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Differential equations of fractional order $\alpha \in (2,3)$ with boundary value conditions in abstract Banach spaces

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Abstract. In this paper, we study boundary value problems for differential equations involving Caputo derivative of order $\alpha \in (2,3)$ in Banach spaces. Some sufficient conditions for the existence and uniqueness of solutions are established by virtue of fractional calculus, a special singular type Gronwall inequality and fixed point method under some suitable conditions. Examples are given to illustrate the main results.

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1. Introduction

In this paper, we will extend the earlier work [1] on fractional boundary value problems (BVP for short), for fractional differential equations of order $\alpha \in (2,3)$ in R to the abstract Banach space X of the type

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,T],\ T \geq 1, \\ y(0) = y_0, \quad y'(0) = y_0^*, \quad y''(T) = y_T, \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (2,3)$, $f : J \times X \to X$ is a given function satisfying some assumptions that will be specified later and y_0, y_0^*, y_T are three elements of X.

Differential equations with fractional order have recently proved to be strong tools in modelling of many physical phenomena; for a good bibliography on this topic we refer to Miller and Ross [12]. As a consequence, there was an intensive development of the theory of differential equations of fractional order. We refer to the monographs of Diethelm [6], Kilbas et al. [11], Lakshmikantham et al. [14], Podlubny [15]. Particulary, Agarwal et al. [2] establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative in R. Very recently, some fractional differential equations and optimal controls in abstract Banach spaces have been studied by Balachandran et al. [3, 4],

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Dong et al. [7], El-Borai [8], Henderson and Ouahab [9], Hernández [10], Mophou and N'Guérékata [13], Wang et al. [17 - 21] and Zhou et al. [23 - 27].

Utilizing fractional calculus, Hölder inequality, a special singular type Gronwall inequality (Lemma 8) and fixed point method, some existence and uniqueness results for the fractional BVP (1) are presented. Compared with the earlier results obtained in [1], there are at least three differences: (i) the work space is not R but the abstract Banach space X; (ii) f is not necessarily jointly continuous and it satisfies some weaker assumptions; (iii) a special singular type Gronwall inequality is used to obtain the priori bounds.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, we give a special singular type Gronwall inequality which can be used to establish the estimate of a fixed point set $\{y = \lambda F y, \lambda \in [0,1]\}$. In Section 4, we give three main results (Theorems 3–5), the first result based on the Banach contraction principle, the second result based on Schaefer's fixed point theorem and the third result based on a nonlinear alternative of Leray-Schauder type. Examples are given in Section 5 to demonstrate the application of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. We denote by $C(J,X)$ the Banach space of all continuous functions from J into X with the norm $\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}$. We also denote the Banach space $C^n(J,X) := \big\{u \in C(J,X) :\ u^{(n)} \in C(J,X)\big\}$ with the norm $\|u\|_{C^n} := \sum_{k=0}^n \|u^{(n)}\|_C$ for $u \in C^n(J,X)$. For measurable functions $m : J \to R$, define the norm $\|m\|_{L^p(J,R)} = \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}$, $1 \le p < \infty$. We denote by $L^p(J,R)$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,R)} < \infty$.

We need some basic definitions and properties of fractional calculus theory which are used further in this paper.

Definition 1 (See [11])**.** *The fractional order integral of the function $h \in L^1([a,b], R_+)$ of order $\alpha \in R_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the Gamma function.

Definition 2 (See [11])**.** *For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 3 (See [11]). *For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h is defined by*

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 1 (See [22], Lemma 2.3). *Let $n > \alpha > n-1$. If $h \in C^n([a,b])$, then*

$$I^{\alpha}({}^cD^{\alpha}h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in R, i = 0,1,2,\cdots,n-1$, where n is the smallest integer greater than or equal to α.

Now, let us recall the definition of a solution of the fractional BVP (1).

Definition 4 (See [2], Definition 3.7). *A function $y \in C^3(J,X)$ with its α-derivative existing on J is said to be a solution of the fractional BVP (1) if y satisfies the equation ${}^cD^{\alpha}y(t) = f(t,y(t))$ a.e. on J, and the conditions $y(0) = y_0,\ y'(0) = y_0^*,\ y''(T) = y_T$.*

For the existence of solutions for the fractional BVP (1), we need the following auxiliary lemma, which is a consequence of Lemma 1.

Lemma 2 (See [2], Lemma 3.8). *Let $\bar{f} : J \to X$ be continuous. A function $y \in C(J,X)$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\bar{f}(s)ds - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3}\bar{f}(s)ds + y_0 + y_0^* t + \frac{y_T}{2}t^2,$$

if and only if y is a solution of the following fractional BVP

$$\begin{cases} {}^cD^{\alpha}y(t) = \bar{f}(t),\ 2 < \alpha < 3,\ t \in J, \\ y(0) = y_0,\ y'(0) = y_0^*,\ y''(T) = y_T. \end{cases} \tag{2}$$

As a consequence of Lemmas 2, we have the following known result, which is useful in what follows.

Lemma 3 (See [1], Lemma 3.4). *Let $f : J \times X \to X$ be continuous. A function $y \in C(J,X)$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s,y(s))ds \\ & + y_0 + y_0^* t + \frac{y_T}{2}t^2, \end{aligned}$$

if and only if y is a solution of the fractional BVP (1).

Lemma 4 (Bochner theorem). *A measurable function $f\colon J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 5 (Mazur lemma). *If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{conv}\mathcal{K}$ is compact.*

Lemma 6 (Ascoli-Arzela theorem). *Let $\mathcal{S} = \{s(t)\}$ be a function family of continuous mappings $s : [a, b] \to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}(n = 1, 2, \cdots, t \in [a, b])$ in $\mathcal{S}$.*

Theorem 1 (Schaefer's fixed point theorem). *Let $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{x \in X : x = \lambda^* Fx \text{ for some } \lambda^* \in [0, 1]\}$ is bounded, then F has fixed points.*

Theorem 2 (Nonlinear alternative of Leray-Schauder type). *Let $\mathcal{C}$ be a nonempty convex subset of X. Let U be a nonempty open subset of $\mathcal{C}$ with $0 \in U$ and $F : \overline{U} \to \mathcal{C}$ compact and continuous operators. Then either*

(i) F has fixed points.

(ii) There exist $y \in \partial U$ and $\lambda^* \in [0, 1]$ with $y = \lambda^* F(y)$.

3. A special singular type Gronwall inequality

In order to apply the Schaefer fixed point theorem to show the existence of solutions, we need a new special singular type Gronwall inequality with a mixed type singular integral operator. It will play an essential role in the study of BVP for fractional differential equations.

We first collect a generalized Gronwall inequality which appeared in our earlier work [16].

Lemma 7 (See [16], Lemma 3.2). *Let $y \in C(J, X)$ satisfy the following inequality:*

$$\|y(t)\| \le a+b\int_0^t \|y(\theta)\|^{\lambda_1} d\theta+c\int_0^T \|y(\theta)\|^{\lambda_2} d\theta+d\int_0^t \|y_\theta\|_B^{\lambda_3} d\theta+e\int_0^T \|y_\theta\|_B^{\lambda_4} d\theta, t \in J,$$

where $\lambda_1, \lambda_3 \in [0, 1], \lambda_2, \lambda_4 \in [0, 1), a, b, c, d, e \ge 0$ are constants and $\|y_\theta\|_B = \sup_{0\le s\le\theta} \|y(s)\|$. Then there exists a constant $M^* > 0$ such that

$$\|y(t)\| \le M^*.$$

Using the above generalized Gronwall inequality, we can obtain a new special singular type Gronwall inequality.

Lemma 8. *Let $y \in C(J, X)$ satisfy the following inequality:*

$$\|y(t)\| \le a + b\int_0^t (t-s)^{\alpha-1}\|y(s)\|^\lambda ds + c\int_0^T (T-s)^{\alpha-3}\|y(s)\|^\lambda ds, \tag{3}$$

where $\alpha \in (2, 3)$, $\lambda \in [0, 1 - \frac{1}{p}]$ for some $1 < p < \frac{1}{3-\alpha}$, $a, b, c \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$\|y(t)\| \le M^*.$$

Proof. Let

$$x(t) = \begin{cases} 1, & \|y(t)\| \le 1, \\ y(t), & \|y(t)\| > 1. \end{cases}$$

It follows from condition (3) and Hölder inequality that

$$\begin{aligned}
\|y(t)\| \le \|x(t)\| &\le (a+1) + b\int_0^t (t-s)^{\alpha-1}\|x(s)\|^{\lambda} ds + c\int_0^T (T-s)^{\alpha-3}\|x(s)\|^{\lambda} ds \\
&\le (a+1) + b\left(\int_0^t (t-s)^{p(\alpha-1)} ds\right)^{\frac{1}{p}} \left(\int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}} ds\right)^{\frac{p-1}{p}} \\
&\quad + c\left(\int_0^T (T-s)^{p(\alpha-3)} ds\right)^{\frac{1}{p}} \left(\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}} ds\right)^{\frac{p-1}{p}} \\
&\le (a+1) + b\left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}} \int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}} ds \\
&\quad + c\left(\frac{T^{p(\alpha-3)+1}}{p(\alpha-3)+1}\right)^{\frac{1}{p}} \int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}} ds.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|y(t)\| \le & \|x(t)\| \\
\le & (a+1) + \left[b\left(\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{\frac{1}{p}} + c\left(\frac{T^{p(\alpha-3)+1}}{p(\alpha-3)+1}\right)^{\frac{1}{p}} \right] \int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}} ds,
\end{aligned}$$

where $0 \le \frac{\lambda p}{p-1} < 1$. By Lemma 7, one can complete the rest of the proof immediately.

For $\frac{\lambda p}{p-1} = 1$, applying the classical Gronwall inequality, one can complete the proof. □

Remark 1. *Although Lemma 8 is a special case of Theorem 4 in [5], it is enough to deal with the possible estimation in Section 4.*

4. Main results

Before stating and proving the main results, we introduce the following hypotheses.

(H1) The function $f : J \times X \to X$ is strongly measurable with respect to t on J and is continuous with respect to u on X.

(H2) There exist a constant $\alpha_1 \in (0, \alpha - 2)$ and real valued function $m(t) \in L^{\frac{1}{\alpha_1}}(J, R)$ such that

$$\|f(t,u_1) - f(t,u_2)\| \le m(t)\|u_1 - u_2\|, \ \textit{for each } t \in J, \textit{ and all } u_1, u_2 \in X.$$

(H3) There exist a constant $\alpha_2 \in (0, \alpha - 2)$ and real valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, R)$ such that

$$\|f(t, y)\| \le h(t), \text{ for each } t \in J, \text{ and all } y \in X.$$

For brevity, let $M = \|m\|_{L^{\frac{1}{\alpha_1}}(J,R)}$, $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J,R)}$.

Our first result is based on the Banach contraction principle.

Theorem 3. *Assume that (H1)–(H3) hold. If*

$$\Omega_{\alpha,T} = \left(\frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + \frac{M}{2\Gamma(\alpha-2)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1-2}{1-\alpha_1})^{1-\alpha_1}} \right) < 1, \tag{4}$$

then system (1) has a unique solution on J.

Proof. For each $t \in J$, we have

$$\begin{aligned}\int_0^t \left\|(t-s)^{\alpha-1} f(s, y(s))\right\| ds &\le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\ &\le \frac{T^{\alpha-\alpha_2} H}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.\end{aligned}$$

Thus, $\|(t-s)^{\alpha-1} f(s, y(s))\|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $y \in C(J, X)$. Then $(t-s)^{\alpha-1} f(s, y(s))$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$ due to Lemma 4. Similarly,

$$\begin{aligned}\int_0^T \left\|(T-s)^{\alpha-3} f(s, y(s))\right\| ds &\le \left(\int_0^T (T-s)^{\frac{\alpha-3}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\ &= \frac{T^{\alpha-\alpha_2-2} H}{(\frac{\alpha-\alpha_2-2}{1-\alpha_2})^{1-\alpha_2}}.\end{aligned}$$

Thus, $\|(T-s)^{\alpha-3} f(s, y(s))\|$ is Lebesgue integrable with respect to $s \in [0, T]$ and $y \in C(J, X)$. Then $(T-s)^{\alpha-3} f(s, y(s))$ is Bochner integrable with respect to $s \in [0, T]$ due to Lemma 4.

Hence, the fractional BVP (1) is equivalent to the following fractional integral equation

$$\begin{aligned} y(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s, y(s)) ds \\ &+ y_0 + y_0^* t + \frac{y_T}{2} t^2, \ t \in J. \end{aligned}$$

Let

$$r \ge \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{HT^{\alpha-\alpha_2}}{2\Gamma(\alpha-2)(\frac{\alpha-\alpha_2-2}{1-\alpha_2})^{1-\alpha_2}} + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2} T^2.$$

Now we define the operator F on $B_r := \{y \in C(J,X) : \|y\| \leq r\}$ as follows

$$(Fy)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s,y(s))ds + y_0 + y_0^* t + \frac{y_T}{2} t^2, \; t \in J. \tag{5}$$

Therefore, the existence of a solution of the fractional BVP (1) is equivalent to that the operator F has a fixed point on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fy \in B_r$ for every $y \in B_r$

For every $y \in B_r$ and any $\delta > 0$, by (H3) and Hölder inequality, we get

$$\begin{aligned}
&\|(Fy)(t+\delta) - (Fy)(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] h(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} h(s)ds \\
&\quad + \frac{(2t+\delta)\delta}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} h(s)ds \\
&\quad + \|y_0^*\|\delta + \frac{\|y_T\|}{2}(2t+\delta)\delta \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}]^{\frac{1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \frac{(2t+\delta)\delta}{2\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\frac{\alpha-3}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\
&\quad + \|y_0^*\|\delta + \frac{\|y_T\|}{2}(2t+\delta)\delta \\
&\leq \frac{H}{\Gamma(\alpha)} \left(\frac{(t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} - \frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} - \frac{t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} \\
&\quad + \frac{H}{\Gamma(\alpha)} \left(\frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} + \frac{(2t+\delta)\delta}{2\Gamma(\alpha-2)} \frac{T^{\alpha-\alpha_2-2} H}{(\frac{\alpha-\alpha_2-2}{1-\alpha_2})^{1-\alpha_2}} \\
&\quad + \|y_0^*\|\delta + \frac{\|y_T\|}{2}(2t+\delta)\delta \\
&\leq \frac{H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} \left[\left((t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}} - t^{\frac{\alpha-\alpha_2}{1-\alpha_2}} - \delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}} \right)^{1-\alpha_2} + \delta^{\alpha-\alpha_2} \right] \\
&\quad + \left(\frac{H}{2\Gamma(\alpha-2)} \frac{T^{\alpha-\alpha_2-2}}{(\frac{\alpha-\alpha_2-2}{1-\alpha_2})^{1-\alpha_2}} + \frac{\|y_T\|}{2} \right) (2T+\delta)\delta + \|y_0^*\|\delta.
\end{aligned}$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero. Therefore, F is continuous on J, i.e., $Fy \in C(J, X)$.
Moreover, for $y \in B_r$ and all $t \in J$, we get

$$\begin{aligned}\|(Fy)(t)\| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + \frac{T^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} h(s) ds \\ & + \|y_0\| + \|y_0^*\| T + \frac{\|y_T\|}{2} T^2 \\ \leq & \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\ & + \frac{T^2}{2\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\frac{\alpha-3}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \\ & + \|y_0\| + \|y_0^*\| T + \frac{\|y_T\|}{2} T^2 \\ \leq & \frac{HT^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{HT^{\alpha-\alpha_2}}{2\Gamma(\alpha-2)(\frac{\alpha-\alpha_2-2}{1-\alpha_2})^{1-\alpha_2}} + \|y_0\| + \|y_0^*\| T + \frac{\|y_T\|}{2} T^2,\end{aligned}$$

which implies that $\|Fy\|_\infty \leq r$. Thus, we can conclude that for all $y \in B_r$, $Fy \in B_r$. i.e., $F : B_r \to B_r$.

Step 2. F is a contraction mapping on B_r.

For $x, y \in B_r$ and any $t \in J$, using (H2) and Hölder inequality, we get

$$\begin{aligned}\|(Fx)(t) - (Fy)(t)\| & \\ \leq & \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s) ds + \frac{T^2 \|x-y\|_\infty}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} m(s) ds \\ \leq & \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^t (m(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\ & + \frac{T^2 \|x-y\|_\infty}{2\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\frac{\alpha-3}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^T (m(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\ \leq & \left(\frac{M}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + \frac{M}{2\Gamma(\alpha-2)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1-2}{1-\alpha_1})^{1-\alpha_1}} \right) \|x-y\|_\infty.\end{aligned}$$

So we obtain

$$\|Fx - Fy\|_\infty \leq \Omega_{\alpha,T} \|x-y\|_\infty.$$

Thus, F is a contraction due to condition (4).

By the Banach contraction principle, we can deduce that F has a unique fixed point which is just the unique solution of the fractional BVP (1). □

Our second result is based on the well known Schaefer's fixed point theorem.

We make the following assumptions:

(H4) There exist constants $\lambda \in [0, 1-\frac{1}{p}]$ for some $1 < p < \frac{1}{3-\alpha}$ and $N > 0$ such that

$$\|f(t,u)\| \leq N(1+\|u\|^{\lambda}) \textit{ for each } t \in J \textit{ and all } u \in X.$$

(H5) For every $t \in J$, the sets $K_1 = \{(t-s)^{\alpha-1}f(s,y(s)) : y \in C(J,X), s \in [0,t]\}$ and $K_2 = \{(t-s)^{\alpha-3}f(s,y(s)) : y \in C(J,X), s \in [0,t]\}$ are relatively compact.

Theorem 4. *Assume that (H1), (H4) and (H5) hold. Then the fractional BVP (1) has at least one solution on J.*

Proof. Transform the fractional BVP (1) into a fixed point problem. Consider the operator $F : C(J,X) \to C(J,X)$ defined as (5). It is obvious that F is well defined due to (H1), Hölder inequality and Lemma 4.

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,X)$. Then for each $t \in J$, we have

$$\|(Fy_n)(t) - (Fy)(t)\| \leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{2\Gamma(\alpha-1)}\right)\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty}.$$

Since f is continuous, we have

$$\|Fy_n - Fy\|_{\infty} \leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}}{2\Gamma(\alpha-1)}\right)\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty} \to 0 \textit{ as } n \to \infty.$$

Step 2. F maps bounded sets into bounded sets in $C(J,X)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $\ell > 0$ such that for each $y \in B_{\eta^*} = \{y \in C(J,X) : \|y\|_{\infty} \leq \eta^*\}$, we have $\|Fy\|_{\infty} \leq \ell$.

For each $t \in J$, by (H4), we get

$$\begin{aligned}
\|(Fy)(t)\| \leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}N(1+\|y(s)\|^{\lambda})ds \\
&+ \frac{T^2}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3}N(1+\|y(s)\|^{\lambda})ds \\
&+ \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2 \\
\leq & \frac{N(1+(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&+ \frac{T^2N(1+(\eta^*)^{\lambda})}{2\Gamma(\alpha-2)}\int_0^T (T-s)^{\alpha-3}ds \\
&+ \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2 \\
\leq & \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right)T^{\alpha}N(1+(\eta^*)^{\lambda}) \\
&+ \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2,
\end{aligned}$$

which implies that

$$\|Fy\|_\infty \leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha-1)}\right) T^\alpha N(1+(\eta^*)^\lambda) + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|T^2}{2} := \ell.$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J,X)$.

Let $0 \leq t_1 < t_2 \leq T$, $y \in B_{\eta^*}$. Using (H4) again, we have

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] N(1+\|y(s)\|^\lambda) ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} N(1+\|y(s)\|^\lambda) ds \\
&\qquad + \frac{t_2^2 - t_1^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} N(1+\|y(s)\|^\lambda) ds \\
&\qquad + \|y_0^*\|(t_2-t_1) + \frac{\|y_T\|}{2}(t_2^2 - t_1^2) \\
&\quad \leq \frac{N(1+(\eta^*)^\lambda)}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right) \\
&\qquad + \frac{(t_2^2 - t_1^2) N(1+(\eta^*)^\lambda)}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} ds \\
&\qquad + \|y_0^*\|(t_2-t_1) + \frac{\|y_T\|}{2}(t_2^2 - t_1^2) \\
&\quad \leq \frac{N(1+(\eta^*)^\lambda)}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + \frac{T^{\alpha-2} N(1+(\eta^*)^\lambda)}{2\Gamma(\alpha-1)}(t_2^2 - t_1^2) \\
&\qquad + \|y_0^*\|(t_2-t_1) + \frac{\|y_T\|}{2}(t_2^2 - t_1^2).
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero, therefore F is equicontinuous.

Now, let $\{y_n\}$, $n = 1, 2, \cdots$ be a sequence on B_{η^*}, and

$$(Fy_n)(t) = (F_1 y_n)(t) + (F_2 y_n)(t) + (F_3 y)(t), \ t \in J.$$

where

$$\begin{aligned}
(F_1 y_n)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds, \ t \in J, \\
(F_2 y_n)(t) &= -\frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s, y_n(s)) ds, \ t \in J, \\
(F_3 y)(t) &= y_0 + y_0^* t + \frac{y_T}{2} t^2, \ t \in J.
\end{aligned}$$

In view of condition (H5) and Lemma 5, we know that $\overline{conv} K_1$ is compact. For any

$t^* \in J$,

$$\begin{aligned}(F_1y_n)(t^*) =&\frac{1}{\Gamma(\alpha)}\int_0^{t^*}(t^*-s)^{\alpha-1}f(s,y_n(s))ds\\ =&\frac{1}{\Gamma(\alpha)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-1}f\left(\frac{it^*}{k},y_n(\frac{it^*}{k})\right)\\ =&\frac{t^*}{\Gamma(\alpha)}\widetilde{\xi_{n1}},\end{aligned}$$

where

$$\widetilde{\xi_{n1}}=\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}(t^*-\frac{it^*}{k})^{\alpha-1}f\left(\frac{it^*}{k},y_n(\frac{it^*}{k})\right).$$

Since $\overline{conv}K_1$ is convex and compact, we know that $\widetilde{\xi_{n1}} \in \overline{conv}K_1$. Hence, for any $t^* \in J$, the set $\{(F_1y_n)(t^*)\}$ is relatively compact. From Lemma 6, every $\{(F_1y_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1y_{n_k})(t)\}$, $k=1,2,\cdots$ on J. Thus, the set $\{F_1y : y \in B_{\eta^*}\}$ is relatively compact.

Set

$$(\overline{F}_2y_n)(t) = -\frac{t^2}{2\Gamma(\alpha-2)}\int_0^t(t-s)^{\alpha-3}f(s,y_n(s))ds, t\in J.$$

For any $t^* \in J$,

$$\begin{aligned}(\overline{F}_2y_n)(t^*) =&-\frac{(t^*)^2}{2\Gamma(\alpha-2)}\int_0^{t^*}(t^*-s)^{\alpha-3}f(s,y_n(s))ds,\\ =&-\frac{(t^*)^2}{2\Gamma(\alpha-2)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-3}f\left(\frac{it^*}{k},y_n(\frac{it^*}{k})\right)\\ =&-\frac{(t^*)^3}{2\Gamma(\alpha-2)}\widetilde{\xi_{n2}},\end{aligned}$$

where

$$\widetilde{\xi_{n2}}=\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}(t^*-\frac{it^*}{k})^{\alpha-3}f\left(\frac{it^*}{k},y_n(\frac{it^*}{k})\right).$$

Since $\overline{conv}K_2$ is convex and compact, we know that $\widetilde{\xi_{n2}} \in \overline{conv}K_2$. Hence, for any $t^* \in J$, the set $\{(\overline{F}_2y_n)(t^*)\}$ is relatively compact. From Lemma 6, every $\{(\overline{F}_2y_n)(t)\}$ contains a uniformly convergent subsequence $\{(\overline{F}_2y_{n_k})(t)\}$, $k=1,2,\cdots$ on J. Particularly, $\{(F_2y_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_2y_{n_k})(t)\}$, $k=1,2,\cdots$ on J. Thus, the set $\{F_2y : y \in B_{\eta^*}\}$ is relatively compact.

Obviously, the set $\{F_3y : y \in B_{\eta^*}\}$ is relatively compact. As a result, the set $\{Fy : y \in B_{\eta^*}\}$ is relatively compact.

As a consequence of Steps 1–3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{y \in C(J,X) : y = \lambda^* F y, \text{ for some } \lambda^* \in [0,1]\}$$

is bounded.

Let $y \in E(F)$, then $y = \lambda^* F y$ for some $\lambda^* \in [0,1]$. Thus, for each $t \in J$, we have

$$\begin{aligned} y(t) =& \lambda^* \bigg(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s)) ds \\ & - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} f(s,y(s)) ds + y_0 + y_0^* t + \frac{y_T}{2} t^2 \bigg). \end{aligned}$$

For each $t \in J$, we have

$$\begin{aligned} \|y(t)\| \le& \|(Fy)(t)\| \\ \le& \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (1 + \|y(s)\|^\lambda) ds \\ & + \frac{T^2 N}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} (1 + \|y(s)\|^\lambda) ds \\ & + \|y_0\| + \|y_0^*\| T + \frac{\|y_T\|}{2} T^2 \\ \le& \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{T^2 N}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} ds \\ & + \|y_0\| + \|y_0^*\| T + \frac{\|y_T\|}{2} T^2 + \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\|^\lambda ds \\ & + \frac{T^2 N}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} \|y(s)\|^\lambda ds \\ \le& \frac{N T^\alpha}{\Gamma(\alpha+1)} + \frac{N T^\alpha}{2\Gamma(\alpha-1)} + \|y_0\| + \|y_0^*\| T + \frac{\|y_T\|}{2} T^2 \\ & + \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\|^\lambda ds \\ & + \frac{T^2 N}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3} \|y(s)\|^\lambda ds. \end{aligned}$$

By Lemma 8, there exists a $M^* > 0$ such that

$$\|y(t)\| \le M^*, \ t \in J.$$

Thus for every $t \in J$, we have

$$\|y\|_\infty \le M^*.$$

This shows that the set $E(F)$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the fractional BVP (1). □

In the following theorem we apply an nonlinear alternative of Leray-Schauder type in which condition (H4) is weakened.

(H4′) There exist a constant $\alpha_3 \in (0, \alpha - 2)$, a real valued function $\phi_f(t) \in L^{\frac{1}{\alpha_3}}(J, R_+)$ and continuous and nondecreasing $\psi : [0, +\infty) \to (0, +\infty)$ such that

$$\|f(t,u)\| \le \phi_f(t)\psi(\|u\|) \ \textit{for each } t \in J \textit{ and all } u \in X.$$

(H6) There exists a constant $N^* > 0$ such that

$$\frac{N^*}{\frac{\psi(N^*)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\vartheta}{\Gamma(\alpha)(\alpha-\alpha_3)^{1-\alpha_3}} + \frac{\psi(N^*)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\vartheta}{2\Gamma(\alpha-2)(\alpha-\alpha_3-2)^{1-\alpha_3}} + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2} > 1, \tag{6}$$

where $\vartheta = \|\phi_f\|_{L^{\frac{1}{\alpha_3}}(J,R_+)}$.

Theorem 5. *Assume that (H1), (H4'), (H5) and (H6) hold. Then the fractional BVP (1) has at least one solution on J.*

Proof. Consider the operator F defined in Theorem 4. It can be easily shown that F is continuous and completely continuous. Repeating the same process in Step 4 in Theorem 4, using (H4′) and Hölder inequality again, for each $t \in J$, we have

$$\begin{aligned}
\|y(t)\| \le & \|(Fy)(t)\| \\
\le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_f(s)\psi(\|y(s)\|)ds \\
& + \frac{T^2}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3}\phi_f(s)\psi(\|y(s)\|)ds \\
& + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2 \\
\le & \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_f(s)ds \\
& + \frac{T^2\psi(\|y\|_\infty)}{2\Gamma(\alpha-2)} \int_0^T (T-s)^{\alpha-3}\phi_f(s)ds \\
& + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2 \\
\le & \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_3}} ds\right)^{1-\alpha_3} \left(\int_0^t (\phi_f(s))^{\frac{1}{\alpha_3}} ds\right)^{\alpha_3} \\
& + \frac{T^2\psi(\|y\|_\infty)}{2\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\frac{\alpha-3}{1-\alpha_3}} ds\right)^{1-\alpha_3} \left(\int_0^T (\phi_f(s))^{\frac{1}{\alpha_3}} ds\right)^{\alpha_3} \\
& + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2 \\
\le & \frac{\psi(\|y\|_\infty)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\vartheta}{\Gamma(\alpha)(\alpha-\alpha_3)^{1-\alpha_3}} + \frac{\psi(\|y\|_\infty)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\vartheta}{2\Gamma(\alpha-2)(\alpha-\alpha_3-2)^{1-\alpha_3}} \\
& + \|y_0\| + \|y_0^*\|T + \frac{\|y_T\|}{2}T^2.
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{\frac{\psi(\|y\|_\infty)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\vartheta}{\Gamma(\alpha)(\alpha-\alpha_3)^{1-\alpha_3}}+\frac{\psi(\|y\|_\infty)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\vartheta}{2\Gamma(\alpha-2)(\alpha-\alpha_3-2)^{1-\alpha_3}}+\|y_0\|+\|y_0^*\|T+\frac{\|y_T\|}{2}T^2}\leq 1.$$

By (H6), there exists an $N^*>0$ such that $\|y\|_\infty \neq N^*$.

Let

$$U=\{y\in C(J,X):\|y\|_\infty<N^*\}.$$

The operator $F:\overline{U}\to C(J,X)$ is continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y=\lambda^*F(y)$, $\lambda^*\in[0,1]$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point $y\in\overline{U}$, which implies that the fractional BVP (1) has at least one solution $y\in C(J,X)$. □

5. Application

Let $X=L^2[0,1]$ be equipped with its natural norm and inner product defined respectively for all $u,v\in L^2[0,1]$ by

$$\|u\|_{L^2[0,1]}=\left(\int_0^1|u(x)|^2dx\right)^{\frac{1}{2}} \text{ and } \langle u,v\rangle=\int_0^1 u(x)\overline{v(x)}dx.$$

Example 1. *Let us consider the first boundary value problem:*

$$\begin{cases} {}^cD^\alpha y(t,x)=\frac{e^{-t}\cos(t,x)|y(t,x)|}{(1+ke^t)(1+|y(t,x)|)},\ x\in[0,1],\ t\in J=[0,T],\ \alpha\in(2,3),\ k>0,\\ y(0,x)=0,\ y'(0,x)=0,\ y''(T,x)=0,\ x\in[0,1],\\ y(t,0)=y(t,1)=0,\ t>0. \end{cases}\tag{7}$$

First of all, denote

$$y(\cdot)(x)=y(\cdot,x),\ \ \cos(\cdot,x)=\cos(\cdot)(x),\ \ f(\cdot,y(\cdot))(x)=\frac{e^{-\cdot}\cos(\cdot,x)|y(\cdot,x)|}{(1+ke^{\cdot})(1+|y(\cdot,x)|)}.$$

Then, the fractional BVP (1) is the abstract formulation of problem (7).

It is easy to see that f satisfies (H1). Moreover, for (H2) and (H3), we take $m(t)=h(t)=\frac{e^{-t}}{1+k}\in L^{\frac{1}{\beta}}(J,R_+)$, where $\beta\in(0,\alpha-2)$.

Choosing a suitable $T\geq 1$ and a big enough $k>0$ and a suitable $\alpha\in(2,3)$ and $\beta\in(0,\alpha-2)$, one can arrive at the following inequality

$$\Omega_{\alpha,T}=\left(\frac{M}{\Gamma(\alpha)}\times\frac{T^{\alpha-\beta}}{(\frac{\alpha-\beta}{1-\beta})^{1-\beta}}+\frac{M}{2\Gamma(\alpha-2)}\times\frac{T^{\alpha-\beta}}{(\frac{\alpha-\beta-2}{1-\beta})^{1-\beta}}\right)<1,$$

where $M=\left\|\frac{e^{-t}}{1+k}\right\|_{L^{\frac{1}{\beta}}(J,R_+)}$.

Obviously, all the assumptions in Theorem 3 are satisfied. Our results can be used to solve problem (7).

Example 2. *Let us consider the second boundary value problem*

$$\begin{cases} {}^cD^\alpha y(t,x) = \frac{t^{\nu+3}|y(t,x)|^\lambda}{1+e^t}, \ t \in J, \ \alpha \in (2,3), \\ \qquad\qquad \nu > -\alpha, \ \lambda \in [0, 1-\frac{1}{p}], \ 1 < p < \frac{1}{3-\alpha}, \\ y(0,x) = 0, \ y'(0,x) = 0, \ y''(T,x) = 0, \ x \in [0,1], \\ y(t,0) = y(t,1) = 0, \ t > 0. \end{cases} \tag{8}$$

Set

$$f_1(t,y) = \frac{t^{\nu+3}|y|^\lambda}{1+e^t}, \ (t,y) \in J \times X, \ \text{with } |y| \leq M_1.$$

Obviously, for all $y \in X := R$ and each $t \in J$, $|f_1(t,y)| \leq \frac{T^{\nu+3}}{2} |y|^\lambda$. Since $\nu > -\alpha$ and $\nu + 3 > -(\alpha - 3)$, it is not difficult to see

$$\int_0^t (t-s)^{\alpha-1} \frac{s^{\nu+3}|y(s)|^\lambda}{1+e^s} ds \leq M_1^\lambda \int_0^t (t-s)^{\alpha-1} s^{\nu+3} ds$$
$$\leq M_1^\lambda \frac{\Gamma(\alpha)\Gamma(\nu+4)}{\Gamma(\alpha+\nu+4)} t^{\alpha+\nu+3} \leq M_1^\lambda \frac{\Gamma(\alpha)\Gamma(\nu+4)}{\Gamma(\alpha+\nu+4)} T^{\alpha+\nu+3},$$

and

$$\int_0^t (t-s)^{\alpha-3} \frac{s^\nu|y(s)|^\lambda}{1+e^s} ds \leq M_1^\lambda \frac{\Gamma(\alpha-2)\Gamma(\nu+4)}{\Gamma(\alpha+\nu+2)} T^{\alpha+\nu+1}.$$

As a result, the sets

$$K_{11} = \left\{ (t-s)^{\alpha-1} \frac{s^\nu |y(s)|^\lambda}{1+e^s} : y \in C(J,X), s \in [0,t] \right\},$$
$$K_{12} = \left\{ (t-s)^{\alpha-3} \frac{s^\nu |y(s)|^\lambda}{1+e^s} : y \in C(J,X), s \in [0,t] \right\},$$

are uniformly bounded and equicontinuous, and for any $t^* \in J$ which implies that K_{11} and K_{12} are compact. Thus, all the assumptions in Theorem 4 are satisfied, our results can be applied to problem (8).

Example 3. *Let us consider the third boundary value problem*

$$\begin{cases} {}^cD^\alpha y(t,x) = \frac{t^{\nu+3}\psi(|y(t,x)|)}{1+e^t}, \ t \in J, \ \alpha \in (2,3), \ \nu > -\alpha, \\ y(0,x) = 0, \ y'(0,x) = 0, \ y''(T,x) = 0, \ x \in [0,1], \\ y(t,0) = y(t,1) = 0, \ t > 0, \end{cases} \tag{9}$$

where $\psi : [0,+\infty) \to (0,+\infty)$ is a continuous and nondecreasing function.

Set $\alpha_3 \in (0, \alpha-2)$, $\phi(t) = \frac{t^{\nu+3}}{1+e^t} \in L^{\frac{1}{\alpha_3}}(J, R_+)$ and $f_2(t,y) = \frac{t^{\nu+3}\psi(y)}{1+e^t}$, $(t,y) \in J \times X$, with $\|y\| \leq M_2$. Obviously, (H1), (H4′) and (H5) hold. By choosing a suitable T, the function ψ and a suitable $\alpha \in (2,3)$ and $\alpha_3 \in (0, \alpha-2)$ such that the following inequality

$$\frac{r}{\frac{\psi(r)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\|\frac{t^{\nu+3}}{1+e^t}\|_{L^{\frac{1}{\alpha_3}}(J,R_+)}}{\Gamma(\alpha)(\alpha-\alpha_3)^{1-\alpha_3}} + \frac{\psi(r)T^{\alpha-\alpha_3}(1-\alpha_3)^{1-\alpha_3}\|\frac{t^{\nu+3}}{1+e^t}\|_{L^{\frac{1}{\alpha_3}}(J,R_+)}}{2\Gamma(\alpha-2)(\alpha-\alpha_3-2)^{1-\alpha_3}}} > 1$$

has at least a positive solution. Thus, all the assumptions in Theorem 5 are satisfied, our results can be applied to problem (9).

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