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Estimating π from the Wallis sequence

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Abstract. The aim of this paper is to define new sequences related to the Wallis sequence having higher rates of convergence. Some sharp inequalities are established.

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1. Introduction and motivation

Perhaps one of the most known sequences related to the constant π is the Wallis sequence

$$W(n) = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1},$$

which converges to $\pi/2$ with the convergence rate estimated by n^{-1}, since

$$\frac{3}{10n} < \frac{\pi}{2} - W(n) < \frac{4}{10n} \quad (n \geq 3),$$

see, e.g., [6, Rel. 2c]. As the Wallis sequence is slowly convergent towards its limit, it is not suitable for approximating the constant π. In consequence, many authors were preoccupied in the recent past to accelerate the Wallis sequence. See [1, p. 258], [2, p. 384], [3, p. 213], [4, p. 5, p. 47], [5, p. 384], [7, p. 14, p. 465], [11] and the references therein.

In particular, Lampret [6, Rel. 2c] used the following version of the Stirling formula

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \exp\frac{\theta_x}{12x} \quad (x > 0), \tag{1}$$

where $\theta_x \in (0,1)$ to prove the following representation

$$\pi = W(n) \left(2 + \frac{1}{n}\right) e^{-1} \left(1 + \frac{1}{2n}\right)^{2n} \exp\left(\frac{\theta'_n}{6n+3} - \frac{\theta_n}{6n}\right) \quad (n \geq 1). \tag{2}$$

As the Stirling formula (1) is now the first approximation of the following Stirling asymptotic series

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$$\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \cdots\right),$$

it results that θ_x tends to 1 as x approaches infinity.

Motivated by Lampret's representation (2), we prove that the best approximation of the form

$$\pi \approx W(n)\left(2+\frac{1}{n}\right)e^{-1}\left(1+\frac{1}{2n}\right)^{2n}\exp\left(\frac{a}{6n+3}-\frac{b}{6n}\right) \quad (a,b\in[0,1]) \tag{3}$$

is obtained for $a=b=1$. For these priviledged values, (3) becomes

$$\pi \approx W(n)\left(2+\frac{1}{n}\right)e^{-1}\left(1+\frac{1}{2n}\right)^{2n}\exp\left(-\frac{1}{12n^2+6n}\right),$$

but we prove that the approximation

$$\pi \approx W(n)\left(2+\frac{1}{n}\right)e^{-1}\left(1+\frac{1}{2n}\right)^{2n}\exp\left(-\frac{1}{12n^2+6n+\frac{6}{5}}\right)$$

gives better results. Moreover, we state and prove the following double inequality

Theorem 1. *For every integer $n \geq 1$, we have*

$$W(n)\left(2+\frac{1}{n}\right)e^{-1}\left(1+\frac{1}{2n}\right)^{2n}\exp\left(-\frac{1}{12n^2+6n}\right)$$
$$< \pi < W(n)\left(2+\frac{1}{n}\right)e^{-1}\left(1+\frac{1}{2n}\right)^{2n}\exp\left(-\frac{1}{12n^2+6n+\frac{6}{5}}\right).$$

In the fourth section of [6], the following estimates were established

$$W_1(n) < W(n) < W_2(n) \quad (n \geq 2), \tag{4}$$

where

$$W_1(n) = \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{4n^2-1}\right)^n \exp\left(\frac{-1}{6n(4n^2-1)} - \frac{1}{80(n^2-1)^2}\right)$$
$$W_2(n) = \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{4n^2-1}\right)^n \exp\left(\frac{-1}{6n(4n^2-1)} + \frac{1}{80(n^2-1)^2}\right).$$

Finally, Lampret [6, Rel. (17a)-(17e)] improved (4) to

$$W_1^*(n) < W(n) < W_2^*(n) \quad (n \geq 2), \tag{5}$$

where

$$W_1^*(n) = \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\exp\varphi(n), \quad W_2^*(n) = \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\exp\psi(n)$$

$$\varphi(n) = \frac{n}{4n^2-1} - \frac{n}{2(4n^2-1)^2} - \frac{1}{6n(4n^2-1)} - \frac{1}{80(n^2-1)^2}$$

and

$$\psi(n) = \frac{n}{4n^2-1} - \frac{n}{3(4n^2-1)^2} - \frac{1}{6n(4n^2-1)} + \frac{1}{80(n^2-1)^2}.$$

Motivated by the estimates (4)-(5), we introduce the following approximation family

$$W(n) \approx \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\exp\mu_n(\alpha,\beta,\delta), \tag{6}$$

with α, β, δ real parameters, where

$$\mu_n(\alpha,\beta,\delta) = \frac{n}{4n^2-1} - \frac{\alpha n}{(4n^2-1)^2} - \frac{\beta}{n(4n^2-1)} + \frac{\delta}{80(n^2-1)^2}$$

and we prove that the best such approximation is obtained for

$$\alpha = -\frac{11}{30}, \quad \beta = \frac{23}{60}, \quad \delta = 0,$$

namely

$$W(n) \approx \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\exp\left(\frac{n}{4n^2-1} + \frac{11n}{30(4n^2-1)^2} - \frac{23}{60n(4n^2-1)}\right).$$

Furthermore, the next approximation is better

$$W(n) \approx \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right) \times \exp\left(\frac{n}{4n^2-1} + \frac{11n}{30(4n^2-1)^2} - \frac{23}{60n(4n^2-1)} - \frac{493}{107520n^3(n^2-1)^2}\right)$$

and we prove the following double inequality.

Theorem 2. *For every integer $n \geq 1$, we have*

$$\frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\exp\left(\frac{n}{4n^2-1} + \frac{11n}{30(4n^2-1)^2} - \frac{23}{60n(4n^2-1)} - \frac{493}{107\,520n^3(n^2-1)^2}\right)$$
$$< W(n) < \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\exp\left(\frac{n}{4n^2-1} + \frac{11n}{30(4n^2-1)^2} - \frac{23}{60n(4n^2-1)}\right).$$

2. Results

We start this section by analyzing the approximations family (3). One way to compare two such approximations is to introduce the relative error sequence w_n by the relations

$$\pi = W(n)\left(2 + \frac{1}{n}\right)e^{-1}\left(1 + \frac{1}{2n}\right)^{2n}\exp\left(\frac{a}{6n+3} - \frac{b}{6n}\right)\exp w_n \quad (n \geq 1) \tag{7}$$

and to consider an approximation (3) the better the faster w_n converges to zero.

Furthermore, a tool for estimating the rate of convergence is the following lemma, which was used in [8-10] to improve some convergences and to construct asymptotic expansions.

Lemma 1. *If $(\omega_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R}, \tag{8}$$

with $k > 1$, then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

For a detailed proof, see e.g. [8]. We see from this lemma that the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ increases together with the value k satisfying (8).

As we are interested to compute a limit of the form (8) for the sequence w_n given by (8), we develop the difference $w_n - w_{n+1}$ as a power series in n^{-1} as

$$w_n - w_{n+1} = \left(-\frac{1}{6}a + \frac{1}{6}b\right)\frac{1}{n^2} + \left(\frac{1}{3}a - \frac{1}{6}b - \frac{1}{6}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right) \tag{9}$$

(this can be made using some computer software such as Maple).

Referring to Lemma 1, the convergence of the sequence w_n to zero is fastest whenever the convergence of the difference $w_n - w_{n+1}$ to zero is the fastest, i.e. when the first two coefficients in (9) vanish, namely when $a = b = 1$.

Proof of Theorem 1. As the sequences

$$a_n = W(n)\left(2 + \frac{1}{n}\right)e^{-1}\left(1 + \frac{1}{2n}\right)^{2n}\exp\left(-\frac{1}{12n^2 + 6n}\right)$$
$$b_n = W(n)\left(2 + \frac{1}{n}\right)e^{-1}\left(1 + \frac{1}{2n}\right)^{2n}\exp\left(-\frac{1}{12n^2 + 6n + \frac{6}{5}}\right)$$

converge to π, it suffices to prove that a_n is strictly increasing and b_n is strictly decreasing.

In this sense, we have $\ln a_{n+1} - \ln a_n = f_1(n)$, $\ln b_{n+1} - \ln b_n = g_1(n)$, where

$$f_1(x) = \ln\frac{4(x+1)^2}{4(x+1)^2 - 1} + \ln\frac{2 + \frac{1}{x+1}}{2 + \frac{1}{x}} + (2x+2)\ln\left(1 + \frac{1}{2x+2}\right) - 2x\ln\left(1 + \frac{1}{2x}\right) - \frac{1}{12(x+1)^2 + 6(x+1)} + \frac{1}{12x^2 + 6x}$$

and

$$g_1(x) = \ln\frac{4(x+1)^2}{4(x+1)^2 - 1} + \ln\frac{2 + \frac{1}{x+1}}{2 + \frac{1}{x}} + (2x+2)\ln\left(1 + \frac{1}{2x+2}\right) - 2x\ln\left(1 + \frac{1}{2x}\right) - \frac{1}{12(x+1)^2 + 6(x+1) + \frac{6}{5}} + \frac{1}{12x^2 + 6x + \frac{6}{5}}.$$

In consequence, it suffices to show that $f_1 > 0$ and $g_1 < 0$. In this sense, we have

$$f_1''(x) = \frac{(4x+3)\left(60x+148x^2+144x^3+48x^4+9\right)}{3x^3(x+1)^3(2x+1)^3(2x+3)^3}$$

and

$$g_1''(x) = -\frac{(4x+3)\,Q(x)}{x^2(x+1)^2(2x+1)^2(2x+3)^2(25x+10x^2+16)^3(5x+10x^2+1)^3},$$

where

$$Q(x) = 12\,288 + 291\,072x + 2962\,853x^2 + 16\,496\,490x^3 + 55\,483\,705x^4 + 118\,363\,500x^5 + 163\,202\,000x^6 + 144\,648\,000x^7 + 79\,458\,000x^8 + 24\,600\,000x^9 + 3280\,000x^{10}.$$

Finally, f_1 is strictly convex and g_1 is strictly concave, with $f_1(\infty) = g_1(\infty) = 0$, so $f_1 > 0$ and $g_1 < 0$. □

Now, in order to find the best approximation of type (6), we associate the relative error sequence z_n by the relations

$$W(n) = \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\exp\mu_n(\alpha,\beta,\delta)\cdot e^{z_n}.$$

By making appeal again to Maple, we get

$$z_n - z_{n+1} = \left(\frac{3}{16}\alpha + \frac{3}{4}\beta - \frac{7}{32}\right)\frac{1}{n^4} + \left(-\frac{3}{8}\alpha - \frac{3}{2}\beta - \frac{1}{20}\delta + \frac{7}{16}\right)\frac{1}{n^5} + \left(\frac{25}{32}\alpha + \frac{45}{16}\beta + \frac{1}{8}\delta - \frac{19}{24}\right)\frac{1}{n^6} + \left(-\frac{45}{32}\alpha - \frac{75}{16}\beta - \frac{2}{5}\delta + \frac{41}{32}\right)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right).$$

The system

$$\begin{cases} \frac{3}{16}\alpha + \frac{3}{4}\beta - \frac{7}{32} = 0 \\ -\frac{3}{8}\alpha - \frac{3}{2}\beta - \frac{1}{20}\delta + \frac{7}{16} = 0 \\ \frac{25}{32}\alpha + \frac{45}{16}\beta + \frac{1}{8}\delta - \frac{19}{24} = 0 \end{cases}$$

determined by the first three coefficients of this power series gives the best values $\alpha = -\frac{11}{30}$, $\beta = \frac{23}{60}$, $\delta = 0$ and the corresponding approximation

$$W(n) \approx \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\exp\left(\frac{n}{4n^2-1} + \frac{11n}{30(4n^2-1)^2} - \frac{23}{60n(4n^2-1)}\right).$$

Proof of Theorem 2. Denoting

$$u(x) = \frac{x}{4x^2-1} + \frac{11x}{30(4x^2-1)^2} - \frac{23}{60x(4x^2-1)} - \frac{493}{107520x^3(x^2-1)^2}$$

and

$$v(x) = u(x) + \frac{493}{107520x^3 (x^2-1)^2},$$

we have to prove that

$$\frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\exp u(n) < W(n) < \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\exp v(n).$$

In fact it suffices to show that the sequence c_n is strictly increasing and d_n is strictly decreasing, where

$$c_n = \frac{\left(1-\frac{1}{2n+1}\right)\exp u(n)}{W(n)}, \quad d_n = \frac{\left(1-\frac{1}{2n+1}\right)\exp v(n)}{W(n)}.$$

We have $\ln c_{n+1} - \ln c_n = f_2(n)$, $\ln d_{n+1} - \ln d_n = g_2(n)$, where

$$f_2(x) = \ln\frac{4(x+1)^2-1}{4(x+1)^2} + \ln\frac{1-\frac{1}{2x+3}}{1-\frac{1}{2x+1}} + u(x+1) - u(x)$$

$$g_2(x) = \ln\frac{4(x+1)^2-1}{4(x+1)^2} + \ln\frac{1-\frac{1}{2x+3}}{1-\frac{1}{2x+1}} + v(x+1) - v(x),$$

with the derivatives

$$f_2'(x) = -\frac{P_2(x)}{8960x^4(x-1)^3(x+1)^4(x+2)^3(2x+3)^3(4x^2-1)^3}$$

$$g_2'(x) = \frac{Q_2(x)}{60x^2(x+1)^2(2x+3)^3(4x^2-1)^3},$$

where

$$P_2(x) = 26\,622 + 115\,362x - 906\,057x^2 - 2812\,130x^3 + 8973\,969x^4 + 32\,403\,996x^5 + 5343\,092x^6 - 67\,006\,464x^7 - 60\,727\,776x^8 + 23\,468\,480x^9 + 57\,479\,616x^{10} + 28\,792\,320x^{11} + 4798\,720x^{12}$$

and

$$Q_2(x) = -621 - 864x + 7024x^2 + 15\,776x^3 + 7888x^4.$$

Since all the coeffcients of the polynomials $P_2(x+1)$ and $Q_2(x+1)$ are positive, we have $f_2'(x) < 0$ and $g_2'(x) > 0$, for $x > 1$.

Now, f_2 is strictly decreasing, g_2 is strictly increasing, with $f_2(\infty) = g_2(\infty) = 0$, so $f_2 > 0$ and $g_2 < 0$ and the conclusion follows. □

Finally, we are convinced that our new approach presented here can be useful for establishing further improvements and refinements in other related problems.

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