pi}_X\boldsymbol{h} \simeq \boldsymbol{f}$ and $\boldsymbol{\pi}_P\boldsymbol{h} \simeq C(g)$. □

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