EUJAP | VOL. 3 | No. 2 | 2007
ORIGINAL SCIENTIFIC PAPER
UDK: 165

THE 1910 *PRINCIPIA*'S THEORY OF FUNCTIONS AND CLASSES AND THE THEORY OF DESCRIPTIONS*

WILLIAM DEMOPOULOS**
The University of Western Ontario

ABSTRACT

It is generally acknowledged that the 1910 *Principia* does not deny the existence of classes, but claims only that the theory it advances can be developed so that any apparent commitment to them is eliminable by the method of contextual analysis. The application of contextual analysis to ontological questions is widely viewed as the central philosophical innovation of Russell's theory of descriptions. *Principia*'s "no-classes theory of classes" is a striking example of such an application. The present paper develops a reconstruction of *Principia*'s theory of functions and classes that is based on Russell's *epistemological* applications of the method of contextual analysis. Such a reconstruction is not eliminativist—indeed, it explicitly assumes the existence of classes—and possesses certain advantages over the no–classes theory advocated by Whitehead and Russell.

Key words: Russell, *Principia Mathematica*, no-class theory, contextual definition, theory of descriptions, ramified type theory, axiom of reducibility, impredicative definition, propositional function

1. Introduction

The 1910[1] *Principia*'s theory of propositional functions and classes is officially a "no-classes theory of classes," a theory according to which classes are eliminable. But it is clear from *Principia*'s solution to the class paradoxes that although the theory it advances holds that classes are eliminable, it does not deny their existence. Whitehead and Russell argue from the supposition that classes *involve* or *presuppose* propositional functions to the conclusion that the paradoxical classes are excluded by the nature of such functions. This supposition rests on the representation of classes by class abstracts and

* The present paper is part of a longer study, 'The 1910 *Principia*'s theory of functions and classes,' which will be published at a later date.

** I am grateful to the Social Sciences and Humanities Research Council of Canada for support of my research. Many thanks to Peter Koellner, Michael Friedman, Solomon Feferman and Peter Clark for stimulating discussions on the topics dealt with here.

[1] This qualification is always to be understood; I do not address the theory of the 1925 edition. With one exception, all my citations are to volume I of the work.

the technique of contextual analysis. But at no stage does *Principia*'s solution to the paradoxes appeal to anything so strong as the denial of classes. Indeed, its account of the paradoxes is a striking vindication of the traditional "logical" conception of class, since it shows the resolution of the difficulties to depend on the close association between properties (or propositional functions) and classes that is the hallmark of that conception.

It is therefore clear that the status of classes in *Principia* is a matter of some subtlety: any apparent commitment to them is eliminable by the method of contextual analysis, but the "reduction" of classes to propositional functions which such an analysis effects does not, by itself, militate against assuming their existence any more than the reduction of propositions about Scott to propositions that do not contain him as a constituent militate against assuming his existence. Nor would the elimination of classes solve the paradoxes. The key step in the solution of the class paradoxes is their "dependence" on propositional functions. Hence whether or not classes are regarded as superfluous, it remains to show that there is a paradox-free notion of *propositional function*. For this and other reasons (to be discussed below) the approach to *Principia* I advocate is one that sets entirely to one side the program of avoiding a commitment to classes or viewing them as superfluous; instead it reconstructs the work within a framework that *admits* classes.

With the exception of *Principia*'s eliminativist sympathies, the reconstruction I propose leaves intact almost all of the work's distinctive features while showing it to contain a theory of our knowledge of classes of considerable elegance. Here and elsewhere, when I speak of our knowledge of classes, this should be understood in terms of Russell's well-known distinction between *knowledge of truths* and *knowledge of things*.[2] My purpose is to expound what I take to be *Principia*'s implicit theory of our knowledge of classes as a species of knowledge of *things*. The theory that emerges bears some similarity to Russell's theory of knowledge of things known by acquaintance and by description, but it is not a straightforward transcription of that theory to the case of classes. The important similarity to bear in mind is that the theory is first and foremost a theory of how classes can be the subjects of propositions that are asserted of them, rather than a theory of our knowledge of the truth of such propositions. I hope to show that such a theory is of interest even if it fails to address everything one might reasonably wish to know concerning our knowledge of classes.

There are at least two desiderata that it would evidently be desirable for a reconstruction of *Principia*'s theory of functions and classes to satisfy:

(1) A reconstruction should motivate ramifying the simple type-theoretical hierarchy of propositional functions. To do so it must show that propositional functions are plausibly represented as satisfying the requirements expressed by ramification. This would

[2] See for example Russell 1912 pp. 44-45 and ch. 5.

follow if it could be shown that the relevant vicious circle principles which constitute Russell's diagnosis of the paradoxes are analytic of the concept of a propositional function; to establish this conclusion is a primary aim of the present paper.

(2) A successful reconstruction should also explain why the axiom of reducibility is intrinsically plausible within the framework of a ramified theory of functions and classes. This goal has proved elusive. It is an unfortunate consequence of the influence of Ramsey (1925) that the usual view of the ramified theory of types is that while it may be essential to Russell's solution of the semantic paradoxes, it is at best unmotivated in the case of the theory of classes. And since (as shown by Myhill (1974)) the theory is too weak to recover arithmetic without an axiom of reducibility, extending the theory by the inclusion of such an axiom has come to be regarded as irremediably ad hoc. The present reconstruction seeks to make the ramified theory *with* reducibility plausible as a theory of our knowledge of functions and classes, whatever its usefulness and plausibility as an account of the semantic paradoxes.[3]

2. Overview

The following overview may provide a useful orientation to the reconstruction I am proposing, even though it assumes familiarity with notions to be explained in the sequel. On my view, the interest of Russell's logicism is almost wholly epistemological: the central point of the ramified theory of functions and classes is to show how, from the assumption that there are certain classes, it is possible to provide an account of our knowledge of them. The class *N* of natural numbers is the paradigm example of a class, knowledge of which, *Principia* seeks to explain. Its approach to our knowledge of *N* appeals to the method of logical construction, an approach that follows the classical Frege-Dedekind definition of *N* in terms of the satisfaction of all inductive functions of zero. However Frege's and Dedekind's methodology elides distinctions of *order*. When the definition is reformulated within the framework of ramified types, its success depends on the presence of a sufficiently rich supply of *predicative* functions. By the axiom of reducibility there is a plethora of such functions, and as has long been recognized, reducibility is unexceptionable if one assumes the existence of classes. When the Occamism of *Principia* is set to one side and classes are admitted, the ramified theory of

[3] I have throughout been guided by Gödel's fundamental paper (1944), and by Warren Goldfarb's (1989) response to it. I have also benefited from Leonard Linsky's (1987), which I have found in many ways insightful. Here I have in mind especially his remark that "[*Principia* presents] an iterative concept of functions, which is formally similar to Cantor's original iterative concept of sets" (p. 37). But I also have a number of difficulties with Linsky's paper: The discussion of propositional functions (on pp. 35-37) relies too heavily on the 1910 Introduction's notion of an "ambiguity" to yield a proper reconstruction of the relevant remarks; Russell's emphasis on the extensionality of classes in his letter to Frege (of 24 July 1902 which Linsky quotes on p. 28) hardly shows him to have explicitly formulated "the mathematical or iterative concept of sets" as Linsky implies; and contrary to Linsky's claim on p. 26, although Frege may have doubted the suitability of Law V as an axiom, until Russell's famous letter he never questioned its consistency.

functions and classes is readily seen to contain a compelling account of our knowledge of classes such as *N*.

The role of reducibility is not dissimilar from that of *Principia*'s axiom of infinity. Arithmetic poses two basically epistemological questions for logicism:

(i) What does our knowledge of the Dedekind infinity of the numbers rest upon?

(ii) How is reasoning by induction justified?

Principia's answers to both questions depend on the provision of logical constructions. But the constructions succeed only if they are provided with a sufficiently rich base on which to operate. In connection with (i), there must be a "simple infinity" of objects *i* of lowest type for the construction of the numbers to return their Dedekind infinity when they are represented as objects of (simple) type ((*i*)), i.e. as classes of classes of individuals. In the case of (ii) there must be enough predicative functions of the reconstructed numbers in order that the class defined by the ramified form of the Frege-Dedekind definition is one for which mathematical induction is recoverable in its full generality. Neither existence assumption is one that *logic* is capable of securing, but if the assumptions they express are correct, the logical constructions succeed in addressing both questions (i) and (ii). Hence far from signifying the failure of the program of a logical foundation for arithmetic, the necessity of such existence assumptions simply confirms the idea that although logic is capable of addressing what is consequent upon various existence assumptions, it cannot settle the existence questions themselves. *Principia*'s success in connection with the epistemology of arithmetic is therefore a kind of triumph for logicism, although in so far as it is forced to give up any claim to having established the apriority of these two existence claims, it is not the triumph a traditional logicist such as Frege desired.

3. Types and orders

My discussion assumes very little by way of the technical elaboration of the theory of types. Briefly, the *simple* type hierarchy of propositional functions begins with the type of individuals and proceeds to functions of individuals, functions of functions of individuals, etc. Ramification imposes a system of orders on the functions of any type. I assume that the simple hierarchy of functions is stratified, but orders may be cumulative or stratified. The discussion is intended to be broad enough to accommodate any reasonable formalization of ramification which embodies the following four constraints on the assignment of (finite) orders: (*i*) individuals have order 0; (*ii*) functions have order at least 1; (*iii*) only propositional functions of an order lower than a function can occur as arguments to the function; (*iv*) a variable bound by a quantifier occurring in a function can range only over functions of an order lower than the order of the function itself. The base of this hierarchy can be expanded to include propositions; provision must then be made for orders of propositions (by contrast with individuals, among which there are no distinctions of order), and the type of propositions must be distinguished from the type of individuals. The hierarchy then continues with propositional functions whose arguments include propositions (and perhaps individuals), functions of such propositions, etc., together with an appropriate system of orders at every level of the expanded hierarchy. Such an expansion is not considered here since our focus is the theory of functions and classes, and this involves only the hierarchy of functions based on the type of individuals. Nor do I consider the possibility (which *Principia* also does not pursue) of extending the hierarchy of types into the transfinite.

Intuitively, the order of a propositional function is a rough measure of the complexity of its quantificational structure. Functions of least complexity involve no quantifications over other functions within their type. Such functions are the functions of order 1, and in *Principia* they are called "predicative functions"; those that are not predicative are called "non-predicative functions." (It should be noted that this terminology is peculiar to *Principia*. On this use of 'predicative,' a predicative function is *not* one whose definition is predicative rather than impredicative, since even a function which occurs primitively and without definition can be predicative in the intended sense.) The predicative and non-predicative functions form a hierarchy of predicable entities based on the type of individuals. The predicative functions form a substructure of this hierarchy which I will sometimes refer to as "the predicative hierarchy." The members of the predicative hierarchy are logically transparent. The non-predicative functions are logically complex by virtue of having a quantificational structure that includes quantification over other functions.

Classes and individuals form a separate hierarchy of objects, one that is simple, not ramified; this hierarchy begins with individuals and proceeds to classes of individuals, classes of classes of individuals, etc. There is no notion of the order of a class, or equivalently, all classes are of the same order.

Principia replaces the traditional logicist's concepts with propositional functions, which are a kind of abstract intensional entity, not to be identified with the open sentences of a language of fixed vocabulary or even the indefinitely extendable one of *Principia*. Consequently, the hierarchy they comprise may be as large as any hierarchy of classes whose membership they are intended to determine. This is in keeping with the admission (*Principia Vol. 2* p. vii) that there are always more propositional functions than there are individuals.[4]

[4] There is a significant secondary literature which seeks to establish a nominalistic interpretation of propositional functions as linguistic expressions. There may be an interesting reconstruction of the theory of types along these lines, but it cannot be advanced, as it sometimes is, as a correct representation of Russell's intentions in the period under study. Not only is there the point made in the text regarding the cardinality of propositional functions relative to that of individuals, but there are explicit statements to the contrary which simply cannot be passed off as mere slips. Consider, for example, the following passage from §V of (Russell 1908 p. 80): "A propositional function of *x* may, as we have seen, be of any order; hence any statement about 'all properties of *x*' is meaningless. (A 'property of *x*' is the same thing as a 'propositional function which holds of *x*.')"

The notion of type is readily extended to variables; the type of variables for individuals differs from that of functions of individuals, and both in turn differ from that for functions of functions of individuals, etc. In *Principia*, there are special symbols for variables for functions of lowest order, but no other orders are explicitly indicated by the symbolism. Because of the required division of variables into types and orders, variables are not universal; they are not unrestricted in their scope. The loss of the "unrestricted variable" is addressed at the level of the language of *Principia* by the device of typical ambiguity, according to which the type of a syntactic expression for a variable is ambiguous: the same symbol may stand for a variable of any type; and within any type, the same symbol can stand for a predicative function of that type. It is therefore ambiguous what proposition or propositional function is expressed by a formula. Since what matters are only *relative* types, the ambiguity as to type can never lead to a violation of the type restrictions.

The device of typical ambiguity yields a considerable notational simplification. But its philosophical point is that it allows Russell to recover a surrogate for the unrestricted variable of his pre-type-theoretic view.[5] As we have just seen, this arises from a combination of two ideas: the variables constituent in a proposition or propositional function are of maximal scope within a type, and the symbols by which variables are expressed are ambiguously interpretable over arbitrary types.

4. The basic logicist model of our knowledge of classes

The classical epistemological problem posed by classes can arise in connection with our knowledge of any infinite class. As such it is importantly different from the problem of determining the composition of the set theoretic universe and from the related concern with the independence questions of set theory. The problem is to explain how classes are "given" to us, a problem that has an obvious parallel with Frege's question, in §62 of *Grundlagen*, "How, then, are numbers given to us, if we cannot have ideas or intuitions of them?" As with numbers, for classes to be acceptable to a logicist it must be possible to provide an account of our knowledge of them that does not rest on ideas or intuitions. The traditional logicist answer is that classes are given to us as the extensions of concepts.[6] This epistemological question and *Principia*'s nuanced development of a logicist answer to it are our central focus. As we will see in greater detail, *Principia*'s implicit theory of our knowledge of classes is carried out within a framework of existence assumptions about propositional functions. The principal justification for these

[5] I owe this observation to Bernie Linsky.

[6] The appeal of this answer was not confined to logicists. It was, for example, strongly endorsed by Weyl in his book on the continuum. For a discussion of Weyl, together with relevant references and citations, see Parsons 2002 §1, esp. pp. 380 – 381.

assumptions is that they support the account of *how* certain classes, which we unquestionably know, *are* known.

Knowledge of a class is mediated by knowledge of a propositional function which determines it. Knowledge of special classes by means of knowledge of the predicative functions that determine them is assumed to be unproblematic, but it constitutes the exception not the rule. The theory of propositional functions which evolved out of Frege's logical investigations allows for a rich variety of forms of propositional function. The general epistemic situation for which an account is developed is the one in which our knowledge of a class is mediated by knowledge of a *non*-predicative function, i.e., by a logical construction based on a small and limited number of known predicative functions. An account of such cases is especially desirable when the non-predicative function has an independent foundational interest of its own, as is the case with the Frege-Dedekind definition of the class of natural numbers. According to this definition, the class of natural numbers is the class of all *u* which satisfy every inductive function of zero, where a function is *inductive* if whenever it is possessed by *u* it is possessed by *u*'s immediate successor. Assuming knowledge of *zero* and *immediate succession*, the definition explains our epistemic access to the class of natural numbers in terms of our grasp of a propositional function whose additional complexity is wholly logical; for, under the hypothesis that *zero* and *immediate succession* are understood, *natural number* is explained in terms of the notion, *every propositional function*. The difficulty which the Frege-Dedekind definition presents is one of preserving its success in determining exactly the class of natural numbers while respecting the theory of types and orders.

5. Semantic and epistemic dependency

Taking an uncritically realist view of propositional functions and classes clarifies the special epistemic role functions play in the theory by marking a sharp separation of two types of dependency of classes on propositional functions. I express these as two dependency theses, of which the first is the

> *Semantic dependency thesis*. All truths concerning classes are reducible to the use of class abstracts. Insofar as an abstract employs a propositional function that holds precisely of the members of the class, the thesis implies that classes presuppose their defining functions.

Since a class abstract is a definite description, one of the form,

> the class of all *u* such that φu,

the semantic dependency thesis is sometimes expressed by the claim that classes are "incomplete symbols." In *Principia*, the thesis is advanced as plausible but not proven.

The core of the present reconstruction is a weaker thesis which is arguably implicit in *Principia*, even if it is not as central to the work as it is to my reconstruction of it. The thesis may be formulated as the

> *Epistemic dependency thesis*. Except for finite classes, our knowledge of a class cannot consist in knowledge of its members but must appeal to a propositional function which the members of the class all satisfy.

The epistemic dependency thesis depends upon the semantic dependency thesis for its generality, since if the latter thesis failed to hold for some infinite class, the class would not be epistemically accessible. Although the converse dependence does not hold, the semantic dependency thesis is unmotivated outside the context of a logical theory of classes. It seems plausible to suppose that every *logical* theory of classes is committed to the epistemic dependency thesis.

By the *no-classes theory of classes* I understand the philosophical proposal that we should infer from the truth of the semantic dependency thesis that classes are superfluous. The reducibility of truths concerning classes to truths involving the use of class abstracts is the basic premise of the no-classes theory, and in *Principia* the superflousness of classes is derived from the semantic dependency thesis and a principle of ontological economy:

> …we shall assume a proposition about a class always to be reduced to a statement about a function which defines the class, *i.e.* about a function which is satisfied by the members of the class and no other arguments. Thus a class is an object derived from a function and presupposing the function, just as, for example, $(u).\varphi u$ presupposes the function $\varphi\hat{u}$. (*Principia* pp. 62-63)
>
> … In the case of descriptions it was possible to *prove* that they are incomplete symbols. In the case of classes, we don't know of any equally definite proof. It is not necessary for our purposes to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols which we introduce as representative of classes yield all the propositions for the sake of which classes might be thought essential. When this has been shown, the mere principle of economy of primitive ideas leads to the non-introduction of classes except as incomplete symbols (*Principia* p. 72).

Indeed, the no-classes theory is so closely linked to its justification in terms of the semantic dependency thesis that the two are often identified. The basic contention of the present reconstruction is that we should refrain from drawing an ontological lesson from the semantic dependency thesis and, in accordance with the epistemic dependency thesis, take our knowledge of classes to be facilitated by our knowledge of propositional functions.

There is a third dependency thesis associated with the logical theory of classes that I have not mentioned:

> (*) A class *constitutes one object* or *forms a unity* because its elements all satisfy a common propositional function.

This thesis does not play a major role in my analysis. To begin with, the logical theory of classes seeks to *explain* the unity of a class by appeal to the class's association with a propositional function. But if we assume the existence of classes, it is unclear how explanatory (*) is. For then, given a class α together with the relation ε of class membership, (*) is satisfiable by the function $\hat{u}\ \varepsilon\ \alpha$ (see *PM* p. 58). By contrast with the *logical* notion of *class*, according to the *mathematical* notion of *set*, a set is not only extensional, it is *constituted* by its members. Classes also satisfy a principle of extensionality (*PM* *20.15), but they are not constituted by their elements. A set may comprise some sort of unity, but on the mathematical conception, this is not explained by its association with a propositional function, but has the status of a primitive fact about sets and the process of collecting. The difficulty raised by appeal to a function like $\hat{u}\ \varepsilon\ \alpha$ is that it doesn't take us any further than the mathematical concept of set toward explaining the unity of α. At most (*) shows that the logical conception of *class* can be justified relative to the mathematical notion of *set*, an observation which I take to be the correct lesson of *Principia*'s discussion (on p. 58) of the assumption that there are classes.

Suppose, however, that the italicized expressions of (*) are replaced with *constitutes one object of thought* and *forms a unity capable of entering into a judgment*, respectively; this yields:

> (**) A class *constitutes one object of thought* or *forms a unity capable of entering into a judgment* because its elements all satisfy a common propositional function.

(**) follows from the epistemic dependency thesis. It also may appear to be trivially satisfiable since, for any given class, we can always express such a function with a name for the class and the symbol for the membership relation. However there is a reasonable basis for rejecting the suggestion that the thesis is so easily satisfied in its epistemic form. To satisfy (**) the class must be "given" to us in order to be provided with a name from which an expression for a function for it can be constructed. But if the class is sufficiently complex, it may not be possible that it should be given to us as the argument assumes: (**) may be satisfiable—there may be a function by which the class can be known—but that there is, is not established by so simple an extension of the argument that sufficed for (*).

6. The concept of a propositional function

In light of the foregoing, what principles should constrain the concept of a propositional function and the notion of class that it supports? There is an aspect of the iterative concept of set that bears on our question. Although it may seem strange to apply an idea from the mathematical concept of set to the reconstruction of a logical theory of classes, it will soon become apparent that such an application is not only legitimate, but is actually mandated by the coherent development of the concept of a propositional function and the theory to which it belongs. The aspect of the iterative concept of set that we require is the independence it ascribes to the elements of a set: According to the iterative concept, a set's elements exist independently of their membership in the set. A parallel supposition constrains the concept of a propositional function:

> *Argument Independence.* The objects on which a propositional function acts exist independently of their connection with the function.

There is some textual basis for *Argument Independence*, at least for the case of functions of individuals. Thus, in the 1910 Introduction an argument is said to be a "constituent" of the proposition which is the value of the function for it as argument. The explanation of constituency which immediately follows has the clear implication that constituents exist independently of the function and are properly objects of quantification, or what Russell calls "complete symbols":

> [O]bjects which are neither propositions nor functions … we shall call *individuals*. Such objects will be constituents of propositions or functions, and will be *genuine* constituents, in the sense that they do not disappear under analysis, as for example classes do, or phrases of the form "the so and so". (*Principia* p. 51)

For sets, two ideas are critical. The first idea, typically associated with the iterative conception, is the elements' independence of any set to which they belong. The second idea is the *dependence* of a set on its elements—the *extensionality* of sets; this is essential to *any* concept of set, but, as we have noted, on the iterative conception extensionality is forced by the requirement that a set is constituted by its elements. By contrast, propositional functions are not constituted by the things of which they are true; they are therefore not required to satisfy an extensional criterion of identity.[7] Propositional functions, unlike sets, fulfill a *representational* role. This comes about as follows.

[7] A point of difference between propositional functions and Fregean concepts is that the latter satisfy a principle of extensionality, a condition Frege took to be weaker than the claim that concepts are constituted by the objects which fall under them. For Frege even classes are not constituted by their elements, as is clear from his remark to Peano that "… one must not view a class as constituted by its objects for then in removing the objects one would remove the class" (1980 p.109).

A central problem which *Principles* sought to address is to explain how it is possible to have knowledge of objects of infinite complexity when our intelligence is capable of grasping only objects of finite complexity. Its answer depends on the theory of denoting concepts:

> With regard to infinite classes, say the class of numbers, it is to be observed that the concept *all numbers*, though not itself infinitely complex, yet denotes an infinitely complex object. This is the inmost secret of our power to deal with infinity. An infinitely complex concept, though there may be such, certainly cannot be manipulated by the human intelligence, but infinite collections, owing to the notion of denoting, can be manipulated without introducing any concepts of infinite complexity. (*Principles* §72)

The solution to this problem in *Principia* differs from that of *Principles* because of *Principia*'s almost total rejection of the theory of denoting on which *Principles* is based. In *Principia*, the relevant entities of finite complexity are propositional functions. By grasping them we are able to have knowledge of infinitely complex objects. But the feature of propositional functions on which their ability to accomplish this task depends is the presence in them of variables; it is this feature that distinguishes functions from sets and classes and it is the presence of variables in functions that facilitates their finitary representation of objects of infinite complexity.[8]

Variables are acceptable in a way in which the denoting concepts of the older theories of denoting are not. Denoting concepts—such as *all men*, *some numbers*, and *the even prime*—differ from one another both formally and materially. However, variables are distinguished from one another only formally. The association of a variable with its range shares with the older notion of denoting the idea that a variable is *symbolic*, or, as I say, *representational*. But other aspects of denoting that are appropriate to denoting concepts are foreign to the relation a variable bears to its range. For example, in the older theory, different denoting concepts are associated with different pluralities of individuals, and within a plurality, with different combinations of its members. Thus *all numbers* denotes a different non-relational combination of the plurality of numbers than does *some numbers*, while *the even prime* denotes an individual rather than any kind of combination of members of a plurality. Nothing like this is true of variables; their distinctive feature is that they are capable of standing indifferently for anything falling within their range. *Principia* continues to use the language of the theory of denoting, saying of $\varphi\hat{u}$ that it ambiguously denotes φa, φb, φc, etc. by virtue of the presence of the variable u. The relation of *ambiguously denoting* or *ranging over* that holds

[8] In this connection compare Russell's remark to Moore that "[in 'On denoting'] I only profess to reduce the problem of denoting to the problem of the variable". (Letter to Moore of 25 October 1905, (Russell 1994 p. xxxv), quoted by Urquhart in his "Editor's introduction" to the volume.) I concur with this conclusion. It will be seen, however, that I do not treat variables as denoting concepts; Russell's cautious formulation suggests that at the time of writing he may not have done so either.

between a variable and its range replaces the earlier relation of denoting which a denoting concept bears to its denotation.

Since it cannot be taken for granted that variables are unrestricted, it is necessary to make allowance for the possibility that a function's component variables have limited ranges. It follows that the domain of possible arguments to a function may also be restricted. Although not constituted by the things of which they are true, the presence of variables in propositional functions means that they satisfy a dependence condition that is weaker than, but evidently related to, the principle of extensionality. I call this condition *Function Dependence* to distinguish it from the complete dependence of a set on its elements, and I formulate it as follows:

> *Function Dependence.* A propositional function presupposes the totality of its possible arguments and any totality over which its constituent quantified variables range.[9]

It should be emphasized that the imposition of *Function Dependence* is justified merely by the requirement that some provision should be made for the possibility that functions are in some way typed; the condition does not, by itself, impose typing.

Let me briefly summarize the conclusions we have reached: The logical notion of class is based on that of a propositional function. And the relevant concept of propositional function is one which is based on principles—*Argument Independence* and *Function Dependence*—which parallel the principles governing the mathematical concept of set: *Argument Independence* is the analog of the thesis that sets are *based* or *founded on* their elements and *Function Dependence* is the analog of the thesis that sets are *constituted* by their elements.

7. Vicious circle principles

As for the vicious circle principles that constituted Russell's mature diagnosis of the paradoxes and led him to ramification, we wish to know whether they are analytic of the notions of *propositional function* and *class* as we have proposed these notions should be understood.

Following on the discussion of Gödel (1944), it is useful to distinguish a general vicious circle principle for "totalities," and two that apply more specifically to functions:

[9] A representative formulation of *Principia* is: "A function … presupposes as part of its meaning the totality of its values, or, what comes to the same thing, the totality of its possible arguments…" (p. 54). And "…the values of a function are presupposed by the function, not vice versa" (p. 39).

> *General VCP.* No totality can contain members involving or presupposing the totality; hence, no totality of all arguments to a function can contain members involving or presupposing the totality of its arguments.
>
> *Strong VCP.* No function can belong to a totality which it involves or presupposes.
>
> *Weak VCP.* No function can have arguments involving or presupposing the function.

All three principles are distinct from the prohibition against impredicative definition for which there are three parallel principles:

> *General Impredicativity.* No totality can contain members definable only in terms of the totality; hence, no totality of all arguments to a function can contain members definable only in terms of the totality of its arguments.
>
> *Strong Impredicativity.* No function can belong to a totality which is involved in its definition.
>
> *Weak Impredicativity.* No function can have arguments definable only in terms of the function.

My immediate concern is with the first group of vicious circle principles; I will take up the "impredicativity principles" later.

Weak VCP follows immediately from *Function Dependence*, but it is also a simple consequence of just *Argument Independence*: If our concept of a propositional function is such that its arguments do not depend on the function itself, then functions cannot be involved in or presupposed by their arguments without contravening *Argument Independence*. By contrast with *Weak VCP*, *Strong VCP* requires the full strength of *Function Dependence*. Turning to *General VCP*, we can show that it holds for those totalities that are classes, and a fortiori, for those totalities of arguments to a function that are classes. Indeed, for such totalities, *General VCP* is a consequence of the *semantic dependency thesis* and *Argument Independence*. For, by the semantic dependency thesis any truth concerning classes is transformable into one in which reference to the class is replaced by reference to a propositional function that determines it. In particular, if there were a truth concerning a class that contained a member involving or presupposing the class, there would be a corresponding truth concerning a propositional function one of whose arguments involved or presupposed the function. But this is impossible by *Argument Independence*.

Turning to the impredicativity principles, in his (1944) Gödel argued that from a realist perspective on the totalities comprised by *sets*, the imposition of *General Impredicativity* is not justified. From such a perspective, a definition is merely an expression that uniquely specifies the defined entity. There is therefore nothing problematic in the notion that our only means of identifying a set might involve a quantification over a totality to which it itself belongs. Gödel argued that for an impredicative definition to be problematic, one must think of the definition as not merely a specification of the entity defined, but as somehow essential to the entity's existence. This would be plausible if, for example, a definition were thought of as an explanation of how to effect a construction of what is defined. For it could then be argued that an entity cannot be assumed in its own construction. But if we conceive of sets as existing independently of our constructions, there is no reason to accept *General Impredicativity*.

In response to this argument of Gödel's, Goldfarb (1989) observed that the defense of impredicative definitions of sets depends on an assumption much less tendentious than realism. A set is an extensional entity whose members are essential to it, but no defining formula is essential to its identity. Sets are not preserved under change of elements, but it is a matter of indifference to the identity of a set whether or not it is specified by a defining formula that contains a quantified variable whose range includes the set itself. The prohibition against impredicative definitions of sets therefore lacks any intuitive basis even if one is not a realist regarding their existence.

Goldfarb's observation is certainly correct. The point might be put by saying that the acceptability of *General Impredicativity* turns on a commitment regarding the *essence* of sets, rather than one regarding their *existence*. The nature of the basis for the extensionality that sets enjoy suffices to show why one might plausibly reject *General Impredicativity* for them. But Gödel's chief contention was that only a form of *anti*-realism could sustain the imposition of *General Impredicativity* on *propositional functions*. As Goldfarb recognizes (p. 31), his observation does not by itself refute this contention of Gödel's, but shows only that since propositional functions are not extensional entities, they *may* be constrained by *General Impredicativity*, not that they *must* be. Goldfarb's proposal for addressing this lacuna is to argue that our access to a propositional function is altogether different from the access we have to a class specified by the function:

> To specify a class is to give a propositional function that is true of all and only the members of the class. The specification must be understood on its own; given such an understanding, it is a further question whether or not a given class is the one specified. … [But] the comprehension axioms for propositional functions that are implicit in [*Principia*] involve not so much the specification of these entities as the presentation of them. I shall therefore use 'presentation' rather than 'specification' in this connection. (Goldfarb 1989 pp. 31-32)

Goldfarb's idea is that with the specification of a class there is a certain "space" between the specification and the class specified. But to "specify" a propositional function is to be "immediately presented with" the entity meant. The suggestion is an interesting one, but it is difficult to see how to make entirely explicit the notion of immediate presentation on which it depends. It may therefore be worthwhile to see if there is an alternative approach which does not appeal to this notion.

The strategy we followed for *General, Strong* and *Weak VCP* was to argue that they are forced by the concept of a propositional function. This strategy appears to be correct for the impredicativity principles controlling functions since a definition which violates *Strong* or *Weak Impredicativity* can be excluded on the ground that the function it purports to express would violate *Strong* and *Weak VCP*. But *General Impredicativity* has a different justification, one that depends on the centrality of propositional functions to the explanation of our knowledge of classes—in effect, on the fact that classes are known as the extensions of propositional functions. According to the logical theory, an expression that defines a class in terms of a totality to which the defining function itself belongs fails to define it by a propositional function. *A logical theory which permitted such a definition would therefore be incapable of supporting* Principia*'s explanation of how classes are known.*

Consider, again, the celebrated Frege-Dedekind definition of the natural numbers as the class determined by the function that holds of those individuals which satisfy all inductive functions of zero. This definition fails to express a propositional function because it allows a function to fall within the range of one of its quantified variables. Consequently, what the definition expresses violates both *Weak* and *Strong VCP*. The *rationale* for *General Impredicativity* is therefore that the definitions it excludes fail to express propositional functions, and hence cannot support the *epistemic dependency thesis*. But the *justification* for imposing *General Impredicativity* can only be achieved by demonstrating the adequacy of the hierarchy of classes that *are* capable of being known by propositional functions which accord with *Strong* and *Weak VCP*. This brings us to the axiom of reducibility.

8. The axiom of reducibility

On the present reconstruction, axioms of reducibility are introduced to ensure the epistemic accessibility of enough classes. The axioms are formulated with typical ambiguity and so occur at every type.[10] Such axioms postulate that the quantified variables of a propositional function range over predicative extensional equivalents of functions of every order within a type. A function is prohibited from falling within the range of one of its quantified variables, but ramification imposes no such prohibition on functions *coextensive* with it. On the reconstruction I am proposing, the point of the 1910 theory is epistemic rather than reductive: the theory is concerned to explain our knowledge of classes, not to explain classes away; and the role of axioms of reducibility is to secure

[10] The simplest example of an axiom of reducibility involves the postulation of predicative equivalents of propositional functions of a single argument; formally, $\exists\chi\forall u(\varphi u \equiv \chi!u)$, where the exclamation mark indicates that $\chi\hat{u}$ is a predicative function.

our knowledge of certain classes in terms of non-predicative functions by the postulation of predicative functions.

This essentially epistemological purpose of reducibility has been missed, even by some of *Principia*'s most astute commentators. Thus it has been noted of Gödel that his

> remarks about the axiom of reducibility show [a] lack of sensitivity to the essentially intensional character of Russell's logic; the fact that every propositional function is *coextensive* with one of lowest order does not imply 'the existence in the data of the kind of objects to be constructed' (1944 p. 141), if the objects in question are … propositional functions rather than classes. … The ramified hierarchy with reducibility would fit with a conception according to which classes are admitted as 'real objects,' but the conception of propositional function is … constructivistic. (Parsons 1990 pp. 112-113)[11]

While the postulation of predicative equivalents of functions of all orders significantly affects the totality of classes that are determined by non-predicative functions, the inclusion of predicative equivalents in the range of such a function's variables leaves intact the account of our knowledge of the propositional function itself. The fact that the functions postulated are merely coextensive with functions of higher order is evidently essential to the success of this strategy.

The theory of functions and classes with reducibility depends crucially on an understanding of generality according to which the use of a variable does not require knowledge of its values. There is no presumption that it should be possible, even in principle, to know any function or individual in the range of a variable to a function:

> a function can be apprehended without its being necessary to apprehend its values severally or individually. If this were not the case, no function could be apprehended at all, since the number of values (true and false) of a function is necessarily infinite and there are necessarily possible arguments with which we are unacquainted. (*Principia* pp. 39-40)

In *Problems of philosophy*, Russell devotes a great deal of space to the defense of this account of general knowledge, seeking to establish that much of what appears to require knowledge of particular instances does not in fact do so. The problem becomes especially pressing in connection with a priori knowledge, since it is supposed to be independent of knowledge of particular facts of experience, and thus, of the existence of particular individuals. Russell argues that we can understand a proposition purporting to express a priori knowledge without knowing the constituents of any instance of

[11] The reconstruction of ramification plus reducibility proposed here is one possible development of Parsons's suggestion, but I do not know if it conforms to what he had in mind or whether he would even agree with it.

the proposition, and that we can do so compatibly with an empiricist commitment to the primacy of acquaintance and to the contention that all knowledge of particular existence is a posteriori. There are even cases of *a posteriori* known general truths that do not depend on knowledge of their instances. Russell's example is the proposition

(§) There are numbers greater than 1000 which no one has ever thought of.

By the nature of the case we cannot know an instance of (§), since that would require knowing the number which is a constituent of the instance, thereby precluding the instance from being a witness to the truth of (§).

This suggests the following view of the significance of Russell's theory of descriptions for his subsequent logical discoveries. The interest of that theory is usually restricted to its application to the semantic issues posed by vacuous descriptions. From such a perspective, analyzing classes away after the fashion of the contextual analysis of vacuous singular terms is naturally viewed as the principal lesson the theory of descriptions has to offer for *Principia*'s account of classes. But by focusing on the elimination of the bearers of vacuous descriptions, it is easy to lose sight of the theory of description's equally important epistemic applications; these arise in cases where there is no question of failure of reference. *Principia*'s theory of knowledge of classes parallels Russell's general theory of knowledge of things; its theory would also collapse if our understanding of generality demanded knowledge of instances. Classes stand to the functions which determine them as things stand to the descriptive functions they satisfy. Just as there are many descriptions of the same thing, so also, the same class is capable of being the extension of many functions. Knowledge of a class by means of a predicative function corresponds, in the theory of knowledge of things, to being acquainted with the thing known and in possession of a logically proper name for it. Knowledge of a class by means of a non-predicative function is the analog of knowledge of a thing by means of a logically complex description. The nature of a non-predicative function follows its canonical linguistic expression in the language of *Principia*; knowledge of the function proceeds from knowledge of its predicative components and the logical form of their combination. In the case of knowledge of things, the transparency of the components of the description by which a thing is specified is *epistemic*: the components of the description (or rather, the components of the propositional function which replaces it) are known by acquaintance. But although the components are known by acquaintance, the description to which they belong may be satisfied by something that transcends our acquaintance. By the epistemic dependency thesis, our access to classes is through propositional functions. A class for which we lack a predicative function is accessible only if it is determined by a logical construction built from known predicative functions. In the case of classes, the relevant transparency of the components of the functions by which they are known is their *logical* transparency, the fact that the basic functional constituents are predicative functions. In analogy with the theory of knowledge of things which transcend our acquaintance, although the basic component propositional functions are predicative, the class determined by the logical construction which they comprise can be one that is not known by means of a predicative function.

It is difficult to motivate the theory consisting of ramified types plus reducibility when it is separated from its epistemological point. Nor is it easy to see in what sense the reducibility axioms represent an advance over the mere postulation of classes. But they represent such an advance in at least two respects: They preserve the logical theory of our knowledge of classes, according to which classes are known as the extensions of propositional functions; and the reducibility axioms also preserve an attractive representation of our knowledge of particular classes in terms of non-predicative functions. In connection with this second advance, the primary example of such an attractive representation is the impredicative definition of the class of natural numbers of Frege and Dedekind. It is not required that we know the values of a variable occupying the argument-place of a function in order to know the class the function determines. But it *is* required that the range of its values *be* sufficiently extensive in order that the function should determine the intended class. The ramified analog of the Frege-Dedekind definition captures the class of natural numbers only through the intervention of the axiom of reducibility. It is for this reason that the correctness of *Principia*'s reformulation of the definition has been sometimes thought to capture the class of natural numbers only "accidentally," by the coincidence that the range of the variable which occurs in it is *as a matter of fact* sufficiently extensive. Such contingency at the foundational level is characteristic of the analysis of *Principia*, and is perhaps most conspicuous in its formulation of its axiom of infinity. As we have just seen, it also arises in connection with reducibility. But contingency is not ad hocness, and whatever its defects, this consequence of the analysis has the virtue of making explicit the postulates that honest logical toil would appear to require.

REFERENCES

Anderson C.A. & W. Savage eds. (1987), *Rereading Russell*, Minneapolis: University of Minnesota Press

Braithwaite R.B. ed. (1925), *The foundations of mathematics and other logical essays*, London: Kegan Paul

Feferman S. et. al. eds. (1944), *The collected papers of Kurt Gödel,* Oxford University Press

Feferman S. et al. eds. (1990), *The collected papers of Kurt Gödel,* Oxford University Press

Frege G. (1980): *Philosophical and mathematical correspondence*, ed. by B. McGuinness, trans. by H. Kaal, Oxford: Blackwell

Gödel K. (1944): "Russell's mathematical logic", in Feferman et. al. 1944, pp. 119-141

Goldfarb W. (1987): "Russell's reasons for ramification", in Anderson & Savage 1987, pp. 24-40

Linsky L. (1987): "Russell's no-classes theory of classes", in Thompson 1987, pp. 21-40

Myhill J. (1974): 'The undefinability of the set of natural numbers in the ramified *Principia*,' Nakhnikian 1974, pp. 19-27

Nakhnikian G. ed. (1974), *Bertrand Russell's philosophy*, London: Duckworth

Parsons C. (1990): "Introduction to Gödel (1944)", in Feferman et al. 1990, pp. 102-118

Parsons C. (2002): "Realism and the debate on impredicativity: 1917 – 1944", in Sieg et. al. 2002, pp. 378-395

Ramsey F.P. (1925): "The foundations of mathematics", in Braithwaite 1925, pp. 1-61

Russell B. (1903): *The principles of mathematics*, London: Allen and Unwin

Russell B. (1905): "On denoting," in Russell 1956, pp. 41-56

Russell B. (1908): 'Mathematical logic as based on the theory of types', in Russell 1956, pp. 59-102

Russell B. (1912): *Problems of philosophy*, Oxford: Oxford University Press

Russell B. (1956): *Logic and knowledge*, London: Allen and Unwin

Russell B. (1994): *The collected papers of Bertrand Russell, Vol. IV, Foundations of logic 1903-05*, Urquhart A. & A.C. Lewis eds, The McMaster University Edition, London and New York: Routledge

Sieg W. et al. eds. (2002), *Reflections on the foundations of mathematics: Essays in honor of Solomon Feferman*, Association for Symbolic Logic

Thompson J.J. ed. (1987), *Being and saying: Essays in honor of Richard Cartwright*, Cambridge MA: MIT

Whitehead & Russell (1910): *Principia mathematica, Vol.* 1, Cambridge: Cambridge University Press

Whitehead & Russell (1912): *Principia mathematica, Vol.* 2, Cambridge: Cambridge University Press

Received: December 19, 2006
Accepted: June 28, 2007

Department of Philosophy
Talbot College, The University of Western Ontario
London, Canada N6A 3K7
wgdemo@uwo.ca