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The extension of Willmore's method into 4-space

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Abstract. We present a new method for finding the Frenet vectors and the curvatures of the transversal intersection curve of three implicit hypersurfaces by extending the method of Willmore into four-dimensional Euclidean space.
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1. Introduction

It is known that all geometric properties of a parametrically given regular space curve can be computed, [3, 10 - 12]. But, if the space curve is given as an intersection curve and if the parametric equation for the curve cannot readily be obtained, then the computations of curvatures and Frenet vectors become harder. For that reason, various methods have been given for computing the Frenet apparatus of the intersection curve of two surfaces in Euclidean 3-space, and of three hypersurfaces in Euclidean 4-space. The intersection of (hyper)surfaces can be either transversal or tangential. In the case of transversal intersection, in which the normal vectors of the (hyper)surfaces are linearly independent, the tangential direction at an intersection point can be computed simply by the vector product of the normal vectors of (hyper)surfaces. Hartmann [7] provides formulas for computing the curvature of the intersection curves. Ye and Maekawa [15] give algorithms for computing the differential geometry properties of both transversal and tangential intersection curves of two surfaces for all types of intersection problems. Aléssio [1] describes how to obtain the unit tangent, principal normal, binormal, curvature and the torsion of two transversally intersecting implicit surfaces using the implicit function theorem. Goldman [6] derives curvature formulas for space curves defined by the intersection of two implicit surfaces and also gives the formula of the first curvature of the curve given as an intersection of n implicit hypersurfaces. Willmore [14] introduces a derivative operator to provide an algorithm for finding the curvature and the torsion of implicitly defined space curves. Recently, Aléssio [2] has studied differential geometry properties of the transversal intersection curve of three implicit hypersurfaces

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in 4-space using the implicit function theorem. He also generalizes the method of Ye and Maekawa into 4-space by taking a different base of the normal hyperplane. Besides, Düldül [4] computes the Frenet apparatus of the transversal intersection curve of three parametric hypersurfaces in Euclidean 4-space.

In this paper, we study differential geometry properties of implicitly defined space curves in four-dimensional Euclidean space. The purpose of this paper is to extend the method of Willmore into 4-space and obtain the Frenet vectors and the curvatures of the intersection curve of three transversally intersecting implicit hypersurfaces. Also, we give the extension of the well known Joachimsthal's theorem.

As it is mentioned above, for such an intersection problem in 4-dimensional Euclidean space, Aléssio proposed two different methods. For that reason, compared with these methods, one can ask:

- What is the benefit of your new method?, or
- Is this method better than the others?

First of all, we do not claim that our method is better than the others. But, our extension may have some advantages with respect to the others in 4-space. According to our method, we do not need to

- test the Jacobian determinants whether they are zero or not,
- compute any Hessian matrices and their derivatives,
- solve any linear equation systems with three unknowns,
- obtain the fourth derivative vector of the intersection curve

at the intersection point. Our extension uses only scalar and ternary products in 4-space. This method can be generalized into n-space (at least for calculating the curvatures k_1 and k_2) for the transversal intersection of $n-1$ implicit hypersurfaces.

In Section 2, we introduce the definition of a ternary product of three vectors and essential reviews of differential geometry of curves and surfaces. Section 3 includes the extension of Willmore's method developed for finding the Frenet apparatus of the intersection curve of three transversally intersecting implicit hypersurfaces. Finally, we give the extended Joachimsthal's theorem. As an application of our method, an example is included at the end of the paper.

2. Preliminaries

Definition 1. *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of four-dimensional Euclidean space E^4. The ternary product (or vector product) of the vectors $\mathbf{x} = \sum_{i=1}^4 x_i \mathbf{e_i}$, $\mathbf{y} = \sum_{i=1}^4 y_i \mathbf{e_i}$, and $\mathbf{z} = \sum_{i=1}^4 z_i \mathbf{e_i}$ is defined by [8, 13]*

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

The ternary product $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ yields a vector that is orthogonal to $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$.

Let $M \subset E^4$ be a regular hypersurface given implicitly by $f(x, y, z, w) = 0$ and let $\alpha : I \subset \mathbf{R} \to M$ be an arbitrary curve with arc length parametrization. If $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given by [5]

$$\begin{cases} \mathbf{t}' &= k_1\mathbf{n}, \\ \mathbf{n}' &= -k_1\mathbf{t} + k_2\mathbf{b}_1, \\ \mathbf{b}_1' &= -k_2\mathbf{n} + k_3\mathbf{b}_2, \\ \mathbf{b}_2' &= -k_3\mathbf{b}_1, \end{cases} \tag{1}$$

where $\mathbf{t}, \mathbf{n}, \mathbf{b}_1$, and $\mathbf{b}_2$ denote the tangent, the principal normal, the first binormal, and the second binormal vector fields, respectively; $k_i, (i = 1, 2, 3)$ the ith curvature functions of the curve α (See [2] for the determination of the Frenet frame). Also, the unit normal vector of M is given by $\mathbf{N} = \frac{\nabla f}{\|\nabla f\|}$, where $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right)$.

Using the Frenet formulas, for the derivatives of the curve α, we have

$$\alpha' = \mathbf{t}, \quad \alpha'' = k_1\mathbf{n}, \quad \alpha''' = -k_1^2\mathbf{t} + k_1'\mathbf{n} + k_1k_2\mathbf{b}_1. \tag{2}$$

3. The extension of Willmore's method into 4-space

Let M_1, M_2, and M_3 be three regular transversally intersecting hypersurfaces given by implicit equations $f_i(x, y, z, w) = 0$, $(i = 1, 2, 3)$ and let $\alpha(s) = \Big(x(s), y(s), z(s), w(s)\Big)$ be their intersection curve with arc length parametrization. Since the unit tangent vector is orthogonal to the normal vectors of hypersurfaces, it is parallel to $\nabla f_1 \otimes \nabla f_2 \otimes \nabla f_3 = \mathbf{h}$, i.e., $\lambda\alpha' = \mathbf{h}$. Then, $\lambda x' = h_1, \lambda y' = h_2, \lambda z' = h_3, \lambda w' = h_4$ and $\lambda^2 = \langle \mathbf{h}, \mathbf{h}\rangle$. If we denote

$$\Delta = \lambda\frac{d}{ds} = \left(h_1\frac{\partial}{\partial x} + h_2\frac{\partial}{\partial y} + h_3\frac{\partial}{\partial z} + h_4\frac{\partial}{\partial w}\right),$$

we may write $\Delta\alpha = \mathbf{h}$. If we apply Δ to $\mathbf{h} = \lambda\mathbf{t}$, we obtain

$$\Delta\mathbf{h} = \lambda\lambda'\mathbf{t} + \lambda^2k_1\mathbf{n}. \tag{3}$$

Then, since the unit tangent vector $\mathbf{t} = \mathbf{h}/\lambda$ is known, we have

$$\lambda' = \langle\Delta\mathbf{h}, \mathbf{t}\rangle/\lambda, \quad \langle\Delta\mathbf{h}, \Delta\mathbf{h}\rangle = (\lambda\lambda')^2 + \lambda^4k_1^2.$$

Hence, the first curvature of the intersection curve is obtained by

$$k_1 = \frac{1}{\lambda^2}\sqrt{\langle\Delta\mathbf{h}, \Delta\mathbf{h}\rangle - (\lambda\lambda')^2}. \tag{4}$$

Substituting the values of λ' and k_1 into (3) yields the principal normal vector of the intersection curve as

$$\mathbf{n} = \frac{1}{\lambda^2k_1}\left(\Delta\mathbf{h} - \lambda\lambda'\mathbf{t}\right). \tag{5}$$

If we use (2), we obtain the curvature vector α''. Operating Δ on (3) gives

$$\Delta^2\mathbf{h} = \left(\lambda\left(\lambda'\right)^2 + \lambda^2\lambda'' - \lambda^3 k_1^2\right)\mathbf{t} + \left(3\lambda^2\lambda' k_1 + \lambda^3 k_1'\right)\mathbf{n} + \left(\lambda^3 k_1 k_2\right)\mathbf{b}_1. \tag{6}$$

The ternary product of $\mathbf{h}$, $\Delta\mathbf{h}$ and $\Delta^2\mathbf{h}$ yields

$$\mathbf{h} \otimes \Delta\mathbf{h} \otimes \Delta^2\mathbf{h} = \lambda^6 k_1^2 k_2 \mathbf{b}_2.$$

Thus, the second binormal vector is obtained by

$$\mathbf{b}_2 = \frac{\mathbf{h} \otimes \Delta\mathbf{h} \otimes \Delta^2\mathbf{h}}{||\mathbf{h} \otimes \Delta\mathbf{h} \otimes \Delta^2\mathbf{h}||} \tag{7}$$

and the first binormal vector is computed by

$$\mathbf{b}_1 = \mathbf{b}_2 \otimes \mathbf{t} \otimes \mathbf{n}. \tag{8}$$

Using $\mathbf{b}_1$ and (6), the second curvature of the intersection curve is given by

$$k_2 = \frac{1}{\lambda^3 k_1}\langle \Delta^2\mathbf{h}, \mathbf{b}_1\rangle. \tag{9}$$

We need to find k_1' to obtain the third derivative of the intersection curve. By taking the scalar product of both hand sides of (6) with $\mathbf{n}$, we obtain

$$\langle \Delta^2\mathbf{h}, \mathbf{n}\rangle = 3\lambda^2\lambda' k_1 + \lambda^3 k_1' \tag{10}$$

from which we can find k_1'. Hence, α''' can be found by (2).

If we apply Δ to (6), we get

$$\Delta^3\mathbf{h} = \{...\}\mathbf{t} + \{...\}\mathbf{n} + \{...\}\mathbf{b}_1 + \lambda^4 k_1 k_2 k_3 \mathbf{b}_2.$$

Finally, the third curvature of the intersection curve is computed by

$$k_3 = \frac{1}{\lambda^4 k_1 k_2}\langle \Delta^3\mathbf{h}, \mathbf{b}_2\rangle. \tag{11}$$

Remark 1. *If the first curvature k_1 is zero at a point, then (5) does not define the principal normal vector. (If k_1 is identically zero, then the intersection curve is a straight line. In this case, the Frenet frame of the curve is not defined.)*

We assume here that k_1 vanishes at a point Q. In this case $\mathbf{h}$ and $\Delta\mathbf{h}$ are linearly dependent at Q. Let $S_i = \left\{\mathbf{h}, \Delta^i\mathbf{h}\right\}$, $(i = 2, 3, ...)$. If r is the first value of i which makes the set S_i linearly independent at Q, then we have $k_1 = k_1' = ... = k_1^{(r-2)} = 0$ and $k_1^{(r-1)} \neq 0$. In this case, we have

$$\Delta^r\mathbf{h} = \mu\mathbf{t} + \lambda^{r+1} k_1^{(r-1)}\mathbf{n},$$

where μ depends on λ and its derivatives up to rth order. Hence, the last equation defines the principal normal vector at Q.

Let $D_j = \left\{ \mathbf{h}, \Delta^r \mathbf{h}, \Delta^{r+j} \mathbf{h} \right\}$, $(j = 1, 2, ...)$, and let s be the first value of j which makes the set D_j linearly independent at Q. In this case, the second and the first binormal vectors of the intersection curve can be computed by

$$\mathbf{b}_2 = \frac{\mathbf{h} \otimes \Delta^r \mathbf{h} \otimes \Delta^{r+s} \mathbf{h}}{||\mathbf{h} \otimes \Delta^r \mathbf{h} \otimes \Delta^{r+s} \mathbf{h}||}$$

and $\mathbf{b}_1 = \mathbf{b}_2 \otimes \mathbf{t} \otimes \mathbf{n}$, respectively (If the second curvature k_2 is identically zero, then the curve is a planar curve in 4-space, that is, $\mathbf{b}_1$ and $\mathbf{b}_2$ are not defined).

If $s = 1$, then $k_2 \neq 0$. In this case, the second curvature is given by

$$k_2 = \frac{\langle \Delta^{r+1} \mathbf{h}, \mathbf{b}_1 \rangle}{r \lambda^{r+2} k_1^{(r-1)}}$$

and the third curvature is obtained by

$$k_3 = \frac{\langle \Delta^{r+2} \mathbf{h}, \mathbf{b}_2 \rangle}{\left(\sum_{i=1}^{r} i \right) \lambda^{r+3} k_1^{(r-1)} k_2}$$

If $s > 1$, then we have $k_2 = k_2' = ... = k_2^{(s-2)} = 0$ and $k_2^{(s-1)} \neq 0$. In this case, substituting these values into $\Delta^{r+s+1} \mathbf{h}$ yields the third curvature k_3.

4. The extended Joachimsthal's theorem

Theorem 1. *Suppose that the hypersurfaces M_1, M_2, and M_3 intersect through a smooth curve $\alpha = \alpha(s)$ in Euclidean 4-space and let θ_{ij} be the angle between the linearly independent unit normal vectors N_i and N_j (restricted to α) of the hypersurfaces. Assume that α is a line of curvature on M_i and M_j. Then, α is a line of curvature on M_k if and only if θ_{ik} and θ_{jk} are constant along α.*

Proof. Let α be a line of curvature on M_1 and M_2. Then,

$$\frac{d}{ds} \mathbf{N}_i = \mathbf{N}_i' = -\lambda_i \mathbf{t}, \quad i = 1, 2 \tag{12}$$

where λ_i are principal curvatures on M_i.

If we differentiate $\langle \mathbf{N}_i, \mathbf{N}_3 \rangle = \cos \theta_{i3}$ with respect to "s" and use (12), we get

$$\langle \mathbf{N}_i, \mathbf{N}_3' \rangle = -\frac{d\theta_{i3}}{ds} \sin \theta_{i3}. \tag{13}$$

($\Rightarrow$) Let α be a line of curvature on M_3. Hence, we also have $\mathbf{N}_3' = -\lambda_3 \mathbf{t}$. Since $\theta_{i3} \neq 0$, π, using (13) we obtain θ_{i3} =constant $(i = 1, 2)$ along α.

($\Leftarrow$) Let $\theta_{i3} (i = 1, 2)$ be constant along α. From (13) we get $\mathbf{N}_3' \perp \mathbf{N}_i$ which yield $\mathbf{N}_3' \parallel \mathbf{t}$, i.e. $\mathbf{N}_3' = \lambda \mathbf{t}$ for some λ. In other words, α is a line of curvature on M_3. $\square$

3D scalar fields (points in three-dimensional space with associated function values) and 3D objects in motion take an important role in many real world applications. Put into 4-space, the 3D scalar field can be considered as an implicit hypersurface defined by the equation $w - f(x,y,z) = 0$. Also, given a surface in 3-space with its implicit equation $g(x,y,z) = 0$ and allowing this surface to move on a trajectory, a continuous family of surfaces is generated, each of which may be specified by a particular value of time, "t". This family of surfaces $G(x,y,z,t) = 0$ may be thought of as a hypersurface in four-dimensional space [9, 16].

Now, let us give an example in which the hypersurfaces are generated by moving 2-surfaces in 3-space.

Example 1. *Let us consider the surfaces*

$$S_1 : y^3 + z = 0, \quad S_2 : y^2 - x = 0, \quad S_3 : y - 1 = 0$$

in Euclidean 3-space. If the surfaces S_1 and S_2 move in the positive y-direction and the surface S_3 moves in the negative y-direction, we get the corresponding hypersurfaces in 4-space as (the time parameter is denoted by w)

$$\begin{aligned} M_1 : f_1(x,y,z,w) =& (y-w)^3 + z = 0, \\ M_2 : f_2(x,y,z,w) =& (y-w)^2 - x = 0, \\ M_3 : f_3(x,y,z,w) =& y + w - 1 = 0. \end{aligned}$$

Let us find the Frenet vectors and the curvatures of the intersection curve at the intersection point $P = (1,0,1,1)$. Since

$$\begin{aligned} \nabla f_1 =& \Big(0, 3(y-w)^2, 1, -3(y-w)^2\Big), \\ \nabla f_2 =& \Big(-1, 2(y-w), 0, -2(y-w)\Big), \\ \nabla f_3 =& (0,1,0,1), \end{aligned}$$

we get

$$\mathbf{h} = \Big(-4(y-w), -1, 6(y-w)^2, 1\Big). \tag{14}$$

If we apply Δ to (14) consecutively, we obtain

$$\Delta\mathbf{h} = \lambda\Bigg(-4(y'-w'), 0, 12(y-w)(y'-w'), 0\Bigg), \tag{15}$$

$$\begin{aligned} \Delta^2\mathbf{h} =& \lambda\lambda'\Bigg(-4(y'-w'), 0, 12(y-w)(y'-w'), 0\Bigg) \\ &+ \lambda^2\Bigg(-4(y''-w''), 0, 12(y'-w')^2 + 12(y-w)(y''-w''), 0\Bigg), \end{aligned} \tag{16}$$

$$\Delta^3\mathbf{h} = \left(\lambda(\lambda')^2 + \lambda^2\lambda''\right)\Big(-4(y'-w'), 0, 12(y-w)(y'-w'), 0\Big)$$
$$+ 3\lambda^2\lambda'\Big(-4(y''-w''), 0, 12(y'-w')^2 + 12(y-w)(y''-w''), 0\Big) \tag{17}$$
$$+ \lambda^3\Big(-4(y'''-w'''), 0, 36(y'-w')(y''-w'') + 12(y-w)(y'''-w'''), 0\Big).$$

Then, we have

$$\mathbf{h}(P) = (4, -1, 6, 1), \lambda(P) = \pm 3\sqrt{6}.$$

So, for $\lambda(P) = 3\sqrt{6}$, *we find the unit tangent vector of the intersection curve as*

$$\mathbf{t}(P) = \frac{\mathbf{h}(P)}{\lambda(P)} = \left(\frac{4}{3\sqrt{6}}, \frac{-1}{3\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{3\sqrt{6}}\right). \tag{18}$$

Using (15) and (18), we obtain

$$\Delta\mathbf{h}(P) = (8, 0, 24, 0)$$

which yields

$$\lambda'(P) = \frac{88}{27}, \quad k_1(P) = \frac{8\sqrt{21}}{243}.$$

Substituting the obtained results into (5) gives the principal normal vector

$$\mathbf{n}(P) = \left(\frac{-17}{6\sqrt{21}}, \frac{11}{6\sqrt{21}}, \frac{5}{2\sqrt{21}}, \frac{-11}{6\sqrt{21}}\right). \tag{19}$$

Hence, we get $\alpha''(P) = \left(\frac{-68}{729}, \frac{44}{729}, \frac{20}{243}, \frac{-44}{729}\right)$. *If we substitute the coordinates of* $\alpha'(P)$ *and* $\alpha''(P)$ *into (16), we have*

$$\Delta^2\mathbf{h}(P) = (0, 0, 48, 0).$$

Thus, the second and the first binormal vectors of the intersection curve are found from (7) and (8) as

$$\mathbf{b}_2(P) = \left(0, \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), \quad \mathbf{b}_1(P) = \left(\frac{3}{2\sqrt{7}}, \frac{3}{2\sqrt{7}}, \frac{-1}{2\sqrt{7}}, \frac{-3}{2\sqrt{7}}\right). \tag{20}$$

Using $\mathbf{b}_1$ *and (9), we find the second curvature as* $k_2(P) = \frac{-3\sqrt{2}}{28}$. *Substituting the known values into (10) yields* $k_1'(P) = \frac{-3308}{6561\sqrt{14}}$. *Then, we obtain the third derivative of the intersection curve from (2) as*

$$\alpha'''(P) = \left(\frac{8384}{59049\sqrt{6}}, \frac{-64568}{59049\sqrt{6}}, \frac{-29624}{137781\sqrt{6}}, \frac{64568}{59049\sqrt{6}}\right).$$

On the other hand, the scalar product of both hand sides of (10) with $\mathbf{t}$ *gives us*

$$\langle \Delta\mathbf{h}, \mathbf{t}\rangle = \lambda\left(\lambda'\right)^2 + \lambda^2\lambda'' - \lambda^3 k_1^2.$$

Hence, we get $\left(\lambda\,(\lambda')^2+\lambda^2\lambda''\right)(P)=912\sqrt{6}$. *If we substitute these results into (17), we find*

$$\Delta^3\mathbf{h}(P)=\left(\frac{2620160}{729},0,\frac{2620160}{243},0\right)$$

which yields $k_3(P)=0$.

5. Conclusion

In this paper, the method of Willmore which provides an algorithm for computing the curvature and torsion of implicitly defined space curves is extended into four-dimensional Euclidean space. We show that all differential geometry properties of the transversal intersection curve of three implicit hypersurfaces can also be obtained by our new method besides the methods of Aléssio. Our new method may have some advantages in relation to the other methods. But, the advantages with respect to the others in 4-space can be reversed depending on the implicit equations of the intersecting hypersurfaces.

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