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Uniform and non-uniform quantization of Gaussian processes

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Abstract. Quantization of a continuous-value signal into a discrete form (or discretization of amplitude) is a standard task in all analog/digital devices. We consider quantization of a signal (or random process) in a probabilistic framework. The quantization method presented in this paper can be applied to signal coding and storage capacity problems. In order to demonstrate a general approach, both uniform and non-uniform quantization of a Gaussian process are studied in more detail and compared with a conventional piecewise constant approximation. We investigate asymptotic properties of some accuracy characteristics, such as a random quantization rate, in terms of the correlation structure of the original random process when quantization cellwidth tends to zero. Some examples and numerical experiments are presented.

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1. Introduction

Analog signals correspond to data represented in continuously variable physical quantities in contrast to the digital representation of data in discrete units. In many applications, both discretization in time (or *sampling*) and in amplitude (or *quantizing*) are exploited. Due to many factors (e.g., measurement errors, essential phenomenon randomness) signals are frequently random and therefore random process models are used. Unlike standard techniques when quantization of a *single random variable* is studied, the presented quantization methods are based on the correlation structure of the model random process and applicable both to coding and archiving problems for *realizations of the process* and also to predicting the necessary capacity of the memory needed for quantized process realizations. Ignoring the quantization problem, we may use various approximation methods to restore an initial random process with a given accuracy (algebraic or trigonometric polynomials, [15]; splines, [1, 10, 16, 20]; wavelets, [17]). Quantization is generally less well understood than linear approximation. One of the reasons is that it is a nonlinear operation. Nevertheless, quantization is a standard procedure for all analog/digital devices. Various quantization problems are considered in the number of

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papers, mostly in applied literature (see, e.g., [4, 5, 21] and references therein). For a comprehensive overview of quantization techniques, we refer to [9] (see also [8] and references therein). Some statistical properties of quantizers are also studied in [7]. Let $[x]$ be the integer part of x. The uniform quantizer $q_\varepsilon(x)$ can be defined as

$$q_\varepsilon(x) = \varepsilon[x/\varepsilon]$$

for a fixed *cellwidth* $\varepsilon > 0$. In the number of quantization levels' optimization problems, the non-uniform quantization is used (see, e.g., [9]). Following Bennett's notation [3], the non-uniform n-level *companding quantizer* (or *compander*) $q_{n,G}(x)$ is defined as follows:

$$q_{n,G}(x) = G^{-1}(q_{n,U}[G(x)]),$$

where $G : \mathbb{R} \to (0,1)$ is onto and increasing, and $q_{n,U}$ is the n-level uniform quantizer on $(0,1)$,

$$q_{n,U}(y) = \begin{cases} \frac{k}{n} - \frac{1}{2n}, & \text{if } y \in \left(\frac{k-1}{n}, \frac{k}{n}\right], \ k = 1, \dots, n-1, \\ 1 - \frac{1}{2n}, & \text{if } y \in \left(\frac{n-1}{n}, 1\right). \end{cases}$$

Note that the inverse function $S := G^{-1}$ provides a transformation from a uniform quantization to a non-uniform one. Therefore, the quantization intervals of the companding quantizer $q_{n,G}$, $I_{1,n} = (-\infty, S(\frac{1}{n})]$, $I_{k,n} = (S(\frac{k-1}{n}), S(\frac{k}{n})]$, $k = 2, \dots, n-1$, and $I_{n,n} = (S(\frac{n-1}{n}), \infty)$, and the corresponding quantization levels' grid, $\{u_k = u_k(n), k = 1, \dots, n\} = \{S(\frac{k}{n} - \frac{1}{2n}), k = 1, \dots, n\}$ (cf. generating densities for regular designs of observation (time) points, see, e.g., [14, 16]). The quality of a quantizer $q(x)$ can be measured for a random process $X(t), t \in [0,T]$, for instance, by the maximum mean square error (MMSE)

$$\varepsilon(X) = \varepsilon(X, q(X)) := \max_{t \in [0,T]} ||X(t) - q(X(t))||,$$

where $\|Y\|^2 := \mathrm{E}(Y^2)$ for a random variable Y. There are quantizers with fixed or variable codeword length (or *rate*). Fixed-rate quantizers use all codes (or *codewords*) of equal length, say r. In variable-rate quantization, the instant rate $r(X)$ is the length of the corresponding codeword, i.e., a random variable, and the average rate

$$R(X) := \mathrm{E}(r(X))$$

is of interest. The MMSE characterizes restoration quality (i.e., accuracy) of a quantizer whereas the rate concerns the storage capacity needed for quantized process realizations. Some problems for the fixed-rate quantization of Gaussian processes and optimal properties of the corresponding Karhunen-Loève expansion are considered in [13]. This approach is related to reproducing kernel Hilbert space (RKHS) techniques and some results for ϵ-entropy for sets in a metric space (see also [11, 12]).

Here we consider the *uniform* and *non-uniform* quantization with (random) variable-rate for the linear space $\mathcal{C}^m[0,T]$ of random processes with continuous quadratic mean (q.m.) derivatives up to the order $m \geq 0$. The space $C^m[0,T]$

of non-random functions with continuous derivatives up to the order m can be considered as a linear subspace of $\mathcal{C}^m[0,T]$ by usual embedding. Let $X(t), t \in [0,T]$, be a random process (or a *signal*) with covariance function $K(t,s),\ t,s \in [0,T]$, and continuous sample paths. Following the standard notation for crossings of a level by a continuous function f (see, e.g., [6]), f is said to have a crossing of the level u at t_0 if in each neighborhood of t_0 there are points t_1 and t_2 such that $(f(t_1)-u)(f(t_2)-u)<0$. Let $N_u(f) := N_u(f,T)$ denote the number of crossings of the level u by f in $[0,T]$. For a realization of the random process $X(t)$, $t \in [0,T]$, denote by $r_\varepsilon(X)$ and $r_{n,G}(X)$ the total number of quantization points in $[0,T]$ (or random *quantization rate*) for uniform $q_\varepsilon(X)$ and non-uniform $q_{n,G}(X)$ quantizers, respectively,

$$r_\varepsilon(X) := \sum_{k\in\mathbb{Z}} N_{u_k}(X), \quad r_{n,G}(X) := \sum_{k=1}^{n} N_{u_k}(X).$$

Note that uniform $r_\varepsilon(X)$ and non-uniform $r_{n,G}(X)$ quantization rates are random variables whenever $X(t), t \in [0,T]$, is a Gaussian process with continuous sample paths (see, e.g., [6]). For the random process $X(t), t \in [0,T]$, define also the *average uniform* and *non-uniform quantization rates* as

$$R_\varepsilon(X) := \mathrm{E}(r_\varepsilon(X)) \ \text{ and } \ R_{n,G}(X) := \mathrm{E}(r_{n,G}(X)), \tag{1}$$

respectively. We investigate the asymptotic properties of $r_\varepsilon(X)$ and $R_\varepsilon(X)$ as $\varepsilon \to 0$ and, the corresponding non-uniform case, $R_{n,G}(X)$ as $n \to \infty$.

For a set of random variables $Y_\varepsilon, \varepsilon > 0$, let $\stackrel{a.s.}{\to}$ denote convergence almost surely and $\stackrel{s}{\to}$ in s-mean, $s \geq 1$, as $\varepsilon \to 0$. Let $\{x\}$ be the fractional part of x, $x = [x] + \{x\}$. For a random variable Y and a quantizer $q(Y)$ (uniform or non-uniform), consider the *quantization error* $e(Y) := Y - q(Y)$. Properties of the quantization errors are considered in a number of applied papers (see, e.g., [9] and references therein). Shykula and Seleznjev [19] derive the stochastic structure of the asymptotic quantization errors for a random variable with values in a finite interval. The asymptotic quantization errors for unbounded quantizers are studied in [18]. Denote by $\lambda_2 = \lambda_2(X)$ the second spectral moment of a stationary process $X(t)$, $t \in [0,T]$.

In this paper, we study (random) variable-rate uniform and non-uniform quantization of a random process with continuous sample paths. The main distinction with a number of mentioned results is exploiting correlation properties of a random process for evaluation of the rate. Moreover, the developed technique could be applied to a wide class of random functions. In order to demonstrate the general approach, the asymptotic properties of the uniform and non-uniform quantizers for a Gaussian process are studied in more details.

The paper is organized as follows. In Section 2, asymptotic properties of uniform quantization rate $r_\varepsilon(X)$ for a Gaussian process $X(t)$, $t \in [0,T]$, as $\varepsilon \to 0$ are investigated. We study also the asymptotic behavior of uniform and non-uniform average quantization rates. For some classes of Gaussian processes and a given accuracy, we compare approximations by a quantized process and by a piecewise constant process. Section 3 provides some numerical experiments and examples of applications of the obtained results. Section 4 proves the statements in the previous sections.

2. Results

For deterministic signals, evaluation of the necessary capacity of the memory for a quantized sequence is a complicated task in general. We develop an average case analysis of this problem assuming a probabilistic model. Henceforth, let $X(t), t \in [0,T]$, be a Gaussian zero mean process with covariance function $K(t,s)$, $t,s \in [0,T]$, and continuous sample paths.

For further references, we introduce the following non-singularity conditions. Let $Y(t)$, $t \in [0,T]$, $Y \in \mathcal{C}^1[0,T]$, be a zero mean Gaussian process with covariance function $B(t,s)$, $t,s \in [0,T]$. Denote by $B_{11}(t,s) = \partial^2 B(t,s)/(\partial t \partial s)$, $t,s \in [0,T]$, the covariance function of the derivative process $Y^{(1)}(t)$, $t \in [0,T]$. Let $B_{01}(t,s)$ be the covariance between $Y(t)$ and $Y^{(1)}(s)$, $B_{01}(t,s) = \partial B(t,s)/\partial s$, $t,s \in [0,T]$. The following conditions on $Y(t)$, $t \in [0,T]$, are related to the curve crossing problems for non-stationary Gaussian processes (see, e.g., [6, Ch. 13]).

($A1$) The joint normal distribution for $Y(t)$ and $Y^{(1)}(t)$ is non-singular for each t, i.e., $B(t,t) > 0$, $B_{11}(t,t) > 0$, $\left|B_{01}(t,t)/\sqrt{B(t,t)B_{11}(t,t)}\right| < 1, t \in [0,T]$.

($A2$) $B_{11}(t,s)$, $t,s \in [0,T]$, is continuous at all diagonal points (t,t).

In the following theorem, we investigate the asymptotic behavior of rates $r_\varepsilon(X)$ as $\varepsilon \to 0$ in various convergence modes, specifically, a.s.-convergence and convergence of the first and the second moments. Denote by

$$v(X) := \int_0^T \left|X^{(1)}(t)\right| dt \quad \text{and} \quad V(X) := \mathrm{E}(v(X))$$

the *variation* and the *average variation* of $X(t), t \in [0,T]$, respectively. Observe that if $X \in \mathcal{C}^2[0,T]$, then $X(t)$ is equivalent to a process $\tilde{X}(t)$ which, with probability one, has continuously differentiable sample paths in $0 \le t \le T$ (see, e.g., Chapter 9.4 in [6]) and, hence, $v(X)$ is well defined. Let $M(X)$ be the number of crossings of zero level by a sample path of the process $X^{(1)}(t)$, $t \in [0,T]$, $X \in \mathcal{C}^2[0,T]$ (i.e., the number of local extremes).

Theorem 1. *Let $X(t), t \in [0,T], X \in \mathcal{C}^2[0,T]$, be a zero mean Gaussian process. If ($A1$) and ($A2$) hold for $X^{(1)}(t)$, $t \in [0,T]$, then*

(i) $|\varepsilon r_\varepsilon(X) - v(X)| \le \varepsilon(M(X)+1)$ *(a.s.) and* $\varepsilon r_\varepsilon(X) \xrightarrow{a.s.} v(X)$ *as* $\varepsilon \to 0$*;*

(ii) $|\varepsilon R_\varepsilon(X) - V(X)| \le \varepsilon(\mathrm{E}(M(X))+1)$ *and* $\varepsilon r_\varepsilon(X) \xrightarrow{1} v(X)$ *as* $\varepsilon \to 0$.

(iii) If additionally $X \in \mathcal{C}^3[0,T]$, then $\mathrm{E}|\varepsilon r_\varepsilon(X) - v(X)|^2 \le \varepsilon^2 \mathrm{E}\big(M(X)+1\big)^2$ *and* $\varepsilon r_\varepsilon(X) \xrightarrow{2} v(X)$ *as* $\varepsilon \to 0$.

Remark 1. *(i) Note that Gaussianity of $X(t), t \in [0,T]$, is a technical assumption and, for example, Theorem 1(i) is valid for a wide class of smooth enough processes with a finite (a.s.) number of local extremes. For the Gaussian case, we evaluate the rate of convergence (cf. the Banach Theorem for $N_u(f)$, the number of crossings*

of a level $u \in \mathbb{R}$ by function f). Particularly, in Theorem 1 (ii) we have, by Cramér and Leadbetter [6] Chapter 13.2,

$$E(M(X)) = \int_0^T \int_{\mathbb{R}} |z| g_t(0,z)\, dt\, dz,$$

where $g_t(u,z)$ is the joint probability density function of Gaussian processes $X^{(1)}(t)$ and $X^{(2)}(t)$, $t \in [0,T]$. For a stationary Gaussian process $X(t)$, $t \in [0,T]$, with covariance function $K(t,s) = k(t-s)$, we get

$$E(M(X)) = \frac{T}{\pi} \sqrt{-\frac{k^{(4)}(0)}{k^{(2)}(0)}}.$$

(ii) Theorem 1 (ii) and (iii) can be generalized to the convergence in mean of higher order for smooth enough random processes.

(iii) The smoothness condition $X \in \mathcal{C}^2[0,T]$ is technical and we claim that mean convergence in Theorem 1(ii) holds even for continuously differentiable (in q.m.) random functions.

As a corollary of Theorem 1 (ii) and (iii) we have the following proposition, where the average quantization rate $R_\varepsilon(X) = \mathrm{E}(r_\varepsilon(X))$ and the variance of $r_\varepsilon(X)$ are studied asymptotically as $\varepsilon \to 0$. Let $\sigma_1(t) = K_{11}(t,t)^{1/2}$ and $\rho_{11}(t,s)$ be the standard deviation and the correlation function of $X^{(1)}(t)$, $t,s \in [0,T]$, respectively. Denote by

$$\gamma(t,s) := 2\pi^{-1}\sigma_1(t)\sigma_1(s)\Big(\big(1-\rho_{11}^2(t,s)\big)^{1/2} + \rho_{11}(t,s)\big(\pi/2 - \arccos \rho_{11}(t,s)\big)\Big).$$

Proposition 1. *Let $X(t), t \in [0,T], X \in \mathcal{C}^2[0,T]$, be a zero mean Gaussian process with covariance function $K(t,s)$. If (A1) and (A2) hold for $X^{(1)}(t)$, $t \in [0,T]$, then*

(i) $\varepsilon R_\varepsilon(X) \to V(X) = \sqrt{2/\pi} \int_0^T \sigma_1(t)\, dt$ as $\varepsilon \to 0$.

(ii) If $X \in \mathcal{C}^3[0,T]$, then

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^2 \mathrm{Var}(r_\varepsilon(X)) &= \mathrm{Var}(v(X)) \\ &= \iint_{[0,T]^2} \gamma(t,s)\, dt\, ds - 2\pi^{-1}\bigg(\int_0^T \sigma_1(t)\, dt\bigg)^2.\end{aligned}$$

The following corollary is an immediate consequence of Proposition 1 (i) for the stationary case.

Corollary 1. *Suppose that $X(t), t \in [0,T], X \in \mathcal{C}^2[0,T]$, is a zero mean stationary Gaussian process, $\lambda_2 = \lambda_2(X)$. Then*

$$R_\varepsilon(X) \sim \varepsilon^{-1} T \sqrt{2\lambda_2/\pi} \text{ as } \varepsilon \to 0.$$

In the following theorem, the average quantization rate $R_{n,G}(X)$ for n-level companding quantizer $q_{n,G}(X)$ is studied asymptotically as $n \to \infty$. Let $p_t(z)$ and $p_t(u,z)$ denote the density function of $X^{(1)}(t)$ and the joint density function of $X(t)$ and $X^{(1)}(t)$, $t \in [0,T]$, respectively. We introduce the following condition on the joint distribution of $X(t)$ and $X^{(1)}(t)$, $t \in [0,T]$.

(*B*) There exists $b(t,z) : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $p_t(u|z) = p_t(u,z)/p_t(z) \le b(t,z)$, $u, z \in \mathbb{R}$, $t \in [0,T]$, and $\int_0^T \int_{\mathbb{R}} b(t,z)|z|p_t(z)\,dz\,dt < \infty$.

Theorem 2. *Let $X(t), t \in [0,T], X \in \mathcal{C}^1[0,T]$, be a zero mean Gaussian process with covariance function $K(t,s)$. If (A1), (A2), and (B) hold for $X(t)$, $t \in [0,T]$, then*

$$n^{-1}R_{n,G}(X) \to \int_0^T \iint_{\mathbb{R}^2} |z|p_t(u,z)\,dz\,dG(u)\,dt \text{ as } n \to \infty.$$

Remark 2. *We claim that Theorems 1 and 2 can be generalized for Gaussian processes with sufficiently smooth mean functions (cf. Chapter 13.2 in [6]).*

The following corollary is an immediate consequence of Theorem 2 for the stationary case.

Corollary 2. *Let $X(t), t \in [0,T], X \in \mathcal{C}^1[0,T]$, be a zero mean stationary Gaussian process with one-dimensional density $f(x)$, $x \in \mathbb{R}$, $\lambda_2 = \lambda_2(X)$. If (A1) and (A2) hold for $X(t)$, $t \in [0,T]$, then*

$$n^{-1}R_{n,G}(X) \to T\sqrt{2\lambda_2/\pi} \int_{\mathbb{R}} f(u)\,dG(u) \text{ as } n \to \infty.$$

The quantized process $L_\varepsilon(t) = q_\varepsilon(X(t)), t \in [0,T]$, can be considered as a piecewise constant approximation of $X(t), t \in [0,T]$, with random number of knots corresponding to crossings of quantization levels. For Gaussian processes, we compare this method with a conventional piecewise constant approximation, $P_m(t)$, $t \in [0,T]$, when discretization in time is used. Let $X(t), t \in [0,T]$, be sampled at distinct design points $T_m := (t_0, t_1, \dots, t_m)$, and the set of all $(m+1)$-point designs be denoted by $D_m := \{T_m : 0 = t_0 < t_1 < \dots < t_m = T\}$. The sequence of sampling points (or *designs*) $T_m := T_m(h) \in D_m$ is generated by a positive continuous density function $h(t)$, $i/m = T^{-1}\int_0^{t_i} h(t)dt$, $i = 0, \dots, m$, and $P_m(t) = P_m(X, T_m)(t) = X(t_i)$, $t \in [t_i, t_{i+1})$, $i = 0, \dots, m-1$. Following [14], we define *asymptotic optimality* of a sequence of sampling designs T_m^* by (in terms of MMSE)

$$\lim_{m\to\infty} ||X - P_m(X, T_m^*)||/\inf_{T_m \in D_m} ||X - P_m(X, T_m)|| = 1.$$

The optimal sequences of designs for spline approximation are studied in [16]. A corresponding result for piecewise constant approximation and the uniform mean norm $\max_{t\in[0,T]} ||X(t)||$ is obtained in the following proposition. For a random process $Y(t)$, $t \in [0,T]$, $Y \in \mathcal{C}^1[0,T]$, we introduce the following condition (*local stationarity*),

$$\lim_{s\to 0} ||Y(t+s) - Y(t)||/|s| = ||Y^{(1)}(t)|| > 0 \text{ uniformly in } t \in [0,T] \tag{2}$$

(see, e.g., [14, 16]). Let

$$\varepsilon_m(X) = \mathrm{MMSE}(X, P_m) := \max_{t\in[0,T]} ||X(t) - P_m(t)||$$

denote the maximum mean square error for a random process $X \in \mathcal{C}^1[0,T]$. Let

$$h^*(t) := c(t)/\int_0^T c(s)\,ds, \quad t \in [0,T],$$

where $c(t) := \|X^{(1)}(t)\|$.

Proposition 2. *Let a random process $X(t)$, $t \in [0,T]$, $X \in \mathcal{C}^1[0,T]$, satisfy (2). Then*

(i) $\lim_{m\to\infty} m\,\varepsilon_m(X) = \max_{t\in[0,T]} \big(c(t)/\,h(t)\big)$.

(ii) The sequence of designs $T_m = T_m(h)$ is optimal iff $T_m = T_m^*$, where $T_m^* = T_m(h^*)$. For the optimal T_m^*,

$$\lim_{m\to\infty} m\,\varepsilon_m(X) = \int_0^T c(t)\,dt. \tag{3}$$

Henceforth, we suppose that the corresponding assumptions for the results used below are fulfilled. For a Gaussian process $X \in \mathcal{C}^2[0,T]$, it follows by the definition and Proposition 1 (i) that $\delta = \delta_q = \mathrm{MMSE}(X, L_\varepsilon) \le \varepsilon$ for the approximation by the quantized process $L_\varepsilon(t)$ with the average number of points (average *uniform* quantization rate)

$$m_q = m_q(\varepsilon) \sim \varepsilon^{-1}\sqrt{2/\pi}\int_0^T c(t)\,dt \text{ as } \varepsilon \to 0. \tag{4}$$

For a fixed accuracy $\delta = \delta_p = \varepsilon_m(X)$, (3) implies that it needs m_p sampling points,

$$m_p = m_p(\delta) \sim \delta^{-1}\int_0^T c(t)\,dt \text{ as } \delta \to 0. \tag{5}$$

Note, that by the definition $\varepsilon = \varepsilon(\delta) \ge \delta \ge \|X(0) - L_\varepsilon(0)\| \sim \varepsilon/\sqrt{3}$ as $\varepsilon \to 0$ (see, e.g., [19]). Thus, from (4) and (5), we have

$$\lim_{\delta\to 0}\frac{m_q}{m_p} = \lim_{\delta\to 0}\frac{\delta}{\varepsilon}\sqrt{\frac{2}{\pi}} \le \sqrt{\frac{2}{\pi}} \approx 0.798. \tag{6}$$

For a stationary Gaussian process $X \in \mathcal{C}^1[0,T]$, we have $\delta = \mathrm{MMSE}(X, L_\varepsilon) = \|X(0) - L_\varepsilon(0)\| \sim \varepsilon/\sqrt{3}$ as $\varepsilon \to 0$, and therefore,

$$\lim_{\delta\to 0}\frac{m_q}{m_p} = \sqrt{\frac{2}{3\pi}} \approx 0.461. \tag{7}$$

A similar result can be obtained for a non-uniform quantization. Hence, for certain classes of Gaussian processes and a given accuracy, the approximation by a quantized

process demands less storage capacity for archiving digital information. Notice here that comparisons with the optimal piecewise constant approximation in (6) and (7) are performed under the assumption that the optimal density $h^*(t)$ (i.e., $\|X^{(1)}(t)\|$) is known although it is not given in practice. Of course, one can also remark that for differentiable random processes, more smooth approximation methods can be used (e.g., Hermite interpolation splines) with higher approximation order (see, e.g, [16]). But in practical applications, e.g., in conventional signal processing techniques, both quantization and sampling are required, i.e., *only* discretized values of random functions are available. Henceforth, the direct methods comparison is not correct an, even in a piecewise linear approximation the function values have to be quantized first.

3. Numerical experiments

In this section, we illustrate the convergence results from Corollaries 1 and 2. Denote by $Q_\varepsilon(X) := \varepsilon R_\varepsilon(X)$ and $Q_{n,G}(X) := n^{-1}R_{n,G}(X)$ the *normalized average uniform* and *normalized average non-uniform quantization rates*, respectively. Recall that $\varepsilon \to 0$ for uniform quantization corresponds to $n \to \infty$ for non-uniform (cf. (1)). We approximate a random process in the example below by piecewise linear with equidistant sampling points and estimate $Q_\varepsilon(X)$, $\varepsilon > 0$, and $Q_{n,G}(X)$, $n \geq 1$, by the standard Monte-Carlo method. Let N denote the number of simulations.

Example 1. *Let Z_1 and Z_2 be two independent standard normal random variables. Then, $X_1(t) = Z_1\cos(t) + Z_2\sin(t)$, $t \in [0, 2\pi]$, $T = 2\pi$, is a stationary Gaussian process with zero mean, covariance function $K_1(t,s) = \cos(t-s)$, $\lambda_2(X_1) = 1$, and one-dimensional standard normal density $f(x)$. For uniform quantization rate, it follows from Corollary 1 that*

$$Q_\varepsilon(X_1) \to 2\pi\sqrt{2/\pi} \approx 5.0133 \text{ as } \varepsilon \to 0. \tag{8}$$

Let $G_1(x)$, $x \in \mathbb{R}$, be onto and increasing function defined as follows:

$$G_1(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ (x+1)/2, & \text{if } -1 < x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

Then for non-uniform quantization rate, it follows from Corollary 2 that

$$Q_{n,G_1}(X_1) \to T\sqrt{2\lambda_2/\pi}\int_{\mathbb{R}} f(u)\,dG_1(u) = 2\pi\sqrt{2/\pi}\int_0^1 f(u)\,du \approx 1.7112 \text{ as } n \to \infty \tag{9}$$

Figure 1 demonstrates convergence in (8) and (9), respectively, as $\varepsilon \to 0$ and $n \to \infty$.

Example 2. *Let $X_2(t), t \in [0, 2\pi], T = 2\pi$, be a stationary Gaussian process with zero mean and covariance function $K_2(t,s) = e^{-(t-s)^2}$, $\lambda_2(X_2) = 2$; one-dimensional density $f(x)$ is standard normal. Let $G_2(x)$, $x \in \mathbb{R}$, be the standard normal distribution function. Similarly, it follows from Corollaries 1 and 2 that*

$$Q_\varepsilon(X_2) \to 2\pi\sqrt{4/\pi} \approx 7.0898 \text{ as } \varepsilon \to 0 \tag{10}$$

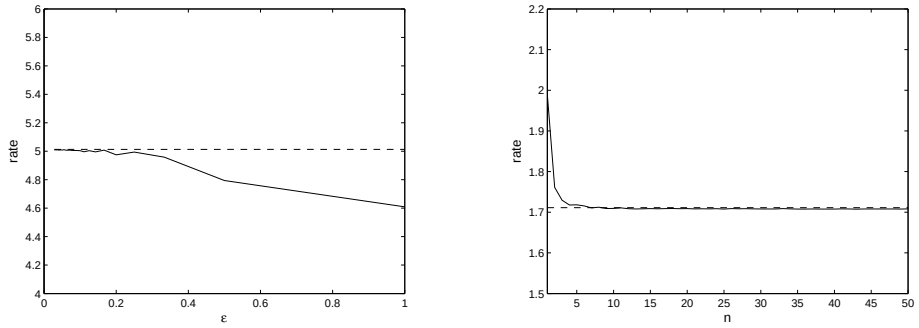

Figure 1: *The normalized average quantization rates $Q_\varepsilon(X_1)$ (left) and $Q_{n,G_1}(X_1)$ (right); the number of simulations $N = 5000$.*

and

$$Q_{n,G_2}(X_2) \to T\sqrt{2\lambda_2/\pi}\int_{\mathbb{R}} f(u)\,dG_2(u) = 2\pi\sqrt{4/\pi}\int_{\mathbb{R}} f^2(u)\,du \approx 2.0 \text{ as } n \to \infty, \quad (11)$$

respectively. Figure 2 demonstrates convergence in (10) and (11), respectively, as $\varepsilon \to 0$ and $n \to \infty$.

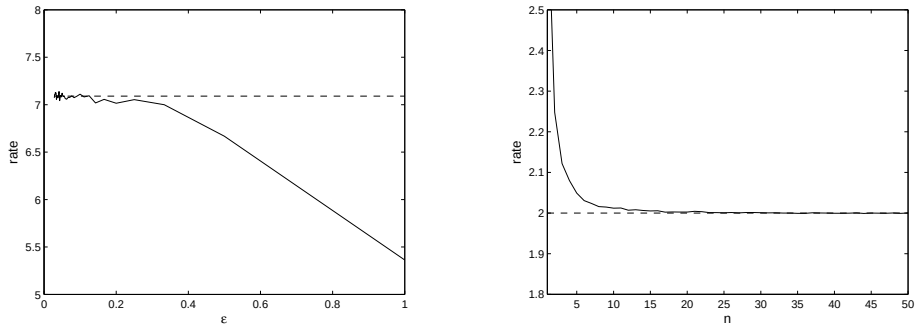

Figure 2: *The estimated normalized average quantization rates $Q_\varepsilon(X_2)$ (left) and $Q_{n,G_2}(X_2)$ (right); the number of simulations $N = 5000$.*

4. Proofs

Without loss of generality, let henceforth $T = 1$.

Proof of Theorem 1. We consider the uniform quantization of the process $X \in \mathcal{C}^2[0,1]$ with the infinite levels' ε-grid, the quantizer $q_\varepsilon(X(t)) = \varepsilon[X(t)/\varepsilon]$, the quantization rate $r_\varepsilon(X) = r_\varepsilon(X,[0,1])$, and the quantization error $e_\varepsilon(X(t)) = X(t) - q_\varepsilon(X(t)) = \varepsilon\{X(t)/\varepsilon\}$, $0 \le e_\varepsilon(X(t)) < \varepsilon$, $t \in [0,1]$. We have

$$v(X) = v(X,[0,1]) = \int_0^1 |X^{(1)}(t)|\,dt. \quad (12)$$

On the other hand, let $\tau = \{0 < \tau_1 < \tau_2 < \ldots < \tau_{M(X)} < 1\}$ be a set of local extremes of the process $X(t)$, $t \in [0,1]$, and $\tau_0 := 0, \tau_{M(X)+1} := 1$. Using the elementary properties of functions of bounded variation, we obtain

$$v(X,[0,1]) = \sum_{i=0}^{M(X)} v(X,[\tau_i,\tau_{i+1}]) = \sum_{i=0}^{M(X)} |X(\tau_{i+1}) - X(\tau_i)|. \tag{13}$$

For a fixed $\varepsilon \in (0,1)$ the total number of quantization points $r_\varepsilon(X)$ is additive on disjoint intervals, i.e., $r_\varepsilon(X) = \sum_{i=0}^{M(X)} r_\varepsilon(X,[\tau_i,\tau_{i+1}])$. Thus, it follows from (12) and (13) that

$$\Big|\varepsilon r_\varepsilon(X) - v(X)\Big| \le \sum_{i=0}^{M(X)} \Big|\varepsilon r_\varepsilon\big(X,[\tau_i,\tau_{i+1}]\big) - |X(\tau_{i+1}) - X(\tau_i)|\Big|. \tag{14}$$

Observe that by definitions

$$\begin{aligned} &X(\tau_{i+1}) - X(\tau_i) = q_\varepsilon(X(\tau_{i+1})) - q_\varepsilon(X(\tau_i)) + e_\varepsilon(X(\tau_{i+1})) - e_\varepsilon(X(\tau_i)),\\ &\varepsilon r_\varepsilon\big(X,[\tau_i,\tau_{i+1}]\big) = \varepsilon\big|\big[X(\tau_{i+1})/\varepsilon\big] - \big[X(\tau_i)/\varepsilon\big]\big| = \big|q_\varepsilon(X(\tau_{i+1})) - q_\varepsilon(X(\tau_i))\big|, \end{aligned}$$

where $i = 0,\ldots,M(X)$. Therefore, for $i = 0,\ldots,M(X)$, we get

$$\begin{aligned} &\varepsilon r_\varepsilon\big(X,[\tau_i,\tau_{i+1}]\big) - |X(\tau_{i+1}) - X(\tau_i)| \\ &\qquad = \begin{cases} e_\varepsilon(X(\tau_{i+1})) - e_\varepsilon(X(\tau_i)), & \text{if } X(\tau_{i+1}) \le X(\tau_i),\\ -(e_\varepsilon(X(\tau_{i+1})) - e_\varepsilon(X(\tau_i))), & \text{if } X(\tau_{i+1}) > X(\tau_i). \end{cases} \end{aligned}$$

Combining these relations with (14), we have

$$\begin{aligned} \Big|\varepsilon r_\varepsilon(X) - v(X)\Big| &\le \sum_{i=0}^{M(X)} \Big|e_\varepsilon(X(\tau_{i+1})) - e_\varepsilon(X(\tau_i))\Big| \\ &\le \varepsilon(M(X)+1),\ \varepsilon \in (0,1), \end{aligned} \tag{15}$$

since $0 \le e_\varepsilon(X(t)) < \varepsilon$, $t \in [0,1]$. Note that $M(X)$ is the number of crossings of zero level by a sample path of the process $X^{(1)}(t), t \in [0,1]$. Belyaev [2] shows that if a zero mean Gaussian process $Z(t) \in \mathcal{C}^k[0,1]$, then the k-th moment of the number of crossings of any fixed level u by $Z(t)$ is finite. Hence, taking into account $(A1)$ and $(A2)$, we have for $u = 0$ and $Z(t) = X^{(1)}(t)$,

$$\mathrm{E}(M(X)) < \infty \ \text{ if } \ X \in \mathcal{C}^2[0,T] \ \text{ and } \ \mathrm{E}\big(M(X)^2\big) < \infty \ \text{ if } X \in \mathcal{C}^3[0,T]. \tag{16}$$

Further, letting $\varepsilon \to 0$ in (15), we get (i) since $M(X) < \infty$ almost surely by (16). Finally, if we take expectations and limit $\varepsilon \to 0$ in (15), then due to (16) assertion (ii) follows. Similarly, we prove assertion (iii). This completes the proof. □

Proof of Proposition 1. We use the following elementary property of moments. Let $X, X_1, X_2, \ldots$ be random variables with finite sth moments, $s \ge 1$. If $X_n \xrightarrow{s} X$

as $n \to \infty$, then $\mathrm{E}\big|X_n\big|^s \to \mathrm{E}\big|X\big|^s$ as $n \to \infty$. This together with Theorem 1 (ii) and (iii) yields

$$\lim_{\varepsilon\to 0} \varepsilon \mathrm{E}(r_\varepsilon(X)) = \mathrm{E}(v(X)) = V(X) = \mathrm{E}\int_0^1 \big|X^{(1)}(t)\big|\,dt, \tag{17}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^2 \mathrm{Var}(r_\varepsilon(X)) &= \mathrm{Var}(v(X)) \\ &= \mathrm{E}\iint_{[0,1]^2} \big|X^{(1)}(t)X^{(1)}(s)\big|\,dt\,ds - \Big(\mathrm{E}(v(X))\Big)^2. \end{aligned}\tag{18}$$

Further, the modification of Fubini's Theorem for random functions implies that we can change the order of integration for the terms on the right-hand side of (17) and (18),

$$\mathrm{E}\int_0^1 \big|X^{(1)}(t)\big|\,dt = \int_0^1 \mathrm{E}\big|X^{(1)}(t)\big|\,dt, \tag{19}$$

$$\mathrm{E}\iint_{[0,1]^2} \big|X^{(1)}(t)X^{(1)}(s)\big|\,dt\,ds = \iint_{[0,1]^2} \mathrm{E}\big|X^{(1)}(t)X^{(1)}(s)\big|\,dt\,ds. \tag{20}$$

Note that for any standard normal random variable Y, $\mathrm{E}|Y| = \sqrt{2/\pi}$. Hence we obtain

$$\mathrm{E}\big|X^{(1)}(t)\big| = \sqrt{2/\pi}\,\sigma_1(t), \quad t \in [0,1]. \tag{21}$$

Combining now (17) with (19) and (21) gives assertion (i).

Let Z_1 and Z_2 be standard normal variables such that $\mathrm{E}(Z_1 Z_2) = \rho$. By orthonormalization, we have

$$\mathrm{E}\big|Z_1 Z_2\big| = 2\pi^{-1}\big(\sqrt{1-\rho^2} + \rho(\pi/2 - \arccos\rho)\big),\ \rho \in [-1,1].$$

Thus, for $Z_1 = X^{(1)}(t)$ and $Z_2 = X^{(1)}(s)$,

$$\mathrm{E}\big|X^{(1)}(t)X^{(1)}(s)\big| = \gamma(t,s). \tag{22}$$

Combining (18) with (20) and (22), we get assertion (ii). This completes the proof. □

Proof of Theorem 2. Recall that the random non-uniform quantization rate $r_{n,G}(X)$ is the total number of quantization points of the process $X(t)$, $t \in [0,1]$, with the levels' grid $u_n = \{u_k = u_k(n)\} = \{S(\frac{k}{n} - \frac{1}{2n}),\ k = 1,\dots,n\}$, $S = G^{-1}$, $r_{n,G}(X) = \sum_{k=1}^n N_{u_k}(X)$. If $(A1)$ and $(A2)$ hold for the process $X(t)$, $t \in [0,T]$, then we have

$$\mathrm{E}(N_{u_k}(X)) = \int_0^1 \int_{\mathbb{R}} |z| p_t(u_k, z)\,dt\,dz, \quad k = 1,\dots,n, \tag{23}$$

(see, e.g., [6, Chapter 13.2]). Summing (23) over all $k = 1, \dots, n$, and multiplying both parts by $n^{-1} = G(u_{k+1}) - G(u_k) = \Delta G(u_k)$, we obtain

$$n^{-1} R_{n,\,G}(X) = \int_0^1 \int_{\mathbb{R}} |z| \Big(\sum_{k=1}^{n} p_t(u_k, z) \Delta G(u_k) \Big)\, dt\, dz$$
$$= \int_0^1 \int_{\mathbb{R}} \Big(\sum_{k=1}^{n} p_t(u_k|z) \Delta G(u_k) \Big) |z| p_t(z)\, dt\, dz. \tag{24}$$

Denote by

$$w_n(t, z) := \sum_{k=1}^{n} p_t(u_k|z) \Delta G(u_k) \geq 0.$$

By elementary properties of the Lebesgue-Stieltjes' integral, for any fixed $t \in [0, T]$ and $z \in \mathbb{R}$ we have

$$\lim_{n \to \infty} w_n(t, z) = \int_{\mathbb{R}} p_t(u|z)\, dG(u). \tag{25}$$

In fact, the convergence in (25) follows, for example, by splitting up the real line $\mathbb{R}$ into bounded, $|u| \leq a_0$, and unbounded, $|u| > a_0$, intervals, where we choose a_0 in such way that $\int_{|u|>a_0} dG(u)$ is sufficiently small. Further, by (B)

$$w_n(t, z) \leq b(t, z),\ n \geq 1, \tag{26}$$

and $\int_0^T \int_{\mathbb{R}} |z| b(t, z) p_t(z) dt dz < \infty$. Hence, from (24)–(26) and the Dominated Convergence Theorem we get the assertion. This completes the proof. □

Proof of Proposition 2. The proof is similar to that of in [16] for spline approximation. By uniform continuity and positiveness of $c(t) = ||X^{(1)}(t)||$ we get

$$c(t + s) = c(t)(1 + r_t(s)),\ r_t(s) \to 0 \text{ as } s \to 0 \text{ uniformly in } t \in [0, 1]. \tag{27}$$

Now it follows from (2) and (27) that

$$||X(t) - X(t_k)|| = |t - t_k|\, c(t_k)(1 + r_{m,k}(t)), \tag{28}$$

where $\max\{|r_{m,k}(t)|, t \in [t_k, t_{k+1}], k = 0, \dots, m-1\} = \mathrm{o}(1)$ as $m \to \infty$. Applying (28) to the maximum mean square error $\varepsilon_m(X)$, we obtain

$$\varepsilon_m(X) = \max_k \max_{t \in [t_k, t_{k+1}]} ||X(t) - X(t_k)||$$
$$= \max_k \max_{t \in [t_k, t_{k+1}]} (|t - t_k|\, c(t_k)(1 + r_{m,k}(t)))$$
$$= \max_k (h_k c(t_k))(1 + \mathrm{o}(1)) \text{ as } m \to \infty, \tag{29}$$

where $h_k := |t_{k+1} - t_k|, k = 0, \dots, m-1$. Further, the Integral Mean-Value Theorem implies $h_k = 1/(h(w_k)m)$ for some $w_k \in [t_k, t_{k+1}]$. By the definition of $h(t), t \in [0, 1]$,

and (27) we have

$$\begin{aligned}\max_k(c(t_k)h_k) &= \frac{1}{m}\max_k(c(t_k)/h(w_k))(1+\mathrm{o}(1))\\ &= \frac{1}{m}\max_k(c(w_k)/h(w_k))(1+\mathrm{o}(1))\\ &= \frac{1}{m}\max_{[0,1]}(c(t)/h(t))(1+\mathrm{o}(1)) \text{ as } m\to\infty. \end{aligned} \tag{30}$$

Combining (29) and (30) gives

$$\lim_{m\to\infty} m\varepsilon_m(X) = \max_{[0,T]}(c(t)/h(t)).$$

Note that

$$\max_{[0,T]}\frac{c(t)}{h(t)} > \int_0^T h(s)\frac{c(s)}{h(s)}\,ds = \int_0^T c(s)\,ds$$

unless $h(t)/c(t) = const$. This completes the proof. □

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