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A new iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contractive mappings and its application

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Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the set of a solution of an equilibrium problem and the set of fixed points of finite family strict pseudo-contraction mappings in a real Hilbert space. Some strong convergence theorems are established using the iterative scheme. In the meantime, we successfully apply these Theorems to find a common element of the set of a solution of an equilibrium problem and the set of fixed points of finite family of non-expansive mappings in a real Hilbert space. The results in this paper improve the corresponding ones of [3, 6] and references therein.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let K be a nonempty closed convex subset of H. A mapping T of K into itself is called a *k-strict pseudo-contraction* mapping, if $\forall\, x, y \in K$, $\|Tx-Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2$, here $0 \le k < 1$, I denotes an identity operator. We use $F(T)$ to denote the set of fixed points of T (i.e. $F(T) = \{x \in K : Tx = x\}$).

In a real Hilbert space, it is clear that a $k-$strict pseudo-contraction mapping T is equivalent to

$$\langle Tx-Ty, x-y\rangle \le \|x-y\|^2 - \frac{1-k}{2}\|(I-T)x-(I-T)y\|^2, \tag{1}$$

i.e.

$$\frac{1-k}{2}\|(I-T)x-(I-T)y\|^2 \le \langle (I-T)x-(I-T)y, x-y\rangle. \tag{2}$$

Remark 1. *Notice that a mapping $T: K \to K$ is called a non-expansive mapping, if for all $x, y \in K$, $\|Tx-Ty\| \le \|x-y\|$. Therefore, a non-expansive mapping T is a $0-$strict pseudo-contractive mapping.*

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Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in K$ such that

$$F(x, y) \geq 0, \quad \forall\, y \in K. \tag{3}$$

We use $EP(F)$ to denote the set of solutions of problem (3). Many problems in physics, optimization and economics require some elements of $EP(F)$; see [2, 4, 5, 8, 10 - 12].

If $F(x, y) = \langle Ax, y - x\rangle$, here $A : K \to K$ is a nonlinear operator, then problem (3) becomes the following classical variational inequality problem:

Find $x \in K$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall\, y \in K. \tag{4}$$

This shows that problem (4) is a special case of problem (3). Several iterative methods have been proposed to solve the equilibrium problem; see [4, 5, 8, 10 - 12].

In 2009, L.C. Ceng et al. [3] constructed an iterative scheme for a $k-$strict pseudo-contractive mapping as follows:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Tu_n, \ n \geq 1, \end{cases} \tag{5}$$

where $\{\alpha_n\}$, $\{r_n\}$ are two nonnegative real number sequences satisfying $\{\alpha_n\} \subset [\alpha, \beta](\alpha, \beta \in (k, 1))$ and $\liminf_{n\to\infty} r_n > 0$. Under approximative conditions, L. C. Ceng et al. proved that $\{x_n\}$ and $\{u_n\}$ converge strongly (or weakly) to an element of $F(T) \bigcap EP(F)$. To be more precise, they proved the following theorems:

Theorem 1. *Let K be a nonempty closed convex subset of H and F a bifunction from $K \times K$ to $\boldsymbol{R}$ satisfying $(A1) - (A4)$. Let T be a $k-$strict pseudo-contraction mapping of K into K such that $F(T) \bigcap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be generated initially by an arbitrary element $x_1 \in K$ and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Tu_n, \quad n \geq 1, \end{cases} \tag{6}$$

where α_n, r_n satisfy the following conditions:

(i) $\{\alpha_n\} \subset (\alpha, \beta)$ *for some* $\alpha, \beta \in (k, 1)$*;*

(ii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $p \in \Omega = F(T) \bigcap EP(F)$, respectively.

Theorem 2. *Let K be a nonempty closed convex subset of H and F a bifunction from $K \times K$ to $\boldsymbol{R}$ satisfying $(A1) - (A4)$. Let T be a $k-$strict pseudo-contraction mapping of K into K such that $F(T) \bigcap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be generated initially by an arbitrary element $x_1 \in K$ and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Tu_n, \quad n \geq 1, \end{cases} \tag{7}$$

where α_n, r_n satisfy the following conditions:

(i) $\{\alpha_n\} \subset (\alpha, \beta)$ *for some* $\alpha, \beta \in (k, 1)$;

(ii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $p \in \Omega = F(T) \bigcap EP(F)$ *if only if*

$$\liminf_{n\to\infty} d(x_n, F(S) \bigcap EP(F)) = 0,$$

where $d(x_n, F(S) \bigcap EP(F)) = 0$ *denotes the metric distance from the point* x_n *to* $F(S) \bigcap EP(F)$.

It is necessary to point out that only a weak convergence theorem is obtained via iterative scheme (6). In order to obtain the strong convergence theorem for a $k-$strict pseudo-contraction mapping T, Jaiboo and Kumam [6] introduced a CQ iterative scheme as follows:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall\, y \in K, \\ y_n = \alpha_n u_n + (1-\alpha_n) T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \ \ n \geq 1, \end{cases} \tag{8}$$

where $x_0 \in H$, $C_1 = C$,α_n, r_n satisfy the following conditions:

(i) $\{\alpha_n\} \subset (\alpha, \beta)$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p = P_{F(T) \bigcap EP(F)} x_0$.

Remark 2. *We notice that the control coefficient excludes the natural choice of* $\alpha_n = 1/n$ *in the above iterative schemes (6) and (8).*

In this paper, we consult a new implicit iterative scheme (given in Section 3) for finding a common element of the set of the solution of an equilibrium problem and the set of fixed points of a finite family of strict pseudo-contraction mappings in a real Hilbert space. The aim is to make the control coefficient include the natural choice of $\alpha_n = 1/n$ in the new iterative scheme and obtain strong convergence theorems without using the metric projection method. Indeed, some strong convergence theorems are established using the iterative scheme. In the meantime, we successfully apply these Theorems to find a common element of the set of the solution of an equilibrium problem and the set of fixed points of a finite family of non-expansive mappings in a real Hilbert space. The results in this paper improve the corresponding ones of [3, 6] and references therein. Moreover, our requirements on the iterative parameters are also different from those in [3, 6].

2. Some lemmas and conclusions

For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbf{R}$. Let K be a closed convex subset of H, for each point $x \in H$, there exists a unique nearest point in K denoted by $P_K x$ such that

$$\|x - P_K x\| \le \|x - y\|, \ \forall\, y \in K.$$

P_K is called the *metric projection* of H to K. It is well-known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \ge \|P_K x - P_K y\|^2$$

for every $x, y \in H$. Moreover, $P_K x$ is characterized by the properties: for $x \in H$, and $z \in K$,

$$z = P_K(x) \Leftrightarrow \langle x - z, z - y\rangle \ge 0, \ \ \forall\, y \in K. \tag{9}$$

For solving the equilibrium problem (3) for a bifunction $F : K \times K \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ for all $x,\ y \in K$;

(A3) for each $x, y, z \in K$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \le F(x, y);$$

(A4) for each $x \in K, y \mapsto F(x, y)$ is convex and lower semi-continuous.

In what follows, we shall make use of the following lemmas.

Lemma 1 (Demicloseness principle, see [7]). *Let H be a real Hilbert space and K a closed convex subset of H. Let $T : K \to K$ be a $k-$strict pseudo-contraction mapping. Then the mapping $I - T$ is demiclosed on K, where I is the identity mapping, that is, $x_n \rightharpoonup x$ in K and $(I - T)x_n \to 0$ implies that $x \in K$ and $(I - T)x = 0$.*

Lemma 2 (See [9]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$, then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 3 (See [2]). *Let K be a nonempty convex subset of H and F be a bifunction of $K \times K$ into $\boldsymbol{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \textit{for all}\, y \in K.$$

Lemma 4 (See [4]). *Assume that F is a bifunction of $K \times K$ into $\boldsymbol{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ as follows:*

$$T_r(x) = \Big\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \forall\, y \in K \Big\}$$

for all $x \in H$. Then the following hold:

(1) T_r *is single-valued;*

(2) T_r *is firmly non-expansive, that is, for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 5 (See [13]). *Let* $\{a_n\}$ *be a sequence of nonnegative real number satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0.$$

If

(i) $\alpha_n \in [0, 1]$, $\sum \alpha_n = \infty$;

(ii) $\limsup \sigma_n \leq 0$;

(iii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$,

then $a_n \to 0$, *as* $n \to \infty$.

Lemma 6. *Let* H *be a real Hilbert space, then:*

$$\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y\rangle, \ \ \forall\, x, y \in H.$$

3. Strong convergence theorems

In this section, we establish some strong convergence theorems. Firstly, we give Proposition 1 and Proposition 2. They appeared implicitly in [14] and they can also be found in [1].

Proposition 1 (See [1, 14]). *Let* K *be a nonempty closed convex subset of* H. *For an arbitrary nonnegative integer* $r \geq 1$, *let* $\{T_i\}_{i=1}^r$ *be a finite family of* k_i-*strict pseudo-contraction mappings of* K *into* K, $0 \leq k_i < 1$. *Then for some nonnegative real number* λ_i, $0 \leq \lambda_i < 1$, $i = 1, 2, \cdots, r$, $\sum_{i=1}^r \lambda_i = 1$, $\sum_{i=1}^r \lambda_i T_i : K \to K$ *is a* $k-$*strict pseudo-contraction, where* $k = \max\{k_i : i = 1, 2, \cdots, r\}$.

Proposition 2 (See [1, 14]). *Let* K *be a nonempty closed convex subset of* H. *For an arbitrary nonnegative integer* $r \geq 1$, *let* $\{T_i\}_{i=1}^r$ *be a finite family of* k_i-*strict pseudo-contraction mappings of* K *into* K *such that* $F = \bigcap_{i=1}^r F(T_i) \neq \emptyset$, $0 \leq k_i < 1$. *Then for some nonnegative real number* λ_i, $0 \leq \lambda_i < 1$, $i = 1, 2, \cdots, r$, $\sum_{i=1}^r \lambda_i = 1$, $F(\sum_{i=1}^r \lambda_i T_i) = \bigcap_{i=1}^r F(T_i)$.

Proof. In order to have Proposition 2 more clear, we proceed to relate the proof based on the method and technique of [14], but this method is different from the one of [1]. Next, we start to prove it. It is clear that $\bigcap_{i=1}^r F(T_i) \subset F(\sum_{i=1}^r \lambda_i T_i)$.

On the other hand, let $p \in F(\sum_{i=1}^r \lambda_i T_i)$, then for $\forall\ p_1 \in \bigcap_{i=1}^r F(T_i)$, by (1) and $k = \max\{k_i : i = 1, 2, \cdots, r\}$,

$$\begin{aligned}\|p - p_1\|^2 &= \langle p - p_1, p - p_1\rangle = \langle \Sigma_{i=1}^r \lambda_i T_i p - p_1, p - p_1\rangle = \Sigma_{i=1}^r \lambda_i \langle T_i p - p_1, p - p_1\rangle \\ &\le \|p - p_1\|^2 - \frac{1-k}{2}\Sigma_{i=1}^r \|T_i p - p\|^2\end{aligned}$$

Hence, $T_i p = p$, $i = 1, 2, \cdots, r$, i.e. $p \in \bigcap_{i=1}^r F(T_i)$ and $\bigcap_{i=1}^r F(T_i) \supset F(\sum_{i=1}^r \lambda_i T_i)$. This completes the proof of Proposition 2. □

Next we give a parallel algorithm for a finite family of strict pseudo-contraction mappings and study its convergence property.

Theorem 3. *Let K be a nonempty closed convex subset of H and F a bifunction from $K \times K$ to $\boldsymbol{R}$ satisfying $(A1) - (A4)$. Let $\{T_i\}_{i=1}^r$ be a finite family of k_i-strict pseudo-contraction mappings of K into K such that $\Omega = EP(F) \bigcap (\bigcap_{i=1}^r F(T_i)) \neq \emptyset$, $0 \le k_i < 1$. Suppose that v and x_1 are arbitrary points in K, for some nonnegative real number λ_i, $0 \le \lambda_i < 1$, $i = 1, 2, \cdots, r$, $\sum_{i=1}^r \lambda_i = 1$, let $\{x_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\ y \in K, \\ x_{n+1} = \alpha_n v + (1 - \alpha_n) y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n z_n, \\ z_n = (1 - \sigma) u_n + \sigma \sum_{i=1}^r \lambda_i T_i u_n, \quad n \ge 1, \end{cases} \tag{10}$$

where $\sigma \in (0, 1 - k)$, $k = \max\{k_i : 1 \le i \le r\}$, coefficients α_n, β_n, r_n satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p \in \Omega$, respectively, where $p = P_\Omega(v)$.

Proof. Firstly, defining a mapping $S = \sum_{i=1}^r \lambda_i T_i$, then by Proposition 1 and Proposition 2 , S is a $k-$strict pseudo-contraction mapping and $F(S) = F(\sum_{i=1}^r \lambda_i T_i) = \bigcap_{i=1}^r F(T_i)$. Let $p = P_\Omega(v) \in \Omega$, since S is a $k-$strict pseudo-contraction mapping, it is true that

$$\begin{aligned}\|z_n - p\|^2 &= (1-\sigma)\|u_n - p\|^2 + \sigma\|Su_n - p\|^2 - \sigma(1-\sigma)\|Su_n - u_n\|^2 \\ &\le (1-\sigma)\|u_n - p\|^2 + \sigma\|u_n - p\|^2 + \sigma k\|Su_n - u_n\|^2 - \sigma(1-\sigma)\|Su_n - u_n\|^2 \\ &\le \|u_n - p\|^2. \end{aligned} \tag{11}$$

On the other hand, from Lemma 4 we have

$$\begin{aligned}\|u_n - p\| &= \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\|, \\ \|y_n - p\| &= \|(1 - \beta_n)(x_n - p) + \beta_n(z_n - p)\| \le \|x_n - p\|. \end{aligned} \tag{12}$$

Consequently, the next inequality holds:

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|v-p\|+(1-\alpha_n)\|y_n-p\| \\ &\le \alpha_n\|v-p\|+(1-\alpha_n)[(1-\beta_n)\|x_n-p\|+\beta_n\|z_n-p\|] \\ &\le \alpha_n\|v-p\|+(1-\alpha_n)\|x_n-p\| \\ &\le \max\{\|v-p\|,\|x_1-p\|\}. \end{aligned} \tag{13}$$

Inequality (13) establishes that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$ and $\{u_n\}$. Take a constant M such that

$$\{\|v\|,\|z_n\|,\|u_n\|,\|x_n\|\} \le M, \quad \forall\, n \ge 1.$$

Claim 1: $\|x_{n+1}-x_n\| \to 0 \quad (n\to\infty)$. For this purpose, let $\gamma_n = 1-(1-\alpha_n)(1-\beta_n)$, $v_n = \frac{x_{n+1}-x_n+\gamma_n x_n}{\gamma_n} = \frac{\alpha_n v+(1-\alpha_n)\beta_n z_n}{\gamma_n}$. Then

$$\begin{aligned} v_{n+1}-v_n &= \left(\frac{\alpha_{n+1}}{\gamma_{n+1}}-\frac{\alpha_n}{\gamma_n}\right)v+\frac{1-\alpha_{n+1}}{\gamma_{n+1}}\beta_{n+1}z_{n+1}-\frac{1-\alpha_n}{\gamma_n}\beta_n z_n \\ &= \left(\frac{\alpha_{n+1}}{\gamma_{n+1}}-\frac{\alpha_n}{\gamma_n}\right)v+\frac{1-\alpha_{n+1}}{\gamma_{n+1}}\beta_{n+1}(z_{n+1}-z_n) \\ &\quad +\left(\frac{1-\alpha_{n+1}}{\gamma_{n+1}}\beta_{n+1}-\frac{1-\alpha_n}{\gamma_n}\beta_n\right)z_n, \end{aligned} \tag{14}$$

which yields that

$$\begin{aligned} \|v_{n+1}-v_n\| \le &\left|\frac{\alpha_{n+1}}{\gamma_{n+1}}-\frac{\alpha_n}{\gamma_n}\right|M+\frac{(1-\alpha_{n+1})\beta_{n+1}\|z_{n+1}-z_n\|}{\gamma_{n+1}} \\ &+\left|\frac{(1-\alpha_{n+1})\beta_{n+1}}{\gamma_{n+1}}-\frac{(1-\alpha_n)\beta_n}{\gamma_n}\right|M, \end{aligned} \tag{15}$$

Now computing $\|z_{n+1}-z_n\|$, from (10), we have

$$\begin{aligned} \|z_{n+1}-z_n\|^2 =&\|(1-\sigma)(u_{n+1}-u_n)+\sigma(Su_{n+1}-Su_n)\|^2 \\ =&(1-\sigma)\|u_{n+1}-u_n\|^2+\sigma\|Su_{n+1}-Su_n\|^2 \\ &-\sigma(1-\sigma)\|u_{n+1}-u_n-(Su_{n+1}-Su_n)\|^2 \\ \le&\|u_{n+1}-u_n\|^2-\sigma(1-\sigma-k)\|u_{n+1}-u_n-(Su_{n+1}-Su_n)\|^2 \\ \le&\|u_{n+1}-u_n\|^2. \end{aligned} \tag{16}$$

By Lemma 4, we have $u_n = T_{r_n}x_n$ and

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, \tag{17}$$

$$F(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0, \tag{18}$$

Taking $y=u_{n+1}$ in (17) and $y=u_n$ in (18), then because F admits monotonicity, we add (17) to (18) and obtain

$$\langle u_n-u_{n+1},u_{n+1}-x_{n+1}-\frac{r_{n+1}}{r_n}(u_n-x_n)\rangle \ge 0.$$

Thus

$$\langle u_n - u_{n+1}, u_{n+1} - u_n + u_n + x_n - x_n - x_{n+1} - \frac{r_{n+1}}{r_n}(u_n - x_n)\rangle \geq 0,$$

which implies that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_n}2M, \tag{19}$$

Substituting (19) and (16) into (15), then

$$\begin{aligned}\|v_{n+1} - v_n\| \leq & \left|\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right| M + \frac{(1-\alpha_{n+1})\beta_{n+1}\|x_{n+1} - x_n\|}{\gamma_{n+1}} \\ & + \frac{(1-\alpha_{n+1})\beta_{n+1}|r_{n+1} - r_n|2M}{r_n\gamma_{n+1}} + \left|\frac{(1-\alpha_{n+1})\beta_{n+1}}{\gamma_{n+1}} - \frac{(1-\alpha_n)\beta_n}{\gamma_n}\right| M.\end{aligned} \tag{20}$$

It follows from (20) that $\limsup_{n\to\infty}\{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} \leq 0$, which shows that $\|v_n - x_n\| = 0$ by Lemma 2. Again from the definition of v_n, we obtain

$$\|x_{n+1} - x_n\| \to 0 (n \to \infty). \tag{21}$$

From the conditions $\lim_{n\to\infty} \alpha_n = 0$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and (10), (21), next conclusion holds:

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{22}$$

Let $p = P_\Omega(v) \in \Omega$. Since

$$\begin{aligned}\|u_n - p\|^2 = \|T_{r_n}x_n - p\|^2 \leq & \langle u_n - p, x_n - p\rangle \\ = & \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

we have

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Notice that $\|z_n - p\|^2 \leq \|u_n - p\|^2$, hence $\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2$ and

$$\begin{aligned}\|x_n - u_n\|^2 \leq & \|x_n - p\|^2 - \|z_n - p\|^2 \\ \leq & (\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|) \\ \leq & (\|x_n - p\| + \|z_n - p\|)\|x_n - z_n\|.\end{aligned}$$

Let $n \to \infty$, then $\|x_n - u_n\| \to 0$ and $\|z_n - u_n\| \leq \|z_n - x_n\| + \|x_n - u_n\| \to 0$. Consequently, from (10) we have that

$$\|u_n - Su_n\| \to 0 \text{ as } n \to \infty. \tag{23}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\{u_{n_i}\}$ converges weakly to a point $q \in K$. By Lemma 1, $q \in F(S) = \bigcap_{i=1}^r F(T_i)$. We claim $q \in EP(F)$, too. In order to see this, by $u_n = T_{r_n} x_n$, we know that

$$F(u_{n_i}, y) + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq 0, \ \forall\, y \in K. \tag{24}$$

It follows from (A2) that

$$\frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq F(y, u_{n_i}), \ \forall\, y \in K. \tag{25}$$

Notice the following facts $\frac{u_n - x_n}{r_n} \to 0$ and $u_{n_i} \rightharpoonup q$ as $n \to \infty$, and for each $x \in K$, $y \mapsto F(x, y)$ is lower semi-continuous. Then let $i \to \infty$ in inequality (25) and the next inequality holds immediately:

$$F(y, q) \leq 0, \ \forall\, y \in K. \tag{26}$$

For all $y \in K$, let $t \in (0,1)$ and $y_t = ty + (1-t)q$, then $y_t \in K (y \in K, \ q \in K)$ and $F(y_t, q) \leq 0$. So by (A1) and (A4), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, q) \leq tF(y_t, y), \ \forall\, y \in K,$$

i.e. $0 \leq F(y_t, y)$, $\forall\, y \in K$. Letting $t \to 0^+$, then

$$F(q, y) \geq 0, \ \forall\, y \in K. \tag{27}$$

Inequality (27) shows that $q \in EP(F)$.

Choosing a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned} \limsup_{n\to\infty} \langle v - p, x_n - p \rangle &= \lim_{n\to\infty} \langle v - p, x_{n_j} - p \rangle \\ &= \lim_{n\to\infty} \langle v - p, x_{n_j} - u_{n_j} \rangle + \lim_{n\to\infty} \langle v - p, u_{n_j} - p \rangle \\ &= \lim_{n\to\infty} \langle v - p, u_{n_j} - p \rangle \end{aligned} \tag{28}$$

Since $\{u_{n_j}\}$ is bounded, from what we have discussed above, we know that there exists a subsequence of $\{u_{n_j}\}$ such that it converges weakly to a point of Ω. Without loss of generality, we may assume that $\{u_{n_j}\}$ converges weakly to $q \in \Omega$. Then for $p = P_\Omega(v)$, from (9) and (28)

$$\limsup_{n\to\infty} \langle v - p, x_n - p \rangle = \lim_{n\to\infty} \langle v - p, u_{n_j} - p \rangle = \lim_{n\to\infty} \langle v - p, q - p \rangle \leq 0. \tag{29}$$

Now, we start to prove that $\{u_n\}$ and $\{x_n\}$ converge strongly to $p = P_\Omega(v)$. It follows from (10), (12) and Lemma 6 that

$$\begin{aligned} \|x_{n+1} - p\|^2 =& \|\alpha_n (v - p) + (1 - \alpha_n)(y_n - p)\|^2 \\ \leq& (1 - \alpha_n)\|y_n - p\|^2 + 2\alpha_n \langle v - p, x_{n+1} - p \rangle \\ \leq& (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle v - p, x_{n+1} - p \rangle. \end{aligned} \tag{30}$$

Applying Lemma 5 and condition (i) to (30) implies that $\{x_n\}$ converges strongly to $p = P_\Omega(v)$, so is $\{u_n\}$ by $\|u_n - p\| \leq \|x_n - p\|$. This completes the proof of Theorem 3. □

If $r=1$, then the next Corollary 1 is obtained immediately by Theorem 3.

Corollary 1. *Let K be a nonempty closed convex subset of H and F a bifunction from $K\times K$ to $\boldsymbol{R}$ satisfying $(A1)-(A4)$. Let $T: K\to K$ be a $k-$strict pseudo-contractive mapping such that $\Omega = F(T)\bigcap EP(F)\neq\emptyset$. Suppose that v and x_1 are two arbitrary points in K. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \ \ \forall\, y\in K,\\ x_{n+1}=\alpha_n v+(1-\alpha_n)y_n,\\ y_n=(1-\beta_n)x_n+\beta_n z_n,\\ z_n=(1-\sigma)u_n+\sigma T u_n, \ \ n\geq 1, \end{cases} \tag{31}$$

where $\sigma\in(0,1-k)$, coefficient $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy

(i) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(iii) $\liminf_{n\to\infty}r_n>0$, $\lim_{n\to\infty}|r_{n+1}-r_n|=0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p\in\Omega$, respectively, where $p=P_{\Omega}(v)$.

4. Application of Theorem 3

Applying Theorem 3, we may consult a parallel algorithm for a finite family of non-expansive mappings and use it to find a common element of the set of the solution of the equilibrium problem (3) and the set of fixed points of a finite family of non-expansive mappings. Please see the following Theorem 4.

Theorem 4. *Let K be a nonempty closed convex subset of H and F a bifunction from $K\times K$ into $\boldsymbol{R}$ satisfying $(A1)-(A4)$. $\{T_i\}_{i=1}^{r}$ are a finite family of non-expansive mappings of K into K such that $\Omega=EP(F)\bigcap(\bigcap_{i=1}^{r}F(T_i))\neq\emptyset$. Suppose that v and x_1 are two arbitrary points in K, for some nonnegative real number λ_i, $0\leq\lambda_i<1$, $i=1,2,\cdots,r$, $\sum_{i=1}^{r}\lambda_i=1$, let $\{x_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \ \ \forall\, y\in K,\\ x_{n+1}=\alpha_n v+(1-\alpha_n)y_n,\\ y_n=(1-\beta_n)x_n+\beta_n z_n,\\ z_n=(1-\sigma)u_n+\sigma \sum_{i=1}^{r}\lambda_i T_i u_n, \ \ n\geq 1, \end{cases} \tag{32}$$

where $\sigma\in(0,1)$, coefficients α_n,β_n,r_n satisfy the following conditions:

(i) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(iii) $\liminf_{n\to\infty}r_n>0$, $\lim_{n\to\infty}|r_{n+1}-r_n|=0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p\in\Omega$, respectively, where $p=P_{\Omega}(v)$.

Proof. Since a non-expansive mapping is a $0-$strict pseudo-contractive mapping, from Proposition 1 we know that $\sum_{i=1}^{r} \lambda_i T_i$ is a $0-$strict pseudo-contractive mapping. Hence Theorem 4 holds immediately by Theorem 3. This completes the proof of Theorem 4. $\square$

If $r = 1$, then the next Corollary 2 holds immediately:

Corollary 2. *Let K be a nonempty closed convex subset of H and F a bifunction from $K \times K$ to $\boldsymbol{R}$ satisfying $(A1)-(A4)$. T is a non-expansive mapping of K into K such that $\Omega = EP(F) \bigcap F(T) \neq \emptyset$. Suppose that v and x_1 are two arbitrary points in K. Let $\{x_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in K, \\ x_{n+1} = \alpha_n v + (1 - \alpha_n) y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n z_n, \\ z_n = (1 - \sigma) u_n + \sigma T u_n, \quad n \geq 1, \end{cases} \tag{33}$$

where $\sigma \in (0, 1)$, coefficients α_n, β_n, r_n satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*

(iii) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p \in \Omega$, respectively, where $p = P_\Omega(v)$.

Example 1. *Let $H = \mathbf{R}$ and $K = [0, 1]$. Let $F(x, y) = y - x, \forall\, x, y \in [0, 1]$, and $T_1 = \frac{1}{2}x^2$, $T_2 x = \frac{1}{3}x^3$ for all $x \in [0, 1]$. It is easy to verify that F satisfies (A1)-(A4) and $EP(F) = F(T_1) = F(T_2) = \{0\}$. Hence, $EP(F) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset$. Also, it is easy to verify that T_1, T_2 are two nonexpansive mappings. Thus, we may use the algorithm from Theorem 4 to find their common element in $EP(F) \bigcap F(T_1) \bigcap F(T_2)$.*

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