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# On proximate order and type function of Laplace-Stieltjes transformations convergent in the right half-plane

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**Abstract.** The proximate order and type-function for analytic functions of finite order represented by Laplace-Stieltjes transformations $F(s)$ convergent only in the right half-plane is introduced and the growth of such functions is investigated and two necessary and sufficient conditions are obtained.
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## 1. Introduction and main results

Consider Laplace-Stieltjes transformations

$$F(s) = \int_0^{+\infty} e^{-sx} d\alpha(x), \qquad s = \sigma + it, \tag{1}$$

where $\alpha(x)$ is a bounded variation on any interval $[0, X]$, $0 < X < +\infty$, and $\sigma$ and $t$ are real variables. We choose a sequence $\{\lambda_n\}_{n=1}^{\infty}$:

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow +\infty, \tag{2}$$

which satisfies the following conditions:

$$\limsup_{n\to+\infty}(\lambda_{n+1} - \lambda_n) < +\infty, \quad \limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = 0, \tag{3}$$

$$\limsup_{n\to+\infty} \frac{\log A_n^*}{\lambda_n} = 0, \tag{4}$$

where

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$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{-ity} d\alpha(y) \right|.$$

In 1963, Yu J.-R. [13] obtained Valiron-Knopp-Bohr formula:

**Theorem 1.** *Suppose that Laplace-Stieltjes transformations (1) satisfy the first formula of (3) and* $\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} < +\infty$*, then*

$$\limsup_{n\to+\infty} \frac{\log A_n^*}{\lambda_n} \leq \sigma_u^F \leq \limsup_{n\to+\infty} \frac{\log A_n^*}{\lambda_n} + \limsup_{n\to+\infty} \frac{\log n}{\lambda_n},$$

*where* $\sigma_u^F$ *is called the abscissa of uniformly convergent of* $F(s)$*.*

By (3), (4) and Theorem 1, we can get $\sigma_u^F = 0$, *i.e.*, $F(s)$ is analytic in the right half-plane. Set

$$M_u(\sigma, F) = \sup_{0 < x < +\infty, -\infty < t < +\infty} \left| \int_0^x e^{-(\sigma+it)y} d\alpha(y) \right|, \quad \sigma > 0$$
$$\mu(\sigma, F) = \max_{n\in N}\{A_n^* e^{-\lambda_n \sigma}\}, \quad \sigma > 0.$$

Dirichlet series was regarded as a special example of Laplace-Stieltjes transformation. Some problems on the growth and the value distribution of analytic functions defined by Dirichlet series have been studied for a long time and lots of important results were obtained in [2, 7, 9, 12]. In 1963, Yu [13] extented the results of [3, 14] and established the Valiron-Knopp-Bohr formulas of the associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace-Stieltjes transformations. Moreover, he first introduced $M_u(\sigma, F), \mu(\sigma, F)$ and the Borel line and the order of analytic functions represented by Laplace-Stieltjes transformations convergent in the complex plane.

Many problems of analytic functions defined by Laplace-Stieltjes transformations have been studied and some important results have been obtained in [1, 8]. Recently, many mathematicians (such as Sun D.C., Gao Z.S., Kong Y.Y., Shang L.N. and others) are very interested in investigating the functions represented by Laplace-Stieltjes transformation convergent in the half-plane or the whole complex plane in the field of complex analysis (see [4 - 6, 10, 11]). Kong Y.Y. and Sun D.C. investigated some problems of analytic functions represented by Laplace-Stieltjes transformations convergent in the half-plane, such as the exponential order and the exponential low order of zero order Laplace-Stieltjes transformations, type-function and proximate order of finite order Laplace-Stieltjes transformation, and their relative transformations, and obtained some interesting results (see [4 - 6]), Shang L.N. and Gao Z.S. investigated the growth of the infinite order entire function represented by Laplace-Stieltjes transformations and the value distribution of finite order and infinite order analytic functions represented by Laplace-Stieltjes transformations convergent in the right half-plane and obtained some theorems (see [10, 11]).

From those results given by these mathematicians, we can see that the results related to the conditions which the sequence $\{\lambda_n\}$ and the growth $\rho$ of Laplace-Stieltjes transformations satisfied.

Therefore, it is natural to ask: *what will happen if we alter the conditions which the sequence* $\{\lambda_n\}$ *and the growth* $\rho$ *of Laplace-Stieltjes transformation satisfied.*

The above problem is investigated and some theorems about the relation between the proximate order and type function and $A_n^*$ of Laplace-Stieltjes transformation are obtained in this paper. To state the results of this paper, we explain some definitions and notations as follows.

The following definition (see [4])

$$\rho = \limsup_{\sigma\to 0^+} \frac{\log^+ \log^+ M_u(\sigma, F)}{\log \frac{1}{\sigma}}, \quad \sigma > 0,$$

is called $\rho$ order of $F(s)$ in $Res = \sigma > 0$, where $\log^+ C = \max\{\log C, 0\}$. If $\rho \in (0, +\infty)$, we say that $F(s)$ is an analytic function of finite order in the right half-plane.

We introduce a proximate order in the case $\rho \in (0, +\infty)$ as follows.

Let $\rho(r)\,(r > r_0)$ be a non-negative, continuous, monotonous function and let it have a left-hand derivative and a right-hand derivative in every $r\,(> r_0)$, such that

$$\lim_{r\to+\infty} \rho(r) = \rho, \qquad \lim_{r\to+\infty} \rho'(r) r \log r = 0, \tag{5}$$

and set $U(r) = r^{\rho(r)}$, which is a strictly increasing function of $r$ in $r \geq r_0' > r_0$. Let

$$t = rU(r), \qquad r = W(t), \quad r > 0,\ t > 0, \tag{6}$$

be two reciprocally inverse functions. From [16], for any positive real number $k$, we have

$$\lim_{r\to+\infty} \frac{U(kr)}{U(r)} = k^{\rho}, \qquad \lim_{t\to+\infty} \frac{W(kt)}{W(t)} = k^{\frac{1}{\rho+1}}. \tag{7}$$

For Laplace-Stieltjes transformation (1), if

$$\limsup_{\sigma\to 0^+} \frac{\log^+ M_u(\sigma, F)}{U(\frac{1}{\sigma})} = 1, \tag{8}$$

$\rho(\frac{1}{\sigma})$ is called the Proximate order of (1) and $U(\frac{1}{\sigma})$ the type function of (1) in $Res = \sigma > 0$.

The results of this paper are stated as follows:

**Theorem 2.** *Suppose that Laplace-Stieltjes transformations (1) of finite order* $\rho(0 < \rho < \infty)$ *satisfy (2), (3), (4) and*

$$\limsup_{n\to+\infty} \frac{\log\log n}{\log \lambda_n} < \frac{\rho}{1+\rho}. \tag{9}$$

*Then*

$$\limsup_{\sigma\to 0^+} \frac{\log^+ M_u(\sigma, F)}{U(\frac{1}{\sigma})} = 1 \Longleftrightarrow \limsup_{n\to+\infty} \frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} = 1, \tag{10}$$

*where* $B = (1+\rho)^{1+\rho}\rho^{-\rho}$ *and* $U(r)$ *are defined by (8).*

**Theorem 3.** *Suppose that Laplace-Stieltjes transformations (1) of finite order $\rho(0 < \rho < \infty)$ satisfy (2), (3), (4) and (9), then*

$$\lim_{\sigma\to 0^+}\frac{\log^+ M_u(\sigma,F)}{U(\frac{1}{\sigma})}=1 \Longleftrightarrow (i) \quad \limsup_{n\to+\infty}\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)}=1;$$

*(ii) There exists a non-decreasing positive integer sequence $\{n_\nu\}$ satisfying*

$$\lim_{\nu\to+\infty}\frac{\log^+ A_{n_\nu}^*}{BU\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}\right)}=1, \qquad \lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1, \tag{11}$$

*where $B$ and $U(r)$ are stated in Theorem 2.*

## 2. Some Lemmas

**Lemma 1** (See [15, 16]). *Let $\alpha$ and $\lambda$ be any positive real numbers, then*

$$\varphi(\sigma)=\alpha U\left(\frac{1}{\sigma}\right)+\lambda\sigma, \quad \sigma>0,$$

*obtain the minimum*

$$\alpha^{\frac{1}{\rho+1}}\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\frac{\lambda}{W(\lambda)}(1+o(1)), \quad \lambda\to+\infty \qquad \text{in} \qquad \sigma=\frac{(\alpha\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1)), \quad \lambda\to+\infty.$$

For the convenience of the reader, we give the process of proof of this lemma as follows.

**Proof**. For the definition of $U\left(\frac{1}{\sigma}\right)$, we have

$$\varphi'(\sigma)=-\alpha U'\left(\frac{1}{\sigma}\right)\cdot\frac{1}{\sigma^2}+\lambda.$$

Then we can get

$$\begin{aligned}\lambda &= \frac{\alpha}{\sigma}U\left(\frac{1}{\sigma}\right)\left[\rho\left(\frac{1}{\sigma}\right)+\rho'\left(\frac{1}{\sigma}\right)\cdot\frac{1}{\sigma}\log\frac{1}{\sigma}\right]\\ &= \frac{\alpha\rho}{\sigma}U\left(\frac{1}{\sigma}\right)(1+o(1)), \quad \lambda\to+\infty,\end{aligned}$$

as $\varphi'(\sigma)=0$.

With the value of $\sigma$ increasd through the above given value, the value of $\varphi'(\sigma)$ changes from a negative one to a positive one. Then, from (6), (7) and the definition of $\varphi(\sigma)$, we can get that $\varphi(\sigma)$ obtains the minimum when

$$\sigma=\frac{(\alpha\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1)), \quad \lambda\to+\infty,$$

and the minimum is

$$\alpha U\left(\frac{W(\lambda)}{(\alpha\rho)^{\frac{1}{\rho+1}}(1+o(1))}\right)+\lambda\frac{(\alpha\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1))$$
$$=\frac{1}{W(\lambda)}\left[\frac{\alpha W(\lambda)U(W(\lambda))}{(\alpha\rho)^{\frac{\rho}{\rho+1}}(1+o(1))}+\lambda(\alpha\rho)^{\frac{1}{\rho+1}}(1+o(1))\right]$$

$$=\frac{\lambda}{W(\lambda)}\left[\frac{\alpha}{(\alpha\rho)^{\frac{\rho}{\rho+1}}(1+o(1))}+(\alpha\rho)^{\frac{1}{\rho+1}}(1+o(1))\right]$$
$$=\alpha^{\frac{1}{\rho+1}}\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\frac{\lambda}{W(\lambda)}(1+o(1)),\quad \lambda\to+\infty.$$

Thus, we complete the proof of Lemma 1. □

**Lemma 2.** *Let $b$ and $\sigma$ be any positive real number, then*

$$\phi(x)=\frac{x}{W(bx)}-\sigma x,$$

*obtain the maximum*

$$\frac{\rho^\rho}{b(\rho+1)^{\rho+1}}U\left(\frac{1}{\sigma}\right)(1+o(1)),\quad \sigma\to 0^+$$

*in*

$$x=\frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}\frac{1}{\sigma}U\left(\frac{1}{\sigma}\right)(1+o(1)),\quad \sigma\to 0^+.$$

**Proof**. From (6), we can get

$$\frac{dt}{t}=\frac{U(r)+rU'(r)}{U(r)}\frac{dr}{r},\qquad \frac{dr}{r}=\frac{tW'(t)}{W(t)}\frac{dt}{t}.$$

Differentiating $U(r)=r^{\rho(r)}$ and applying (5) and the above two equalities, we can have

$$\frac{tW'(t)}{W(t)}=\frac{U(r)}{U(r)+rU'(r)}=\frac{1}{\rho+1}+o(1),\quad t\to+\infty.$$

By (6), (7) and the above equality, we can have

$$\phi'(x)=\frac{W(bx)-bxW'(bx)}{W^2(bx)}-\sigma$$
$$=\frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}\frac{1}{W(x)}(1+o(1))-\sigma,\quad x\to+\infty.$$

Then we can get that

$$W(x)=\frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}\frac{1}{\sigma}(1+o(1)),\quad x\to+\infty$$

as $\phi'(x) = 0$, *i.e.*,

$$x = \frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}\frac{1}{\sigma}(1+o(1))U\left(\frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}\frac{1}{\sigma}(1+o(1))\right)$$
$$= \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}\frac{1}{\sigma}U\left(\frac{1}{\sigma}\right)(1+o(1)), \quad \sigma \to 0^+,$$

as $\phi'(x) = 0$.

With the value of $x$ increased through the above given value, the value of $\phi'(x)$ changes from a positive one to a negative one. Then, from (6), (7) and the definition of $\phi(x)$, we can get that $\phi(x)$ obtains the maximum when

$$x = \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}\frac{1}{\sigma}U\left(\frac{1}{\sigma}\right)(1+o(1)), \quad \sigma \to 0^+,$$

and the maximum is

$$\frac{\left(\frac{\rho}{\rho+1}\right)^{\rho+1}\frac{1}{\sigma}U\left(\frac{1}{\sigma}\right)(1+o(1))}{bW\left[\left(\frac{\rho}{\rho+1}\right)^{\rho+1}\frac{1}{\sigma}U\left(\frac{1}{\sigma}\right)(1+o(1))\right]} - \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}U\left(\frac{1}{\sigma}\right)(1+o(1))$$
$$= \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho}U\left(\frac{1}{\sigma}\right)(1+o(1)) - \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}U\left(\frac{1}{\sigma}\right)(1+o(1))$$
$$= \frac{1}{b}\frac{\rho^\rho}{(\rho+1)^{\rho+1}}U\left(\frac{1}{\sigma}\right)(1+o(1)).$$

Thus, we complete the proof of this lemma. □

**Lemma 3.** *Let $A > 0$ and $\{\lambda_{n_\nu}\}$ be a strictly increasing sequence tending to $\infty(\nu \to \infty)$ and satisfy $\lambda_{n_1} > Ar_0'U(r_0')$ where $r_0'$ is stated as in Section 1. If $\lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}} = 1$, then there exists a monotone decreasing positive sequence $\{\sigma_\nu\}$ convergent to 0 satisfying*

$$\lambda_{n_\nu} = A\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right), \qquad \lim_{\nu\to\infty}\frac{\frac{1}{\sigma_{\nu+1}}U\left(\frac{1}{\sigma_{\nu+1}}\right)}{\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right)} = 1.$$

**Proof**. Let $t(\sigma) = A\frac{1}{\sigma_v}U\left(\frac{1}{\sigma_v}\right)$, then $t(\sigma)$ is a continuous function as $\sigma > 0$, and is increasing with $\sigma$ reduced as $0 < \sigma < \frac{1}{r_0'}$. Hence for $\lambda_{n_1} > Ar_0'U(r_0') = t(\frac{1}{r_0'}) > 0$, there exists $\sigma_1$ for $0 < \sigma_1 < \frac{1}{r_0'}$ and it satisfies

$$\lambda_{n_1} = t(\sigma_1) = A\frac{1}{\sigma_1}U\left(\frac{1}{\sigma_1}\right).$$

Since $\lambda_{n_2} > \lambda_{n_1}$, there exists $\sigma_2$ for $\sigma_2 < \sigma_1$ and it satisfies

$$\lambda_{n_2} = t(\sigma_2) = A\frac{1}{\sigma_2}U\left(\frac{1}{\sigma_2}\right).$$

$\vdots$

Therefore, we can get a positive, decreasing sequence $\{\sigma_\upsilon\}(\upsilon \to +\infty)$ satisfying

$$\lambda_{n_\upsilon} = t(\sigma_\upsilon) = A\frac{1}{\sigma_\upsilon}U\left(\frac{1}{\sigma_\upsilon}\right).$$

By the definition of $U(\frac{1}{\sigma})$ and $\lambda_{n_\upsilon} \to +\infty(\upsilon \to +\infty)$, we can get $\sigma_\upsilon \to 0(\upsilon \to +\infty)$.

Hence, from the condition of this lemma and $\lambda_{n_\nu} = A\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right)$, we can get

$$\lim_{\nu\to\infty}\frac{\frac{1}{\sigma_{\nu+1}}U\left(\frac{1}{\sigma_{\nu+1}}\right)}{\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right)} = 1.$$

Thus, we complete the proof of this lemma. □

## 3. The proof of Theorem 2

**Proof**. Firstly, we prove the sufficiency of the theorem. For any $\varepsilon > 0$, $\exists N_1 \in N_+ := N \setminus \{0\}$, as $n > N_1$, we have

$$\log^+ A_n^* < (1+\varepsilon)BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right),$$

i.e.,

$$\lambda_n < (1+\varepsilon)B\frac{\lambda_n}{\log^+ A_n^*}U\left(\frac{\lambda_n}{\log^+ A_n^*}\right).$$

Since $r = W(t)$ and $t = rU(r)$ are two reciprocally inverse functions and monotone increasing functions, then we can get

$$W\left(\frac{\lambda_n}{B(1+\varepsilon)}\right) \le \frac{\lambda_n}{\log^+ A_n^*}.$$

Hence we have

$$\log^+ A_n^* \le \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\varepsilon)}\right)}.$$

Thus, there exists a positive constant $D$, such that

$$A_n^* < D\exp\left[\frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\varepsilon)}\right)}\right], \quad n = 0, 1, 2, \cdots. \tag{12}$$

Let

$$I_k(x; it) = \int_{\lambda_k}^{x} e^{-ity}d\alpha(y), \quad \lambda_k \le x \le \lambda_{k+1}$$

for any $t \in R$, then we have

$$|I_k(x; it)| \le A_k^*. \tag{13}$$

Therefore, for any $x : \lambda_k \leq x \leq \lambda_{k+1}, \sigma > 0$, we can get

$$\begin{aligned}\int_0^x e^{-(\sigma+it)y} d\alpha(y) &= \sum_{k=1}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}} e^{-\sigma y} d_y I_k(y; it) + \int_{\lambda_n}^{x} e^{-\sigma y} d_y I_k(y; it) \\ &= \sum_{k=1}^{n-1} \left[ e^{-\lambda_{k+1}\sigma} I_k(\lambda_{k+1}; it) + \sigma \int_{\lambda_k}^{\lambda_{k+1}} e^{-\sigma y} I_k(y; it) dy \right] \\ &\quad + e^{-x\sigma} I_n(x; it) + \sigma \int_{\lambda_n}^{x} e^{-\sigma y} I_n(y; it) dy.\end{aligned} \tag{14}$$

Thus, for any $\sigma > 0$ and any $t \in R$, we have

$$\begin{aligned}\left| \int_0^x e^{-(\sigma+it)y} d\alpha(y) \right| &\leq \sum_{k=1}^{n-1} A_k^* (e^{-\lambda_{k+1}\sigma} + |e^{-\lambda_{k+1}\sigma} - e^{-\lambda_k \sigma})| \\ &\quad + A_n^* (e^{-\sigma x} + |e^{-\sigma x} - e^{-\lambda_n \sigma}|) \leq \sum_{k=1}^{n} A_k^* e^{-\lambda_k \sigma}.\end{aligned} \tag{15}$$

From (12) and (15), we can get

$$\begin{aligned}M_u(\sigma, F) &\leq \sum_{n=0}^{\infty} A_n^* e^{-\lambda_n \sigma} \leq D \sum_{n=0}^{\infty} \exp\left[ \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\varepsilon)}\right)} - \lambda_n \sigma \right] \\ &\leq D \sup_{n \geq 0} \left\{ \exp\left[ \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\varepsilon)}\right)} - \lambda_n (1-\varepsilon)\sigma \right] \right\} \sum_{n=0}^{\infty} e^{-\lambda_n \varepsilon \sigma}.\end{aligned} \tag{16}$$

From (9), there exists $\rho_1 \in (0, \rho)$ such that

$$\limsup_{n\to\infty} \frac{\log\log n}{\log \lambda_n} < \frac{\rho_1}{1+\rho_1}. \tag{17}$$

Thus there exists $N_2 \in N_+$ such that

$$\lambda_n > (\log n)^{\frac{\rho_1+1}{\rho_1}} > 1, \quad n > N_2. \tag{18}$$

Hence we can get

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-\lambda_n \varepsilon \sigma} &\leq N_2 + 1 + \sum_{n=N_2+1}^{+\infty} n^{-\varepsilon\sigma (\log n)^{\frac{\rho_1+1}{\rho_1}}} \\ &\leq N_2 + 1 + \sum_{n=N_2+1}^{T} n^{-\varepsilon\sigma} + \sum_{n=T+1}^{+\infty} n^{-2} \\ &\leq D_1 + \int_{N_2}^{T} \frac{dx}{x^{\varepsilon\sigma}} = D_2 + \frac{1}{1-\varepsilon\sigma} T^{1-\varepsilon\sigma},\end{aligned}$$

where $T = \left[e^{(\frac{2}{\varepsilon\sigma})^{\rho_1}}\right]$ and $D_1, D_2$ are two real constants.

Therefore, by Lemma 2, we have

$$M_u(\sigma, F) \leq D \exp\left[(1+\varepsilon)U(\frac{1}{(1-\varepsilon)\sigma})(1+o(1))\right]\left(D_2 + \frac{1}{1-\varepsilon\sigma}T^{1-\varepsilon\sigma}\right),$$

thus we get

$$\log^+ M_u(\sigma, F) \leq (1+3\varepsilon)U(\frac{1}{\sigma})(1+o(1)). \tag{19}$$

Hence we have

$$\limsup_{\sigma\to 0^+} \frac{\log^+ M_u(\sigma, F)}{U(\frac{1}{\sigma})} \leq 1.$$

Suppose that the above inequality is right, we have

$$\limsup_{\sigma\to 0^+} \frac{\log^+ M_u(\sigma, F)}{U(\frac{1}{\sigma})} = \beta < 1. \tag{20}$$

Set $\varepsilon_1 > 0$ and $\beta + 3\varepsilon_1 < 1$, then there exists $\sigma_0 > 0$

$$\log^+ M_u(\sigma, F) < (\beta + \varepsilon_1)U(\frac{1}{\sigma}). \quad (0 < \sigma < \sigma_0).$$

On the other hand, let

$$I(x; \sigma + it) = \int_0^x e^{-(\sigma+it)y} d\alpha(y).$$

From (3), there exists $K > 0$ satisfying $0 < \lambda_{n+1} - \lambda_n \leq K (n = 1, 2, \cdots)$. For $\sigma(> 0)$ sufficiently reaching 0, it follows $e^{K\sigma} < \frac{3}{2}$.

When $x > \lambda_n$, we have

$$\begin{aligned}\int_{\lambda_n}^x e^{-ity} d\alpha(y) &= \int_{\lambda_n}^x e^{\sigma y} d_y I(y; \sigma + it)\\ &= I(y; \sigma + it)e^{\sigma y}|_{\lambda_n}^x - \sigma \int_{\lambda_n}^x e^{\sigma y} I(y; \sigma + it)dy.\end{aligned}$$

For any $\sigma > 0$ and any $x \in (\lambda_n, \lambda_{n+1}]$, it follows that

$$\begin{aligned}\left|\int_{\lambda_n}^x e^{-ity} d\alpha(y)\right| &\leq M_u(\sigma, F)[|e^{\sigma x} + e^{\sigma\lambda_n}| + |e^{\sigma x} - e^{\sigma\lambda_n}|]\\ &\leq 2M_u(\sigma, F)e^{(\lambda_n+K)\sigma} \leq 3M_u(\sigma, F)e^{\lambda_n\sigma}.\end{aligned} \tag{21}$$

From (21), we have

$$\log^+ A_n^* < (\beta + 2\varepsilon_1)U(\frac{1}{\sigma}) + \lambda_n\sigma. \tag{22}$$

When $n$ is sufficiently large, from Lemma 1, we have

$$\log^+ A_n^* \leq (\beta + 2\varepsilon_1)^{\frac{1}{\rho+1}} \frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}} \frac{\lambda_n}{W(\lambda_n)}(1+\varepsilon_1) = (B(\beta + 2\varepsilon_1))^{\rho+1} \frac{\lambda_n}{W(\lambda_n)}(1+\varepsilon_1),$$

i.e.,

$$W(\lambda_n) \le \frac{\lambda_n}{\log^+ A_n^*}(B(\beta+2\varepsilon_1))^{\rho+1}(1+\varepsilon_1).$$

For $x > x_0 = r_0' U(r_0')$, the function $W(x)$ is monotone increasing, then we have

$$\begin{aligned}\lambda_n &\le \frac{\lambda_n}{\log^+ A_n^*}(B(\beta+2\varepsilon_1))^{\rho+1}(1+\varepsilon_1)U\left(\frac{\lambda_{n+1}}{\log^+ A_n^*}(B(1+\varepsilon))^{\rho+1}(1+\varepsilon)\right)\\ &\le \frac{\lambda_n}{\log^+ A_n^*}(B(\beta+2\varepsilon_1))(1+\varepsilon_1)^{\rho+1}(1+o(1))U\left(\frac{\lambda_n}{\log^+ A_n^*}\right).\end{aligned}$$

Therefore we can get

$$\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} \le (\beta+3\varepsilon_1)^{\rho+2}(1+o(1)).$$

Hence

$$\limsup_{n\to+\infty}\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} \le \beta < 1.$$

Hence we get a contradiction to the condition of the theorem. Thus, sufficiency of the theorem is completed.

The necessity of the theorem can be easily proved similarly to the proof of sufficiency. □

## 4. The proof of Theorem 3

**Proof**. We first prove sufficiency of Theorem 3. From conditions (i),(ii) of Theorem 3, for any $\varepsilon \in (0,1)$ and for sufficiently large $\nu$, we have

$$\log^+ A_{n_\nu}^* > (1-\varepsilon)BU\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}\right),$$

i.e.,

$$\frac{\lambda_{n_\nu}}{(1-\varepsilon)B} > \frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}\right).$$

Since $r = W(t)$ and $t = rU(r)$ are two reciprocally inverse functions and monotone increasing functions, then we can get

$$W\left(\frac{\lambda_{n_\nu}}{(1-\varepsilon)B}\right) > \frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*},$$

i.e.,

$$\log^+ A_{n_\nu}^* > \frac{\lambda_{n_\nu}}{W\left(\frac{\lambda_{n_\nu}}{(1-\varepsilon)B}\right)}.$$

We take a positive real sequence $\{\sigma_\nu\}$ satisfying

$$\lambda_{n_\nu} = \left(\frac{\rho}{\rho+1}\right)^{\rho+1}(1-\varepsilon)B\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right)(1+\varepsilon) = \rho(1-\varepsilon^2)\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right).$$

From Lemma 3, we have $\sigma_\nu \downarrow 0$, then for any sufficiently small $\sigma > 0$, there exists $\nu \in N_+$ such that $\sigma_{\nu+1} \le \sigma \le \sigma_\nu$. By Lemma 2 and Lemma 3, we have

$$\begin{aligned}\log^+ \mu(\sigma, F) &\ge \log^+ A^*_{n_\nu} - \lambda_{n_\nu}\sigma \ge \log^+ A^*_{n_\nu} - \lambda_{n_\nu}\sigma_\nu \\ &\ge \frac{\lambda_{n_\nu}}{W\left(\frac{\lambda_{n_\nu}}{(1-\varepsilon)B}\right)} - \lambda_{n_\nu}\sigma_\nu \\ &= (1-\varepsilon)(1+o(1))U\left(\frac{1}{\sigma_\nu}\right) = (1+o(1))\frac{\sigma_\nu}{\sigma_{\nu+1}}U\left(\frac{1}{\sigma_{\nu+1}}\right) \\ &\ge (1+o(1))U\left(\frac{1}{\sigma_{\nu+1}}\right) \ge (1+o(1))U\left(\frac{1}{\sigma}\right).\end{aligned}$$

From (21), we have $\log^+ M_u(\sigma, F) \ge \log^+ \mu(\sigma, F) + \log\frac{1}{3}$. Then from this and the above inequality, we can get

$$\liminf_{\sigma\to 0}\frac{\log^+ M_u(\sigma, F)}{U(\frac{1}{\sigma})} \ge \liminf_{\sigma\to 0}\frac{\log^+ \mu(\sigma, F) + \log\frac{1}{3}}{U(\frac{1}{\sigma})} \ge 1.$$

Combining Theorem 2, we get

$$\lim_{\sigma\to 0}\frac{\log^+ M_u(\sigma, F)}{U(\frac{1}{\sigma})} = 1.$$

We prove the necessity of Theorem 3 in the following.

If $\lim_{\sigma\to 0^+}\frac{\log^+ M_u(\sigma,F)}{U(\frac{1}{\sigma})} = 1$, by Theorem 2, we can easily get (i) of Theorem 3. Then we will prove (ii) of Theorem 3 in the following. We take a positive decreasing sequence $\{\varepsilon_i\}(0 < \varepsilon_i < 1)$, $\varepsilon_i \to 0(i \to \infty)$. Let

$$E_i = \left\{n : \frac{\log^+ A^*_n}{BU\left(\frac{\lambda_n}{\log^+ A^*_n}\right)} > 1 - \varepsilon_i\right\}, \tag{23}$$

it follows that $\forall i, E_i \ne \Phi$ and $E_i \subset E_{i-1}$. For each $i$, we arrange $n(\in E_i)$ in an increasing sequence $\{n^{(i)}_\nu\}^\infty_{\nu=1}$, then we consider the two cases as follows.

**Case 1.** Suppose that $\lim_{\nu\to+\infty}\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_\nu}} = 1$ for any $i$. Then there exists $N_i \in E_i(i \in N_+)$, when $n^{(i)}_\nu \ge N_i$, we have

$$\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_\nu}} \le 1 + \varepsilon_k. \tag{24}$$

Note $E_{i+1} \subset E_i$, take $N_{i+1} > N_i$, denote by $E_i'$ the subset of $E_i$

$$E_i' = \{n \in E_i : N_i \leq n \leq N_{i+1}\},$$

thus the elements of $E_i'$ satisfy (23) and (24).

Therefore, let $E = \bigcup_{i=1}^{\infty} E_i'$ and arrange $n(\in E_i')$ in an increasing sequence $\{n_\nu\}$, (ii) is proved.

**Case 2.** If there exists $i \in N_+$ satisfying $\lim_{\nu\to+\infty} \frac{\lambda_{n_{\nu+1}^{(i)}}}{\lambda_{n_\nu^{(i)}}} \neq 1$, then since $\lambda_{n_{\nu+1}^{(i)}} > \lambda_{n_\nu^{(i)}}$, we get $\lim_{\nu\to+\infty} \frac{\lambda_{n_{\nu+1}^{(i)}}}{\lambda_{n_\nu^{(i)}}} > 1$. Hence there exists $\{n_{\nu_k}^{(i)}\} \subseteq \{n_\nu^{(i)}\}$ (still marked with $\{n_\nu^{(i)}\}$) and $\delta \in (0, \frac{1}{2}(1+\frac{1}{\rho})^{-\rho})$, and it follows that

$$\frac{\lambda_{n_{\nu+1}^{(i)}}}{\lambda_{n_\nu^{(i)}}} > 1 + \delta. \quad \nu = 1, 2, \cdots.$$

Let

$$\begin{aligned} &n_1' = n_1^{(i)},\ n_2' = n_3^{(i)},\ \cdots,\ n_\nu' = n_{2\nu-1}^{(i)},\ \cdots \\ &n_1'' = n_1^{(i)},\ n_2'' = n_4^{(i)},\ \cdots,\ n_\nu'' = n_{2\nu}^{(i)},\ \cdots, \end{aligned}$$

where $\{n_\nu'\}, \{n_\nu''\}$ are two increasing positive integer sequences, and

$$n_\nu'' < n_{\nu+1}', \quad \lambda_{n_\nu''} > (1+\delta)\lambda_{n_\nu'}, \quad \nu = 1, 2, \cdots.$$

Take $\gamma = \frac{1}{2}\varepsilon_i > 0$ and from (23), for any sufficiently large $\nu$, when $n \notin E_i$ satisfies $n_\nu' < n < n_\nu''$, we can get

$$\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} \leq 1 - \varepsilon_i < 1 - \gamma,$$

thus

$$\log^+ A_n^* < \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1-\gamma)}\right)},$$

i.e.,

$$\log(A_n^* e^{-\lambda_n \sigma}) < \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1-\gamma)}\right)} - \lambda_n \sigma.$$

For $\sigma$ sufficiently reaching $0^+$ and from Lemma 2, it follows that

$$\log(A_n^* e^{-\lambda_n \sigma}) \leq (1-\gamma)(1+o(1))U\left(\frac{1}{\sigma}\right), \quad n_\nu' < n < n_\nu''. \tag{25}$$

Take $\mu > 0$ and

$$\frac{1+\mu}{1+\delta} < 1 - \eta, \quad 0 < \eta < 1.$$

Let $\sigma_\nu = \left[W\left(\frac{\lambda_{n''_\nu}}{B(1+\mu)}\right)\right]^{-1}$, then we have $\sigma_\nu \downarrow 0$ and

$$\lambda_{n''_\nu} = (1+\mu)B\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right). \tag{26}$$

For above $\mu > 0$ and from Theorem 3 (i), there exists a positive integer $n_0 \in N_+$,

$$\log(A_n^* e^{-\lambda_n \sigma}) < \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\mu)}\right)} - \lambda_n \sigma. \quad n \geq n_0. \tag{27}$$

When $n \geq n''_\nu > n_0$, then $\lambda_n \geq \lambda_{n''_\nu}$. Since $W(t)$ is an increasing function, from (26) and (27), we have

$$\log(A_n^* e^{-\lambda_n \sigma_\nu}) < \lambda_n \Big(\frac{1}{W\left(\frac{\lambda_{n''_\nu}}{B(1+\mu)}\right)} - \sigma_\nu\Big) = 0. \tag{28}$$

From Lemma 2 and for sufficiently large $\nu$, when $n_0 \leq n \leq n'_\nu$, it follows that $\lambda_n \leq \lambda_{n'_\nu} < \frac{1}{1+\delta}\lambda_{n''_\nu}$, then we have

$$\begin{aligned}
\log(A_n^* e^{-\lambda_n \sigma_\nu}) &\leq \frac{\frac{1}{1+\delta}\lambda_{n''_\nu}}{W\left(\frac{\frac{1}{1+\delta}\lambda_{n''_\nu}}{B(1+\mu)}\right)} - \frac{1}{1+\delta}\lambda_{n''_\nu}\sigma_\nu \qquad (29)\\
&= \frac{1+\mu}{1+\delta}B\frac{1}{\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right)\left[\frac{1}{W\left(\frac{1}{(1+\delta)\sigma_\nu}U\left(\frac{1}{\sigma_\nu}\right)\right)} - \sigma_\nu\right]\\
&\leq \frac{1-\eta}{1+o(1)}B\left[(1+\delta)^{\frac{1}{1+\rho}} - 1 + o(1)\right]U\left(\frac{1}{\sigma_\nu}\right)\\
&\leq \frac{1-\eta}{1+o(1)}\left[\frac{\delta B}{1+\rho} + o(1)\right]U\left(\frac{1}{\sigma_\nu}\right)\\
&= \frac{1-\eta}{1+o(1)}\left[\delta(1+\frac{1}{\rho})^{\rho} + o(1)\right]U\left(\frac{1}{\sigma_\nu}\right)\\
&\leq (1-\eta)(1+o(1))U\left(\frac{1}{\sigma_\nu}\right),
\end{aligned}$$

when $n \geq n_0$, from (25), (28) and (29), we have

$$\log(A_n^* e^{-\lambda_n \sigma_\nu}) < (1-\beta)(1+o(1))U\left(\frac{1}{\sigma_\nu}\right), \quad 0 < \beta = \min\{\eta, \gamma\} < 1.$$

Hence we have

$$\mu(\sigma_\nu, F) \leq C\exp\left[(1-\beta)(1+o(1))U\left(\frac{1}{\sigma_\nu}\right)\right], \tag{30}$$

where $C$ is a positive real number.

From (19), for any $\varepsilon > 0$ we have

$$M_u(\sigma_\nu, F) \le \sum_{n=0}^{\infty} A_n^* e^{-\lambda_n \sigma_\nu} \le \mu((1-\varepsilon)\sigma_\nu, F) \sum_{n=0}^{\infty} e^{-\varepsilon\sigma_\nu\lambda_n},$$

from the process of proving Theorem 2 and (30), we have

$$M_u(\sigma_\nu, F) \le C_1 \exp\left[(1-\beta)(1+o(1))U\left(\frac{1}{\sigma_\nu}\right)\right]\left[C_2 + \frac{1}{1-\varepsilon\sigma_\nu} T^{1-\varepsilon\sigma_\nu}\right],$$

where $T = \left[e^{(\frac{2}{\varepsilon\sigma_\nu})^{\rho_1}}\right]$ and $C_1, C_2$ are two constants.

Therefore, when $\nu$ is sufficiently large, we have

$$\begin{aligned}\log^+ M_u(\sigma_\nu, F) &\le (1-\beta)(1+o(1))U\left(\frac{1}{\sigma_\nu}\right) + (1-\varepsilon)(\frac{2}{\varepsilon\sigma_\nu})_1^\rho + C_3 \\ &\le (1-\frac{\beta}{2})(1+o(1))U(\frac{1}{\sigma_\nu}),\end{aligned}$$

where $C_3$ is a constant.

Therefore, we get

$$\limsup_{\nu\to\infty} \frac{\log^+ M_u(\sigma_\nu, F)}{U\left(\frac{1}{\sigma_\nu}\right)} \le 1 - \frac{\beta}{2}.$$

This is contradictory to the condition of Theorem 3. Then the necessity of Theorem 3 is proved.

Therefore, we complete the proof of Theorem 3. □

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