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Congruences of three multipartition functions

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Abstract. By means of the multi–section series method, six congruence relations and their corresponding generating functions are established for three functions of multipartitions.

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1. Introduction and motivation

For a natural number m and two indeterminates q, x with $|q| < 1$, the q–shifted factorials of infinite order are defined by

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n) \quad \text{and} \quad E_m = \prod_{k=1}^{\infty}(1-q^{km}).$$

The multi–parameter expression for the former will be abbreviated as

$$[\alpha,\beta,\cdots,\gamma;q]_\infty = (\alpha;q)_\infty(\beta;q)_\infty\cdots(\gamma;q)_\infty.$$

Let $p(n)$ be the unrestricted partition function defined by the generating function

$$\sum_{n=0}^{\infty} p(n)q^n \;=\; \frac{1}{(q;q)_\infty} = \frac{1}{E_1}.$$

Ramanujan [14] discovered the following three congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}; \end{aligned}$$

and the two beautiful identities (cf. Hardy [12, Chapter VI])

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$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} = 5\frac{E_5^5}{E_1^6},$$
$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8} = 7\frac{E_7^3}{E_1^4} + 49q\frac{E_7^7}{E_1^8}.$$

For complete proofs of these generating functions and their implications, the reader can consult the book by Chu and DiClaudio [8, Chapter H].

By means of the generating function approach devised by Atkin [2] and then employed by Hirschhorn–Hunt [13], Chan and Cooper [5] recently proved a family of congruences for $c(n)$ modulo powers of 2. The two simplest cases of them are the following congruences

$$c(2n+1) \equiv 0 \pmod 2,$$
$$c(4n+3) \equiv 0 \pmod 4;$$

where the multipartition function $c(n)$ is defined by the power series expansion

$$\sum_{n\geq 0} c(n)q^n = \frac{1}{(q;q)_\infty^2(q^3;q^3)_\infty^2} = \frac{1}{E_1^2E_3^2}.$$

Inspired by these relations, in this paper we shall investigate congruence properties for the integer sequences $\{C(n), \mathfrak{C}(n), D(n)\}$ defined respectively by the generating functions

$$\sum_{n=0}^{\infty} C(n)q^n = \frac{1}{(q;q)_\infty(q^3;q^3)_\infty} = \frac{1}{E_1E_3},$$
$$\sum_{n=0}^{\infty} \mathfrak{C}(n)q^n = (-q;q)_\infty^2(-q^3;q^3)_\infty^2 = \frac{E_2^2E_6^2}{E_1^2E_3^2},$$
$$\sum_{n=0}^{\infty} D(n)q^n = \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty^4} = \frac{E_3^4}{E_1^4};$$

which can be interpreted in terms of *multipartitions* (cf. Andrews [1]). We shall show that these sequences satisfy the following congruence relations

$$C(4n+2) \equiv 0 \pmod 2,$$
$$C(4n+3) \equiv 0 \pmod 2;$$
$$\mathfrak{C}(2n+1) \equiv 0 \pmod 2,$$
$$\mathfrak{C}(4n+3) \equiv 0 \pmod 8;$$
$$D(2n+1) \equiv 0 \pmod 4,$$
$$D(4n+3) \equiv 0 \pmod{36};$$

and establish product expressions for their corresponding generating functions.

Our approach will essentially be the multi–section series method, that has been used by Chu and Wang [9] to investigate Rogers–Ramanujan identities. Denote the

m-th root of unity by $\omega_m = \exp(\frac{2\pi i}{m})$. Then for the nonnegative integer r with $0 \le r < m$ and the formal power series defined by $\phi(x) := \sum_{n\ge 0}\Omega(n)x^n$, there holds the following multi–section series modulo m (cf. Comtet [10, p. 84])

$$\sum_{n\ge 0}\Omega(mn+r)x^{mn+r} = \frac{1}{m}\sum_{k=1}^{m}\omega_m^{-kr}\phi(x\omega_m^k). \tag{1}$$

This will be combined with Jacobi's triple product identity (cf. Bailey [3, §8.6])

$$[q, x, q/x; q]_\infty = \sum_{k=-\infty}^{\infty}(-1)^k q^{\binom{k}{2}} x^k \tag{2}$$

and three three-term relations from the theory of theta functions. The first one is the fundamental relation originally due to Weierstrass (cf. Whittaker–Watson [15, p. 451]) which can equivalently be reformulated as follows. For the five complex parameters a, b, c, d, e subject to $a^2 = bcde$, there holds the theta function identity (cf. Chu [6, Theorem 1.1]):

$$\langle a/b, a/c, a/d, a/e; q\rangle_\infty \;-\; \langle b, c, d, e; q\rangle_\infty \;=\; b\,\langle a, a/bc, a/bd, a/be; q\rangle_\infty, \tag{3}$$

where the modified Jacobi theta function is defined by

$$\langle \alpha, \beta, \cdots, \gamma; q\rangle_\infty = \langle \alpha; q\rangle_\infty\langle \beta; q\rangle_\infty \cdots \langle \gamma; q\rangle_\infty \;\text{ and }\; \langle x; q\rangle_\infty = (x; q)_\infty(q/x; q)_\infty.$$

The other two relations read as the following equations [7, Proposition 3]

$$\langle -x, -y; q\rangle_\infty + \langle x, y; q\rangle_\infty = 2(-q; q)_\infty^2\langle -xy, -qx/y; q^2\rangle_\infty, \tag{4}$$
$$\langle -x, -y; q\rangle_\infty - \langle x, y; q\rangle_\infty = 2x(-q; q)_\infty^2\langle -qxy, -q^2x/y; q^2\rangle_\infty. \tag{5}$$

The rest of the paper will be organized as follows. The next section will prove the two congruence relations for $C(4n+2)$ and $C(4n+3)$ as well as their generating functions, that will be utilized to review the congruences due to Chan and Cooper [5]. The second section will be devoted to the congruences for $\mathfrak{C}(2n+1)$, $\mathfrak{C}(4n+3)$ and their generating functions. In the third section, we shall establish the congruences for $D(2n+1)$ and $D(4n+3)$ as well as their generating functions, where as preliminaries, new proofs will be presented for the two relations concerning the theta function $z(q)$ that appeared in both Berndt [4, p. 110] and Chan–Cooper [5].

2. Congruences for partition function $C(n)$

For the multipartition function $C(n)$ defined by the power series expansion

$$\sum_{n\ge 0} C(n)q^n = \frac{1}{(q; q)_\infty(q^3; q^3)_\infty} = \frac{1}{E_1E_3}$$

we shall prove, in this section, the following theorem.

Theorem 1. *There hold the following two congruence relations*

$$C(4n+2) \equiv 0 \pmod 2, \tag{6a}$$
$$C(4n+3) \equiv 0 \pmod 2; \tag{6b}$$

and the quartic–section series generating functions

$$\sum_{n\geq 0} C(4n+2)q^n = 2\frac{E_2^4E_6^{12}E_8^2}{E_1^6E_3^9E_4E_{12}^4} + 4q\frac{E_2^2E_4^5E_{12}^4}{E_1^6E_3^5E_8^2}, \tag{7a}$$
$$\sum_{n\geq 0} C(4n+3)q^n = 4\frac{E_4^4E_6^2E_{12}^5}{E_1^5E_3^6E_{24}^2} + 2q\frac{E_2^{12}E_6^4E_{24}^2}{E_1^9E_3^6E_4^4E_{12}}. \tag{7b}$$

Proof. Denote by $f(q)$ the generating function

$$f(q) := \sum_{n\geq 0} C(n)q^n = \frac{1}{(q;q)_\infty(q^3;q^3)_\infty}.$$

Applying formula (1), we have the following quartic-section series

$$\sum_{n\geq 0} C(4n+2)q^{4n+2} = \frac{1}{4}\sum_{k=1}^{4}(-1)^k f(qi^k) = \frac{1}{4}\Big\{f(q)+f(-q)-f(qi)-f(-qi)\Big\}.$$

Now we are going to examine the four terms inside the braces. Firstly, the sum of the first two terms can be manipulated as

$$\begin{aligned} f(q)+f(-q) =& \frac{1}{(q;q)_\infty(q^3;q^3)_\infty} + \frac{1}{(-q;-q)_\infty(-q^3;-q^3)_\infty} \\ =& \frac{(-q;q^2)_\infty(-q^3;q^6)_\infty + (q;q^2)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty(q^2;q^4)_\infty(q^6;q^6)_\infty(q^6;q^{12})_\infty}. \end{aligned}$$

According to (4), the numerator in the last fraction can be factorized into

$$\langle -q,-q^3;q^6\rangle_\infty + \langle q,q^3;q^6\rangle_\infty = 2(-q^6;q^6)_\infty^2\langle -q^4,-q^4;q^{12}\rangle_\infty$$

which leads to the following simplified expression

$$\begin{aligned} f(q)+f(-q) =& \frac{2(-q^6;q^6)_\infty^2\langle -q^4;q^{12}\rangle_\infty^2}{(q^2;q^2)_\infty(q^2;q^4)_\infty(q^6;q^6)_\infty(q^6;q^{12})_\infty} \\ =& 2\frac{(-q^2;q^2)_\infty(-q^6;q^6)_\infty^3(-q^4;q^4)_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty(-q^{12};q^{12})_\infty^2}. \end{aligned}$$

Replacing q by qi in the last equality yields another one

$$f(qi)+f(-qi) = \frac{2(q^2;-q^2)_\infty(q^6;-q^6)_\infty^3(-q^4;q^4)_\infty^2}{(-q^2;-q^2)_\infty(-q^6;-q^6)_\infty(-q^{12};q^{12})_\infty^2}.$$

Combining the last two equations and then simplifying the result, we have

$$\sum_{n\geq 0} C(4n+2)q^{4n+2} = \frac{1}{4}\Big\{f(q)+f(-q)-f(qi)-f(-qi)\Big\} = \frac{E_8^4E_{24}^2}{2E_4^6E_{12}^5} \tag{8a}$$

$$\times\begin{Bmatrix} \big[q^4,-q^2,-q^2;q^4\big]_\infty \big[q^{12},-q^6,-q^6;q^{12}\big]_\infty^2 \\ -\big[q^4,\ q^2,\ q^2;q^4\big]_\infty \big[q^{12},\ q^6,\ q^6;q^{12}\big]_\infty^2 \end{Bmatrix}. \tag{8b}$$

Observing that the right-most fraction displayed (8a) is, in fact, a series in base q^4, the coefficient $C(4n+2)$ will be even if we can show that each coefficient of q^{4m+2} with $m\in\mathbb{N}_0$ for the series inside the braces displayed in (8b) is divisible by 4. In view of the Jacobi triple product identity (2), this is equivalent to the divisibility by 4 for each coefficient of q^{2m+1} in the triple sum

$$\sum_{i,j,k} q^{i^2+3j^2+3k^2}\Big\{1-(-1)^{i+j+k}\Big\} = 2\sum_{i+j+k\equiv_2 1} q^{i^2+3j^2+3k^2}, \tag{8c}$$

where $a\equiv_2 b$ denotes the congruence $a\equiv b \pmod 2$ for brevity. This is justified by the fact that the number of integer solutions of Diophantine equation

$$2m+1 = i^2+3j^2+3k^2$$

is even (including the case when there is no solution), which contributes to factor 2. Therefore, we have proved the congruence relation displayed in (6a).

Another congruence relation displayed in (6b) can be shown analogously. In fact, applying (1) gives the following quartic-section series expression

$$\sum_{n\geq 0} C(4n+3)q^{4n+3} = \frac{1}{4}\sum_{k=1}^{4} i^{-3k} f(qi^k) = \frac{1}{4}\Big\{f(q)-f(-q)+if(qi)-if(-qi)\Big\}.$$

Rewriting the sum of the first two terms inside the braces

$$f(q)-f(-q) = \frac{(-q;q^2)_\infty(-q^3;q^6)_\infty-(q;q^2)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty(q^2;q^4)_\infty(q^6;q^6)_\infty(q^6;q^{12})_\infty}$$

and then factorizing, by means of (5), the numerator

$$\langle -q,-q^3;q^6\rangle_\infty - \langle q,q^3;q^6\rangle_\infty = 2q(-q^2;q^4)_\infty^2(-q^{12};q^{12})_\infty^2$$

we derive the following equality

$$f(q)-f(-q) = \frac{2q\,(-q^2;q^4)_\infty^2(-q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty(q^2;q^4)_\infty(q^6;q^{12})_\infty},$$

which leads, under replacement $q\to qi$, to another expression

$$f(qi)-f(-qi) = \frac{2qi\,(q^2;q^4)_\infty^2(-q^{12};q^{12})_\infty^2}{(-q^2;-q^2)_\infty(-q^6;-q^6)_\infty(-q^2;q^4)_\infty(-q^6;q^{12})_\infty}.$$

Combining the last two equations and then simplifying the result, we get

$$\sum_{n\geq 0} C(4n+3)q^{4n+3} = \frac{1}{4}\Big\{f(q)-f(-q)+if(qi)-if(-qi)\Big\} = \frac{qE_8^2E_{24}^4}{2E_4^5E_{12}^6} \tag{9a}$$

$$\times\left\{\begin{aligned} &\left[q^4,-q^2,-q^2;q^4\right]_\infty^2\left[q^{12},-q^6,-q^6;q^{12}\right]_\infty \\ &\quad-\left[q^4,\ q^2,\ q^2;q^4\right]_\infty^2\left[q^{12},\ q^6,\ q^6;q^{12}\right]_\infty\end{aligned}\right\}. \tag{9b}$$

By carrying out the same procedure as that for $C(4n+2)$, we can check that the congruence $C(4n+3)\equiv 0 \pmod 2$ is equivalent to the fact that each coefficient of q^{2m+1} is a multiple of 4 in the following triple sum

$$\sum_{i,j,k} q^{i^2+j^2+3k^2}\Big\{1-(-1)^{i+j+k}\Big\} = 2\sum_{i+j+k\equiv_2 1} q^{i^2+j^2+3k^2}. \tag{9c}$$

This is guaranteed by the fact that the number of integer solutions of the following Diophantine equation is also even

$$2m+1 = i^2+j^2+3k^2.$$

Furthermore, we can compute the generating functions for $C(4n+2)$ and $C(4n+3)$. According to the Jacobi triple product identity (2), it is not hard to check

$$\sum_{k\equiv_2 0} q^{k^2} = \left[q^8,-q^4,-q^4;q^8\right]_\infty = \frac{E_8^5}{E_4^2E_{16}^2} \tag{10a}$$

$$\sum_{k\equiv_2 1} q^{k^2} = 2q\left[q^8,-q^8,-q^8;q^8\right]_\infty = 2q\frac{E_{16}^2}{E_8}; \tag{10b}$$

and

$$\sum_{i+j\equiv_2 0} q^{i^2+j^2} = \left[q^4,-q^2,-q^2;q^4\right]_\infty^2 = \frac{E_4^{10}}{E_2^4E_8^4} \tag{11a}$$

$$\sum_{i+j\equiv_2 1} q^{i^2+j^2} = 4q\left[q^4,-q^4,-q^4;q^4\right]_\infty^2 = 4q\frac{E_8^4}{E_4^2}. \tag{11b}$$

In view of (8c) and (9c), these equalities can be utilized to simplify the triple sums displayed in (8b) and (9b) as follows

$$\begin{aligned}\mathrm{Eq(8b)} &= 2\sum_{i\equiv_2 1} q^{2i^2}\sum_{j+k\equiv_2 0} q^{6j^2+6k^2} + 2\sum_{i\equiv_2 0} q^{2i^2}\sum_{j+k\equiv_2 1} q^{6j^2+6k^2};\\ &= 4q^2\frac{E_{24}^{10}E_{32}^2}{E_{12}^4E_{16}E_{48}^4} + 8q^6\frac{E_{16}^5E_{48}^4}{E_8^2E_{24}^2E_{32}^2};\end{aligned}$$

$$\begin{aligned}\mathrm{Eq(9b)} &= 2\sum_{k\equiv_2 0} q^{6k^2}\sum_{i+j\equiv_2 1} q^{2i^2+2j^2} + 2\sum_{k\equiv_2 1} q^{6k^2}\sum_{i+j\equiv_2 0} q^{2i^2+2j^2}\\ &= 8q^2\frac{E_{16}^4E_{48}^5}{E_8^2E_{24}^2E_{96}^2} + 4q^6\frac{E_8^{10}E_{96}^2}{E_4^4E_{16}^4E_{48}}.\end{aligned}$$

Substituting them respectively into (8a-8b) and (9a-9b), canceling the common q-factors across the equations and finally replacing q^4 by q, we get the two generating functions displayed in (7a) and (7b).

Summing up, we have completed the proof of Theorem 1. $\square$

Now we are ready to review the two congruence relations due to Chan and Cooper [5]. For the multipartition function $c(n)$ defined by the power series expansion

$$\sum_{n\geq 0} c(n)q^n = \frac{1}{(q;q)_\infty^2 (q^3;q^3)_\infty^2} = \frac{1}{E_1^2 E_3^2}.$$

Chan and Cooper [5] found recently the following interesting result, which has been the primary motivation for the authors to carry on the present research.

Theorem 2 (See [5]). *There hold the congruence relations*

$$c(2n+1) \equiv 0 \pmod 2, \tag{12a}$$
$$c(4n+3) \equiv 0 \pmod 4; \tag{12b}$$

and the generating function

$$\sum_{n=0}^{\infty} c(2n+1)q^n = 2\frac{E_2^4 E_6^4}{E_1^6 E_3^6}. \tag{13}$$

We shall prove that this theorem is obtainable from our Theorem 1, which is partially supported by the observation that the generating function for $c(n)$ results in the square of that for $C(n)$. The informed reader will notice that Chan and Cooper [5] have successfully established some additional congruences modulo powers of 2 and the corresponding generating functions, that we do not intend to pursue here.

Proof. Note that $c(n)$ can be expressed as the following convolution

$$c(n) = \sum_{k=0}^{n} C(k)C(n-k). \tag{14}$$

Splitting the sum with respect to k into two sums according to the parity of k and then inverting the summation order for the second sum, we get

$$\begin{aligned} c(2n+1) &= \sum_{k=0}^{n} C(2k)C(2n+1-2k) + \sum_{k=0}^{n} C(2k+1)C(2n-2k) \\ &= 2\sum_{k=0}^{n} C(2k)C(2n+1-2k) \end{aligned}$$

which confirms the first congruence relation.

Similarly, classifying the summation index k according to its residues modulo 4 and then pairing the four sums into two ones, we have the expression

$$\begin{aligned} c(4n+3) = &\sum_{k=0}^{n} C(4k)C(4n+3-4k) + \sum_{k=0}^{n} C(4k+1)C(4n+2-4k) \\ &+ \sum_{k=0}^{n} C(4k+2)C(4n+1-4k) + \sum_{k=0}^{n} C(4k+3)C(4n-4k) \\ = &\, 2\sum_{k=0}^{n} C(4k+2)C(4n+1-4k) + 2\sum_{k=0}^{n} C(4k+3)C(4n-4k). \end{aligned}$$

In view of Theorem 1, each term in the last two sums is divisible by 2. Therefore $c(4n+3)$ is a multiple of 4, which proves the second congruence.

Generating function (13) follows directly from (1) and (3). In fact, we have

$$\begin{aligned} 2\sum_{n=0}^{\infty} c(2n+1)q^{2n+1} &= \frac{1}{(q;q)_\infty^2(q^3;q^3)_\infty^2} - \frac{1}{(-q;-q)_\infty^2(-q^3;-q^3)_\infty^2} \\ &= \frac{E_4^2E_{12}^2}{E_2^4E_6^4}\Big\{(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2 - (q;q^2)_\infty^2(q^3;q^6)_\infty^2\Big\}. \end{aligned}$$

Rewriting the difference in the braces, through (3), we can factorize it into

$$\begin{aligned} \langle -q,-q^3,-q^5,-q^3;q^6\rangle_\infty - \langle q,q^3,q^5,q^3;q^6\rangle_\infty &= q\langle -q^6,-1,-q^2,-q^2;q^6\rangle_\infty \\ &= 4q\frac{E_4^2E_{12}^2}{E_2^2E_6^2}. \end{aligned} \tag{15}$$

It is a routine to check that this expression leads to the generating function for $\{c(2n+1)\}$ stated in Theorem 2. $\square$

We remark that in view of convolution (14), the generating function expression (13) can also be derived from the product of $f(q)+f(-q)$ and $f(q)-f(-q)$, that appeared in the proof of Theorem 1.

3. Congruences for partition function $\mathfrak{C}(n)$

Similarly to the partition function $c(n)$ treated by Chan and Cooper [5], we may define the following *reciprocal partition function*

$$\sum_{n=0}^{\infty} \mathfrak{C}(n)q^n \;=\; (-q;q)_\infty^2(-q^3;q^3)_\infty^2 = \frac{E_2^2E_6^2}{E_1^2E_3^2}.$$

Interestingly enough, $\mathfrak{C}(n)$ admits similar congruences and generating functions.

Theorem 3. *There hold the following congruence relations*

$$\mathfrak{C}(2n+1) \equiv 0 \pmod 2, \tag{16a}$$
$$\mathfrak{C}(4n+3) \equiv 0 \pmod 8; \tag{16b}$$

and the generating functions

$$\sum_{n=0}^{\infty}\mathfrak{C}(2n+1)q^n = 2\frac{E_2^4E_6^4}{E_1^4E_3^4}, \tag{17a}$$

$$\sum_{n=0}^{\infty}\mathfrak{C}(4n+3)q^n = 8\frac{E_2^2E_4^4E_6^{14}}{E_1^6E_3^{10}E_{12}^4}+8q\frac{E_2^{14}E_6^2E_{12}^4}{E_1^{10}E_3^6E_4^4}. \tag{17b}$$

Proof. It suffices to show both generating function expressions.

Generating function (17a) follows directly from (1) and (3). In fact, we have the bisection series

$$\begin{aligned}2\sum_{n=0}^{\infty}\mathfrak{C}(2n+1)q^{2n+1} &= (-q;q)_\infty^2(-q^3;q^3)_\infty^2-(q;-q)_\infty^2(q^3;-q^3)_\infty^2\\ &= \frac{E_4^2E_{12}^2}{E_2^2E_6^2}\Big\{(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2-(q;q^2)_\infty^2(q^3;q^6)_\infty^2\Big\}.\end{aligned}$$

Factorizing the difference in the braces through (15), and then replacing q^2 by q, we get the generating function for $\mathfrak{C}(2n+1)$ stated in Theorem 3.

Recalling (17a), the quartic–section series generating function can be expressed as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{C}(4n+3)q^{2n+1} &= (-q;q)_\infty^4(-q^3;q^3)_\infty^4-(q;-q)_\infty^4(q^3;-q^3)_\infty^4\\ &= \frac{E_4^4E_{12}^4}{E_2^6E_6^6}\left\{\begin{array}{l}\left[q^2,-q,-q;q^2\right]_\infty^2\left[q^6,-q^3,-q^3;q^6\right]_\infty^2\\ \quad-\left[q^2,\ q,\ q;q^2\right]_\infty^2\left[q^6,\ q^3,\ q^3;q^6\right]_\infty^2\end{array}\right\}.\end{aligned}$$

According to Jacobi's triple product identity (2), the difference inside the braces can be expressed as quadruplicate sums

$$\begin{aligned}2\sum_{i+j+k+\ell\equiv_2 1}q^{i^2+j^2+3k^2+3\ell^2} =&2\sum_{i+j\equiv_2 1}q^{i^2+j^2}\sum_{k+\ell\equiv_2 0}q^{3k^2+3\ell^2}\\ &+2\sum_{i+j\equiv_2 0}q^{i^2+j^2}\sum_{k+\ell\equiv_2 1}q^{3k^2+3\ell^2}\\ =&8q\frac{E_8^4E_{12}^{10}}{E_4^2E_6^4E_{24}^4}+8q^3\frac{E_4^{10}E_{24}^4}{E_2^4E_8^4E_{12}^2},\end{aligned}$$

where the last line has been justified by (11a) and (11b). Therefore, we have found

$$\sum_{n=0}^{\infty}\mathfrak{C}(4n+3)q^{2n+1} = 8q\frac{E_4^2E_8^4E_{12}^{14}}{E_2^6E_6^{10}E_{24}^4}+8q^3\frac{E_4^{14}E_{12}^2E_{24}^4}{E_2^{10}E_6^6E_8^4}$$

which is under $q^2\to q$ equivalent to generating function (17b). $\square$

4. Congruences for partition function $D(n)$

Following Chan and Cooper [5], define the modular function by

$$z(q)=q\prod_{k=0}^{\infty}\frac{(1-q^{12k+2})^2(1-q^{12k+10})^2}{(1-q^{12k+4})^2(1-q^{12k+8})^2}=q\frac{E_2^2E_{12}^4}{E_4^4E_6^2}. \tag{18}$$

Firstly, we present a new proof for the following modular equations.

Lemma 4.

$$1-z(q)\ =\frac{E_1E_6^7}{E_2E_3^3E_4^2E_{12}^2}, \tag{19a}$$

$$1-3z(q)=\frac{E_1^3E_{12}}{E_3E_4^3}. \tag{19b}$$

Under the replacement $q\to-q$, the last equations can equivalently be restated as

$$1+z(q)\ =\frac{E_2^2E_3^3E_{12}}{E_1E_4^3E_6^2}, \tag{20a}$$

$$1+3z(q)=\frac{E_2^9E_3E_{12}^2}{E_1^3E_4^6E_6^3}. \tag{20b}$$

It should be pointed out that (20a) has explicitly appeared in Berndt [4, P110; Lemma 5.3], while (19b) can be found in Cooper [11, Eq 2.9].

Proof. According to the definition of $z(q)$, it is not hard to reformulate $1-z(q)$ as

$$\begin{aligned}1-z(q)=&1-q\frac{E_2^2E_{12}^4}{E_4^4E_6^2}=\frac{E_{12}^2}{E_4^2}\left\{\frac{E_4^2}{E_{12}^2}-q\frac{E_2^2E_{12}^2}{E_4^2E_6^2}\right\}\\=&\frac{E_{12}^2}{E_4^2}\Big\{\langle q^4,q^4;q^{12}\rangle_\infty-q\langle q^2,q^2;q^{12}\rangle_\infty\Big\}\\=&\frac{E_{12}^2}{E_4^2}\Big\{\langle q^4,q^2,-q^2,-q^2;q^6\rangle_\infty-q\langle q^5,q,-q,-q;q^6\rangle_\infty\Big\}.\end{aligned}$$

Specifying $\{a,b,c,d,e\}$ with $\{q^5,q,q^3,-q^3,-q^3\}$ respectively in the three–term relation (3), we can factorize the difference inside the braces into

$$\langle q^4,q^2,-q^2,-q^2;q^6\rangle_\infty-q\langle q^5,q,-q,-q;q^6\rangle_\infty=\langle q,q^3,-q^3,-q^3;q^6\rangle_\infty=\frac{E_1E_6^7}{E_2E_3^3E_{12}^4}$$

which leads to the first modular equation (19a).

Our approach to the second identity (19b) is not so direct. We start from the equation

$$2\left\{1-\frac{E_1^3E_{12}}{E_3E_4^3}\right\}=2\frac{E_2^3E_{12}}{E_4^3E_6}\left\{\frac{E_4^3E_6}{E_2^3E_{12}}-\frac{E_1^3E_6}{E_2^3E_3}\right\} \tag{21a}$$

$$=\frac{E_2^3E_{12}}{E_4^3E_6}\Big\{\langle -1,-q^2,-q^2,-q^2;q^6\rangle_\infty-2\langle q^3,q,q,q;q^6\rangle_\infty\Big\}. \tag{21b}$$

Replacing $\{a,b,c,d,e\}$ in (3) by $\{-q^3,-1,-q^2,-q^2,-q^2\}$ respectively, we have first the factorization

$$\langle -1,-q^2,-q^2,-q^2;q^6\rangle_\infty-\langle q^3,q,q,q;q^6\rangle_\infty=\langle -q^3,-q,-q,-q;q^6\rangle_\infty.$$

Then the difference inside the braces displayed in (21b) can be reformulated as

$$\begin{aligned}\langle -1,-q^2,-q^2,-q^2;q^6\rangle_\infty\ &-\ 2\langle q^3,q,q,q;q^6\rangle_\infty\\ &=\langle -q^3,-q,-q,-q;q^6\rangle_\infty-\langle q^3,q,q,q;q^6\rangle_\infty\\ &=\frac{(-q;q^2)^3_\infty}{(-q^3;q^6)_\infty}-\frac{(q;q^2)^3_\infty}{(q^3;q^6)_\infty}\\ &=\frac{(q^2;q^4)_\infty}{(q^6;q^{12})_\infty}\Big\{\langle -q,q\omega;q^2\rangle_\infty-\langle q,-q\omega;q^2\rangle_\infty\Big\},\end{aligned}$$

where $\omega:=\omega_3=e^{\frac{2\pi i}{3}}$ is the cubic root of unity for brevity. By means of (5), we can factorize the last difference into the expression

$$\begin{aligned}\langle -q,q\omega;q^2\rangle_\infty-\langle q,-q\omega;q^2\rangle_\infty\ &=2q(-q^2;q^2)^2_\infty\langle q^4\omega,q^4/\omega;q^4\rangle_\infty\\ &=6q\frac{(q^{12};q^{12})^2_\infty}{(q^2;q^2)^2_\infty}=6q\frac{E_{12}^2}{E_2^2}.\end{aligned}$$

Recalling (21a) and (21b), we confirm finally the second equation (19b) as follows

$$1-\frac{E_1^3E_{12}}{E_3E_4^3}=3\frac{E_2^2E_{12}^4}{E_4^4E_6^2}=3z(q).$$

This completes the proof of the two equations in Lemma 4. □

By means of the two modular equations displayed in Lemma 4, we are going to establish two congruence relations and the corresponding generating functions for the partition function $D(n)$ defined by

$$\sum_{n=0}^{\infty}D(n)q^n\ =\ \frac{(q^3;q^3)^4_\infty}{(q;q)^4_\infty}=\frac{E_3^4}{E_1^4}.$$

Theorem 5. *There hold the following congruence relations*

$$D(2n+1)\equiv 0\pmod 4,\tag{22a}$$
$$D(4n+3)\equiv 0\pmod{36};\tag{22b}$$

and the corresponding generating functions

$$\sum_{n=0}^{\infty}D(2n+1)q^n\ =4\frac{E_2^5E_3^3E_6}{E_1^9},\tag{23a}$$
$$\sum_{n=0}^{\infty}D(4n+3)q^n\ =36\frac{E_2^{14}E_3^8}{E_1^{20}E_6^2}+108q\frac{E_2^6E_3^4E_6^6}{E_1^{16}}.\tag{23b}$$

Proof. It is obvious that the two congruences are implied by the corresponding generating functions. Multiplying (20a) and (20b) and then dividing the resulting equation by $z(q)$, we get the equality

$$q\Big\{1/z(q)+4+3z(q)\Big\}=\frac{E_2^9E_3^4}{E_1^4E_4^5E_6^3E_{12}}.$$

Therefore, we can reformulate the following generating function

$$\sum_{n=0}^{\infty}D(n)q^n=\frac{E_3^4}{E_1^4}=q\Big\{1/z(q)+4+3z(q)\Big\}\frac{E_4^5E_6^3E_{12}}{E_2^9}.$$

From the definition, the power series expansion of $z(q)$ contains only odd powers of q. Hence the bisection series with odd indices reads as follows

$$\sum_{n=0}^{\infty}D(2n+1)q^{2n}=4\frac{E_4^5E_6^3E_{12}}{E_2^9}.$$

Under the replacement $q^2\to q$, this directly gives equation (23a).

Rewriting the equality displayed in (20b) as

$$\frac{E_3}{E_1^3}=\Big\{1+3z(q)\Big\}\frac{E_4^6E_6^3}{E_2^9E_{12}^2}$$

we can reformulate equation (23a) as

$$\sum_{n=0}^{\infty}D(2n+1)q^n=4\frac{E_4^{18}E_6^{10}}{E_2^{22}E_{12}^6}\Big\{1+3z(q)\Big\}^3.$$

There consequently holds the bisection series generating function

$$\sum_{n=0}^{\infty}D(4n+3)q^{2n+1}=4\frac{E_4^{18}E_6^{10}}{E_2^{22}E_{12}^6}\Big\{9z(q)+27z^3(q)\Big\}.$$

Under the replacement q^2 by q, this equation is clearly equivalent to (23b). □

Before concluding the paper, we would like to point out the existence of another quartic series expression for the generating function of $\mathfrak{C}(4n+3)$ that can be derived using the modular function approach. Multiplying (19a) with (20b) and then dividing by $z(q)$, we get the equation

$$\frac{1}{E_1^2E_3^2}=q\frac{E_4^4E_{12}^4}{E_2^6E_6^6}\Big\{1/z(q)+2-3z(q)\Big\}.$$

From this, we can derive the following alternative generating function for $\mathfrak{C}(4n+3)$

$$\sum_{n=0}^{\infty}\mathfrak{C}(4n+3)q^n=8\frac{E_2^{12}E_6^4}{E_1^{10}E_3^6}-24q\frac{E_2^4E_6^{12}}{E_1^6E_3^{10}}. \tag{24}$$

Comparing the last equation with (17b) leads to the following curious relation

$$\frac{E_4^4 E_6^{12}}{E_3^4 E_{12}^4} + q\frac{E_2^{12} E_{12}^4}{E_1^4 E_4^4} = \frac{E_2^{10} E_6^2}{E_1^4} - 3q\frac{E_2^2 E_6^{10}}{E_3^4}, \tag{25}$$

which is equivalent to the algebraic equation

$$1 - \frac{1-z(q)}{1+3z(q)} = z(q)\left\{1 + 3\frac{1-z(q)}{1+3z(q)}\right\}.$$

These last two equations show that the new expression (24) is the same as (17b).

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