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Coupled fixed point theorems in partially ordered metric spaces and application

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Abstract. In this paper, we prove some coupled fixed point theorems for contractive mappings in partially ordered complete metric spaces under certain conditions to extend and complement the recent fixed point theorems according to Lakshmikantham and Ćirić [V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. **70**(2009), 4341–4349] and Luong and Thuan [N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. **74**(2011), 983–992]. As an application, we give a result of existence and uniqueness for the solutions of a class of nonlinear integral equations.

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1. Introduction and preliminaries

As a well-known classical and valuable theorem in nonlinear analysis the Banach contraction principle [3] has been initially introduced by S. Banach and extended by so many authors in various ways (see, for instance [2, 6, 7, 10, 12 - 14]). It should be noted that Berinde and Borcut [4], Bhaskar and Lakshmikantham [5], Luong and Thuan [9], Lakshmikantham and Ćirić [8], Agarwal et al. [1] and Samet [11] have recently proved some new results for contractions in partially ordered metric spaces. In [8], Lakshmikantham and Ćirić introduced the notions of mixed g-monotone property and coupled coincidence point and proved coupled coincidence and coupled common fixed point theorems for mappings with mixed g-monotone property which are generalizations of the recent fixed point theorems according to Bhaskar and Lakshmikantham [5].

Definition 1 (See [8])**.** *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say F has the mixed g-monotone property if F is monotone*

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g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \quad \textit{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \quad \textit{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Note that if g is the identity mapping, then F is said to have the mixed monotone property (see also [5]).

Definition 2 (See [8]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if*

$$F(x, y) = g(x), \quad F(y, x) = g(y).$$

We remark that if g is the identity mapping, then (x, y) is called a *coupled fixed point* of the mapping F.

Definition 3 (See [8]). *Let X be a non-empty set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say F and g are commutative if*

$$g(F(x, y)) = F(g(x), g(y)) \quad \textit{for all } x, y \in X.$$

From now on, we say that a partially ordered set X which is endowed by a metric d has the property $(*)$ if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \longrightarrow y$, then $y \leq y_n$ for all n.

Now we have the following coupled fixed point theorems as the main results of Lakshmikantham and Ćirić [8] and Luong and Thuan [9], respectively.

Theorem 1 (See [8]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \to t+} \varphi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leq \varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right)$$

for all $x, y, u, v \in X$ which $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and suppose also either

(a) F is continuous or (b) X has the property $(*)$.

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \textit{and} \quad g(y_0) \geq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y) \quad \textit{and} \quad g(y) = F(y, x),$$

that is, F and g have a coupled coincidence.

Theorem 2 (See [9]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \leq F(x_0, y_0) \quad \textit{and} \quad y_0 \geq F(y_0, x_0).$$

Suppose there exist $\varphi, \psi : [0, \infty) \longrightarrow [0, \infty)$ such that

$$\varphi(d(F(x,y), F(u,v))) \leq \frac{1}{2}\varphi(d(x,u) + d(y,v)) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$ where

(i) φ is continuous and non-decreasing,

(ii) $\varphi(t) = 0$ if and only if $t = 0$,

(iii) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$,

and the function ψ satisfies $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0+} \psi(t) = 0$. Suppose either

(a) F is continuous or (b) X has the property $(*)$.

Then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x),$$

that is, F has a coupled fixed point in X.

The motivation of this work is to extend and develop the recent coupled fixed point theorems ([8, 9]) and apply the obtained results to investigate the existence of unique solutions to a class of nonlinear integral equations.

2. Main results

In order to proceed with developing of our work and obtain our results we need the following definition inspired by the definition of commutativity.

Definition 4. *Let (X, d) be a metric space, and let $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say that F and g are w-commutative if*

$$\lim_{n \to \infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

whenever $g(x_n)$ and $g(y_n)$ are convergent sequences in X such that

$$g(x_{n+1}) = F(x_n, y_n) \ \textit{and} \ g(y_{n+1}) = F(y_n, x_n).$$

Note that if F and g are commutative, then they are w-commutative in metric space (X, d).

Now let Φ denote all functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ which satisfy

(i) φ is lower semi-continuous and non-decreasing,

(ii) $\lim_{n\to\infty}\varphi(t_n)=0$ if and only if $\lim_{n\to\infty}t_n=0$ for $t_n\in[0,\infty)$,

(iii) $\varphi(t+s)\le\varphi(t)+\varphi(s)$ for all $t,s\in[0,\infty)$,

and for $\varphi\in\Phi$, Ψ_φ denote all functions $\psi:[0,\infty)\longrightarrow[0,\infty)$ which satisfy

(iv) $\limsup_{n\to\infty}\psi(t_n)<\frac{1}{2}\varphi(r)$ if $\lim_{n\to\infty}t_n=r>0$,

(v) $\lim_{n\to\infty}\psi(t_n)=0$ if $\lim_{n\to\infty}t_n=0$ for $t_n\in[0,\infty)$.

Now we are ready to prove our main result as follows.

Theorem 3. *Let $(X,\le)$ be a partially ordered set and there is a metric d on X such that (X,d) is a complete metric space. Suppose $F:X\times X\longrightarrow X$ and $g:X\longrightarrow X$ are such that F has the mixed g-monotone property and there exist two elements $x_0,y_0\in X$ with*

$$g(x_0)\le F(x_0,y_0)\quad \textit{and}\quad g(y_0)\ge F(y_0,x_0).$$

Assume there exist $\varphi\in\Phi$ and $\psi\in\Psi_\varphi$ such that

$$\varphi(d(F(x,y),F(u,v)))\le\psi(d(g(x),g(u))+d(g(y),g(v)))$$

for all $x,y,u,v\in X$ with $g(x)\le g(u)$, $g(y)\ge g(v)$. Suppose $F(X\times X)\subseteq g(X)$, g is continuous and w-commutative with F and suppose also either

(a) F is continuous or (b) X has the property $(*)$.

Then there exist $x,y\in X$ such that

$$g(x)=F(x,y)\quad \textit{and}\quad g(y)=F(y,x),$$

that is, F and g have a coupled coincidence point.

Proof. Consider $x_0,y_0\in X$ followed by assumptions. Since $F(X\times X)\subseteq g(X)$, then there exist $x_1,y_1\in X$ such that

$$F(x_0,y_0)=g(x_1)\quad\text{and}\quad F(y_0,x_0)=g(y_1).$$

Again we can choose $x_2,y_2\in X$ such that $F(x_1,y_1)=g(x_2)$ and $F(y_1,x_1)=g(y_2)$. Continuing the procedure above we have the sequence $\{x_n\}$ and $\{y_n\}$ recursively as follows.

$$F(x_n,y_n)=g(x_{n+1})\quad\text{and}\quad F(y_n,x_n)=g(y_{n+1})$$

for all $n\in\{0,1,2,...\}$. Now by induction, we prove that

$$g(x_n)\le g(x_{n+1})\quad\text{and}\quad g(y_{n+1})\le g(y_n) \tag{1}$$

for all $n\ge0$. Since $g(x_0)\le g(x_1)$ and $g(y_1)\le g(y_0)$, so the initial step of the induction is true. Suppose that (1) holds. Then using the mixed g-monotone property of F and (1) we obtain

$$g(x_{n+1})=F(x_n,y_n)\le F(x_{n+1},y_n)\le F(x_{n+1},y_{n+1})=g(x_{n+2}),$$

and consequently $g(x_{n+1}) \leq g(x_{n+2})$. Similarly, we can show that $g(y_{n+2}) \leq g(y_{n+1})$. Generally, we conclude that

$$g(x_n) \leq g(x_{n+1}) \quad \text{and} \quad g(y_{n+1}) \leq g(y_n) \quad \text{for all} \;\; n \geq 0.$$

Hence for each $n \in \mathbb{N}$

$$\begin{aligned} \varphi(d(g(x_n), g(x_{n+1}))) &= \varphi(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\ &\leq \psi(d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))). \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} \varphi(d(g(y_n), g(y_{n+1}))) &= \varphi(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &\leq \psi(d(g(y_{n-1}), g(y_n)) + d(g(x_{n-1}), g(x_n))). \end{aligned}$$

This together with (2) and sub-additivity of φ implies that

$$\varphi(\delta_{n+1}) \leq 2\psi(\delta_n) \quad \text{for all } n, \tag{3}$$

where $\delta_n := d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))$. Now we show that $\{g(x_n)\}, \{g(y_n)\}$ are Cauchy sequences. To do this, first assume that $\delta_n = 0$ for some n. Since $\varphi \in \Phi, \psi \in \Psi_\varphi$ by using (3) we have

$$g(x_m) = g(x_n) \;\text{ and }\; g(y_m) = g(y_n) \;\text{ for all } m \geq n$$

which shows that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences and there is nothing to prove. Otherwise, let $\delta_n > 0$ for all n. Then for any n we obtain

$$\varphi(\delta_{n+1}) \leq 2\psi(\delta_n) < \varphi(\delta_n). \tag{4}$$

Since φ is non-decreasing, so (4) implies that $\{\delta_n\}$ is a nonnegative decreasing sequence in $\mathbb{R}$. So we have $\lim_{n\to\infty} \delta_n = r$ for some $r \geq 0$. If $r > 0$, then following the properties of φ, ψ we get

$$\varphi(r) \leq \limsup_{n\to\infty} \varphi(\delta_{n+1}) \leq 2\limsup_{n\to\infty} \psi(\delta_n) < \varphi(r),$$

which is a contradiction. Hence $r = 0$ and so we have

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) = 0. \tag{5}$$

To show that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences, it suffices to prove that subsequences $\{g(x_{2n})\}$ and $\{g(y_{2n})\}$ are Cauchy sequences followed by (5). Suppose the opposite, that at least either $\{g(x_{2n})\}$ or $\{g(y_{2n})\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that for some subsequences $\{n(k)\}, \{m(k)\}$ of integers with $n(k) > m(k) \geq k$ we have

$$d(g(x_{2n(k)}), g(x_{2m(k)})) + d(g(y_{2n(k)}), g(y_{2m(k)})) \geq \epsilon \tag{6}$$

for $k \in \{1, 2, ...\}$. Suppose $r_k = d(g(x_{2n(k)}), g(x_{2m(k)})) + d(g(y_{2n(k)}), g(y_{2m(k)}))$ and $n(k)$ is the smallest integer with $n(k) > m(k) \geq k$ for which (6) holds. This means that

$$\begin{aligned}\epsilon \leq r_k \leq & d(g(x_{2m(k)}), g(x_{2n(k)-2})) + d(g(x_{2n(k)-2}), g(x_{2n(k)-1})) \\ & + d(g(x_{2n(k)-1}), g(x_{2n(k)})) + d(g(y_{2m(k)}), g(y_{2n(k)-2})) \\ & + d(g(y_{2n(k)-2}), g(y_{2n(k)-1})) + d(g(y_{2n(k)-1}), g(y_{2n(k)})) \\ < & \delta_{2n(k)-1} + \delta_{2n(k)} + \epsilon.\end{aligned}$$

Letting $k \longrightarrow \infty$ and using (5) we obtain that $\lim_{k\to\infty} r_k = \epsilon$. On the other hand, we have

$$\begin{aligned}r_k \leq & d(g(x_{2n(k)}), g(x_{2n(k)+1})) + d(g(x_{2n(k)+1}), g(x_{2m(k)+1})) \\ & + d(g(x_{2m(k)+1}), g(x_{2m(k)})) + d(g(y_{2n(k)}), g(y_{2n(k)+1})) \\ & + d(g(y_{2n(k)+1}), g(y_{2m(k)+1})) + d(g(y_{2m(k)+1}), g(y_{2m(k)})) \\ \leq & \delta_{2n(k)+1} + \delta_{2m(k)+1} + d(g(x_{2n(k)+1}), g(x_{2m(k)+1})) \\ & + d(g(y_{2n(k)+1}), g(y_{2m(k)+1})).\end{aligned}$$

Therefore,

$$r_k \leq \delta_{2n(k)+1} + \delta_{2m(k)+1} + d(g(x_{2n(k)+1}), g(x_{2m(k)+1})) + d(g(y_{2n(k)+1}), g(y_{2m(k)+1})).$$

Since φ is sub-additive, so by using (1) we have the following

$$\begin{aligned}\varphi(d(g(x_{2n(k)+1}), g(x_{2m(k)+1}))) &= \varphi(d(F(x_{2m(k)}, y_{2m(k)}), F(x_{2n(k)}, y_{2n(k)}))) \\ &\leq \psi(d(g(x_{2m(k)}), g(x_{2n(k)})) + d(g(y_{2m(k)}), g(y_{2n(k)}))).\end{aligned}$$

So we obtain

$$\varphi(d(g(x_{2n(k)+1}), g(x_{2m(k)+1}))) \leq \psi(r_k). \tag{7}$$

Similarly, we can show that $\varphi(d(g(y_{2n(k)+1}), g(y_{2m(k)+1}))) \leq \psi(r_k)$. This together with (7) implies that

$$\varphi(d(g(x_{2n(k)+1}), g(x_{2m(k)+1})) + d(g(y_{2n(k)+1}), g(y_{2m(k)+1}))) \leq 2\psi(r_k),$$

and so we have

$$\varphi(r_k) \leq \varphi(\delta_{2n(k)+1}) + \varphi(\delta_{2m(k)+1}) + 2\psi(r_k)$$

for all k. Since φ is lower semi-continuous, by taking the limit as $k \longrightarrow \infty$ we get

$$\begin{aligned}\varphi(\epsilon) &\leq \limsup_{k\to\infty} \varphi(r_k) \\ &\leq \lim_{k\to\infty} \varphi(\delta_{2n(k)+1}) + \lim_{k\to\infty} \varphi(\delta_{2m(k)+1}) + 2\limsup_{k\to\infty} \psi(r_k) < \varphi(\epsilon)\end{aligned}$$

which is a contradiction. Thus $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in complete metric space X and hence

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y$$

for some $x, y \in X$. Since g is continuous, so we get

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \text{ and } \lim_{n\to\infty} g(g(y_n)) = g(y).$$

Now using the fact that g is w-commutative with F we have the following two possible cases.

(i) If F is continuous, then we obtain

$$\begin{aligned}
\lim_{n\to\infty} d(g(x), g(g(x_{n+1}))) &= \lim_{n\to\infty} d(g(x), g(F(x_n, y_n))) \\
&= \lim_{n\to\infty} d(g(x), F(g(x_n), g(y_n))) \\
&= d(g(x), F(x, y)), \\
\lim_{n\to\infty} d(g(y), g(g(y_{n+1}))) &= \lim_{n\to\infty} d(g(y), g(F(y_n, x_n))) \\
&= \lim_{n\to\infty} d(g(y), F(g(y_n), g(x_n))) \\
&= d(g(y), F(y, x)),
\end{aligned}$$

which implies that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

(ii) If X has the property $(*)$, then for any n we get $g(x_n) \leq x$ and $y \leq g(y_n)$ which yields the following.

$$\begin{aligned}
\varphi(d(g(x), F(x, y))) &\leq \varphi(d(g(x), g(g(x_{n+1})))) + \varphi(d(g(g(x_{n+1})), F(x, y))) \\
&= \varphi(d(g(x), g(g(x_{n+1})))) + \varphi(d(g(F(x_n, y_n)), F(x, y))).
\end{aligned}$$

Since F and g are w-commutative, by letting $n \longrightarrow \infty$ we have

$$\begin{aligned}
\varphi(d(g(x), F(x, y))) &\leq \limsup_{n\to\infty} \varphi(d(F(g(x_n), g(y_n)), F(x, y))) \\
&\leq \limsup_{n\to\infty} \psi(d(g(x_n), g(x)) + d(g(y_n), g(y))) = 0
\end{aligned}$$

Hence, we obtain $g(x) = F(x, y)$. On the other hand,

$$\begin{aligned}
\varphi(d(g(y), F(y, x))) &\leq \varphi(d(g(y), g(g(y_{n+1})))) + \varphi(d(g(g(y_{n+1})), F(y, x))) \\
&= \varphi(d(g(y), g(g(y_{n+1})))) + \varphi(d(g(F(y_n, x_n)), F(y, x))).
\end{aligned}$$

Again since F and g are w-commutative, by taking the limit as $n \longrightarrow \infty$ we have

$$\begin{aligned}
\varphi(d(g(y), F(y, x))) &\leq \limsup_{n\to\infty} \varphi(d(F(g(y_n), g(x_n)), F(y, x))) \\
&\leq \lim_{n\to\infty} \psi(d(g(y_n), g(y)) + d(g(x_n), g(x))) = 0.
\end{aligned}$$

Hence, we get $g(y) = F(y, x)$. Therefore, F and g have a coupled coincidence point (x, y). □

Remark 1. *In Theorem 3, let g be the identity mapping. Then substituting* $\frac{1}{2}\varphi(x) - \psi(\frac{x}{2})$ *for* $\psi(x)$ *implies the main result of Luong and Thuan in [9] (Theorem 2 of the present paper). Note that the function* $\frac{1}{2}\varphi(x) - \psi(\frac{x}{2})$ *satisfies all the conditions of* *our result. In order to verify this, since in Theorem 2 φ is a continuous function and* $\lim_{t\to r}\psi(t) > 0$ *for all* $r > 0$*, we obtain*

$$\limsup_{n\to\infty}(\frac{1}{2}\varphi(t_n) - \psi(\frac{t_n}{2})) \le \frac{1}{2}\varphi(r) - \liminf_{n\to\infty}\psi(\frac{t_n}{2}) < \frac{1}{2}\varphi(r),$$

for all $t_n \in [0,\infty)$ *such that* $\lim_{n\to\infty} t_n = r > 0$*. On the other hand,*

$$\lim_{t\to 0+}\psi(t) = 0 \quad \textit{implies} \quad \lim_{n\to\infty}\psi(t_n) = 0$$

if $\lim_{n\to\infty} t_n = 0$ *for* $t_n \in [0,\infty)$*.*

Remark 2. *In Theorem 3, let φ be the identity mapping. Then it is easy to see that replacing $\psi(x)$ by $\varphi(\frac{x}{2})$ yields the main result of Lakshmikantham and Ćirić in [8].*

Now we prove the uniqueness of the coupled fixed point by defining a partial ordering on $X \times X$. We remark that if $(X, \le)$ is a partial ordered set then $X \times X$ can be endowed with the following partial ordering:

$$\text{for } (x,y),(u,v) \in X \times X, \quad (x,y) \le (u,v) \Longleftrightarrow x \le u, y \ge v.$$

Theorem 4. *Let all the conditions of Theorem 3 be fulfilled and for every $(x,y),(u,v)$ in $X \times X$, there exists a (z,t) in $X \times X$ such that $(g(z), g(t))$ is comparable to both $(g(x), g(y))$ and $(g(u), g(v))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique (x,y) in $X \times X$ such that*

$$x = g(x) = F(x,y) \ \textit{ and } \ y = g(y) = F(y,x).$$

Proof. From Theorem 3 the set of coupled coincidence point is non-empty. Assume that (x,y) and (u,v) are two coupled coincidence points, that is,

$$g(x) = F(x,y), \quad g(y) = F(y,x),$$
$$g(u) = F(u,v), \quad g(v) = F(v,u).$$

By hypotheses, there exists a $(s,t) \in X \times X$ such that $(g(s), g(t))$ is comparable to both $(g(x), g(y))$ and $(g(u), g(v))$. Take $s_0 = s$ and $t_0 = t$ and define sequences $\{s_n\}$ and $\{t_n\}$ as follows

$$g(s_{n+1}) = F(s_n, t_n) \ \text{ and } \ g(t_{n+1}) = F(t_n, s_n)$$

for all $n \ge 0$. Since $(g(s), g(t))$ is comparable to $(g(x), g(y))$, we may assume that $(g(s_0), g(t_0)) = (g(s), g(t)) \le (g(x), g(y))$ (the other case is similar). Now, by using the induction and the mixed g-monotone property of F we obtain $(g(s_n), g(t_n)) \le (g(x), g(y))$ for all n. Hence we get

$$\begin{aligned}\varphi(d(g(x), g(s_{n+1}))) &= \varphi(d(F(x,y), F(s_n,t_n))) \\ &\le \psi(d(g(s_n), g(x)) + d(g(t_n), g(y))),\end{aligned}$$

and

$$\begin{aligned}\varphi(d(g(y), g(t_{n+1}))) &= \varphi(d(F(y,x), F(t_n,s_n))) \\ &\le \psi(d(g(t_n), g(y)) + d(g(s_n), g(x))).\end{aligned}$$

Thus, we have $\varphi(\delta_{n+1}) \leq 2\psi(\delta_n)$, where $\delta_n = d(g(x), g(s_n)) + d(g(y), g(t_n))$. Inspired by the proof of Theorem 3 we can conclude that δ_n converges to r for some $r \geq 0$. If $r > 0$, then we have

$$\varphi(r) \leq \limsup_{n\to\infty} \varphi(\delta_{n+1}) \leq 2\limsup_{n\to\infty} \psi(\delta_n) < \varphi(r),$$

which shows a contradiction. Hence we get

$$\lim_{n\to\infty} d(g(x), g(s_n)) = \lim_{n\to\infty} d(g(y), g(t_n)) = 0. \tag{8}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(g(u), g(s_n)) = \lim_{n\to\infty} d(g(v), g(t_n)) = 0.$$

This together with (8) implies that

$$g(x) = g(u) \text{ and } g(y) = g(v). \tag{9}$$

Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by w-commutativity of F and g we get

$$g(g(x)) = g(F(x, y)) = F(g(x), g(y)), \quad g(g(y)) = g(F(y, x)) = F(g(y), g(x)).$$

By replacing $g(x)$ and $g(y)$ by p and q, respectively, we get

$$g(p) = F(p, q) \text{ and } g(q) = F(q, p).$$

Hence (p, q) is a coupled coincidence point. So u and v can be replaced by p and q in (9), respectively, which implies that $g(p) = g(x)$ and $g(q) = g(y)$, that is,

$$p = g(p) = F(p, q) \text{ and } q = g(q) = F(q, p).$$

Thus (p, q) is a coupled fixed point of F and g. To prove the uniqueness, let (z, w) be another coupled fixed point. Then (9) implies that $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. □

Remark 3. *Comparing the conditions in Theorem 4 and the conditions in Theorem 2.4 of Luong and Thuan [9], we see that our result is a generalization of Theorem 2.4 in [9].*

Theorem 5. *Let all the conditions of Theorem 3 be fulfilled and for x_0, y_0 in Theorem 3 let us further suppose that $g(x_0), g(y_0)$ are comparable. Then we have*

$$g(x) = F(x, x) \textit{ for some } x \in X.$$

Proof. Following the aspects of the proof of Theorem 3, without loss of generality let $g(x_0) \leq g(y_0)$. By taking $g(x_{n+1}) = F(x_n, y_n)$, $g(y_{n+1}) = F(y_n, x_n)$ for $n \geq 0$ and using the mixed g-monotone property of F we conclude that

$$g(x_n) \leq g(y_n) \text{ for all } n \geq 0.$$

Then sub-additivity φ implies that

$$\begin{aligned}\varphi(d(x,y)) &\leq \varphi(d(x, g(x_{n+1}))) + \varphi(d(g(x_{n+1}), g(y_{n+1}))) + \varphi(d(g(y_{n+1}), y))\\ &= \varphi(d(x, g(x_{n+1}))) + \varphi(d(F(x_n, y_n), F(y_n, x_n))) + \varphi(d(g(y_{n+1}), y))\\ &\leq \varphi(d(x, g(x_{n+1}))) + \psi(2d(g(x_n), g(y_n)))) + \varphi(d(g(y_{n+1}), y)).\end{aligned}$$

By taking the limit as $n \longrightarrow \infty$ since, $g(x_n) \longrightarrow x$ and $g(y_n) \longrightarrow y$, we obtain

$$\varphi(d(x,y)) \leq \limsup_{n\to\infty} \psi(2d(g(x_n), g(y_n))).$$

Now if $x \neq y$, then

$$\varphi(d(x,y)) < \varphi(d(x,y)),$$

which is a contradiction. So $x = y$ and $g(x) = F(x,x)$. □

Remark 4. *Note that by substituting the identity mapping for g in the preceding theorem Theorem 2.6 in [9] will be concluded.*

3. Application

In this section, Theorem 5 is used to guarantee the existence of a unique solution of the following integral equation

$$g(u(t)) = \int_a^b (K_1(s,t) - K_2(s,t))(f_1(s, u(s)) + f_2(s, u(s)))ds + h(t), \quad (10)$$

where g is a strictly increasing function and $t \in I = [a,b]$.
Suppose that

$$\begin{aligned} K_1(s,t), K_2(s,t) &\geq 0, \quad \text{for all } t, s \in [a,b],\\ 0 \leq f_1(t,x) - f_1(t,y) &\leq \lambda\psi(g(x) - g(y)), \\ -\mu\psi(g(x) - g(y)) &\leq f_2(t,x) - f_2(t,y) \leq 0,\end{aligned} \quad (11)$$

for some $\lambda, \mu > 0$ and $x, y \in \mathbb{R}$, $y \leq x$, $t \in [a,b]$ and ψ is an increasing function such that

$$\begin{aligned}\limsup_{n\to\infty} \psi(t_n) < \tfrac{1}{2}\alpha r \ \text{ if } \ \lim_{n\to\infty} t_n = r > 0,\\ \lim_{n\to\infty} \psi(t_n) = 0 \ \text{ if } \ \lim_{n\to\infty} t_n = 0\end{aligned} \quad (12)$$

for $t_n \in [0,\infty)$ and some $\alpha > 0$ such that $\alpha\beta \leq \frac{1}{2}$ where

$$\beta = \max\{\lambda, \mu\} \sup_{t\in I} \int_a^b (K_1(s,t) + K_2(s,t))ds.$$

Note that by taking $\psi(t) = \gamma t$ for $0 < \gamma < \frac{1}{2}\alpha$ condition (12) holds.

Definition 5. *We say that an element $(u, v) \in C(I) \times C(I)$ is a coupled lower and upper solution of the integral equation (10) if $u(t) \le v(t)$ and*

$$g(u(t)) \le \int_a^b K_1(s,t)(f_1(s,u(s)) + f_2(s,v(s)))ds - \int_a^b K_2(s,t)(f_1(s,v(s)) + f_2(s,u(s)))ds + h(t)$$

and

$$g(v(t)) \ge \int_a^b K_1(s,t)(f_1(s,v(s)) + f_2(s,u(s)))ds - \int_a^b K_2(s,t)(f_1(s,u(s)) + f_2(s,v(s)))ds + h(t)$$

for all $t \in I = [a, b]$.

Theorem 6. *Consider integral equations (10). Suppose that $K_i, f_i \in C(I \times I)$ for $i = 1, 2$ such that conditions (11) and (12) are satisfied. Also, let $h, g \in C(I)$ and g be a strictly increasing function on $C(I)$. Then the existence of a coupled lower and upper solution for (10) provides the existence of a unique solution (10) in $C(I)$.*

Proof. Define a partially ordering on $C(I)$ as follows.

$$u, v \in C(I), \quad u \le v \Longleftrightarrow u(t) \le v(t), \text{ for all } t \in I = [a, b].$$

Using the metric

$$d(u, v) = \sup_{t \in I} |u(t) - v(t)|, \quad u, v \in C(I),$$

$C(I)$ is clearly a complete metric space. It is easy to verify that condition (b) in Theorem 3 holds on the complete metric space $C(I)$. We also define a partially ordering on $C(I) \times C(I)$ by

$$(x, y), (u, v) \in C(I) \times C(I), \quad (x, y) \le (u, v) \Longleftrightarrow x(t) \le u(t), y(t) \ge v(t)$$

for all $t \in I = [a, b]$. We easily see that for every $(x, y), (u, v) \in C(I) \times C(I)$,

$$(g(x), g(y)) \le (g(\max\{x, u\}), g(\min\{y, v\})),$$
$$(g(u), g(v)) \le (g(\max\{x, u\}), g(\min\{y, v\})).$$

So $(\max\{x, u\}, \min\{y, v\}) \in C(I) \times C(I)$ is comparable to both (x, y) and (u, v). Now we define $F : C(I) \times C(I) \longrightarrow C(I)$ by

$$F(x,y)(t) = \int_a^b K_1(s,t)(f_1(s,x(s)) + f_2(s,y(s)))ds - \int_a^b K_2(s,t)(f_1(s,y(s)) + f_2(s,x(s)))ds + h(t)$$

for all $t \in I = [a, b]$.

At first, we prove that F has the mixed g-monotone property. To do this, let $x_1, x_2 \in C(I)$ and $g(x_1) \leq g(x_2)$, that is, $g(x_1(t)) \leq g(x_2(t))$ for all $t \in [a, b]$. So $x_1(t) \leq x_2(t)$ for all $t \in [a, b]$. Then using condition (11) for any $y \in C(I)$ and all $t \in [a, b]$ we obtain

$$\begin{aligned} F(x_1, y)(t) - F(x_2, y)(t) = &\int_a^b K_1(s,t)(f_1(s, x_1(s)) - f_1(s, x_2(s)))ds \\ &- \int_a^b K_2(s,t)(f_2(s, x_1(s)) - f_2(s, x_2(s)))ds \leq 0, \end{aligned}$$

which implies that $F(x_1, y) \leq F(x_2, y)$. Similarly, if $y_1, y_2 \in C(I)$ and $g(y_1) \leq g(y_2)$, then $F(x, y_2) \leq F(x, y_1)$ for any $x \in C(I)$. Now, let $\alpha > 0$ be as given in (12). Then for $x, y, u, v \in C(I)$ such that $g(x) \geq g(u)$ and $g(y) \leq g(v)$ we get

$$\begin{aligned} \alpha d(F(x,y), F(u,v)) = &\alpha \sup_{t \in I} \bigg| \int_a^b K_1(s,t)[f_1(s, x(s)) - f_1(s, u(s)) \\ &+ f_2(s, y(s)) - f_2(s, v(s))]ds \\ &+ \int_a^b K_2(s,t)[f_2(s, u(s)) - f_2(s, x(s)) \\ &+ f_1(s, v(s)) - f_1(s, y(s))]ds \bigg|. \end{aligned}$$

This together with condition (11) implies that

$$\begin{aligned} \alpha d(F(x,y), F(u,v)) \leq &\alpha \sup_{t \in I} \bigg| \int_a^b K_1(s,t)[\lambda\psi(g(x(s)) - g(u(s))) \\ &+ \mu\psi(g(v(s)) - g(y(s)))]ds \\ &+ \int_a^b K_2(s,t)[\mu\psi(g(x(s)) - g(u(s))) \\ &+ \lambda\psi(g(v(s)) - g(y(s)))]ds \bigg| \\ \leq &\alpha\beta[\psi(d(g(x), g(u))) + \psi(d(g(y), g(v)))] \\ \leq &2\alpha\beta\psi[d(g(x), g(u)) + d(g(y), g(v))]. \end{aligned}$$

So we obtain

$$\alpha d(F(x,y), F(u,v)) \leq \psi[d(g(x), g(u)) + d(g(y), g(v))].$$

Now by taking $\varphi(t) = \alpha t$ and assuming $(r, s) \in C(I) \times C(I)$ as a coupled lower and upper solution of (10) we have $g(r) \leq F(r, s)$ and $g(s) \geq F(s, r)$. Therefore, since $g(r) \leq g(s)$, so all conditions in Theorem 5 are satisfied and there exists a unique $u \in C(I)$ such that $g(u(t)) = F(u, u)(t)$ for all $t \in [a, b]$, that is, the integral equation (10) has a unique solution in $C(I)$. $\square$

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References

[1] R. P. AGARWAL, M. A. EL-GEBEILY, D. O'REGAN, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87**(2008), 1–8.

[2] R. P. AGARWAL, M. MEEHAN, D. O'REGAN, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[3] S. BANACH, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3**(1922), 133–181.

[4] V. BERINDE, M. BORCUT, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. **74**(2011), 4889–4897.

[5] T. G. BHASKAR, V. LAKSHMIKANTHAM, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65**(2006), 1379–1393.

[6] A. BRANCIARI, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, IJMMS **29**(2002), 531–536.

[7] J. DUGUNDJI, A. GRANAS, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[8] V. LAKSHMIKANTHAM, L. ĆIRIĆ, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70**(2009), 4341–4349.

[9] N. V. LUONG, N. X. THUAN, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74**(2011), 983–992.

[10] A. MEIR, E. KEELER, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28**(1969), 326–329.

[11] B. SAMET, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal. **72**(2010), 4508–4517.

[12] D. R. SMART, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1974.

[13] T. SUZUKI, *A generalized Banach contraction principle which characterizes metric completeness*, Proc. Amer. Math. Soc. **136**(2008), 1861–1869.

[14] E. ZEIDLER, *Nonlinear Functional Analysis and Its Applications I: Fixed Point Theorems*, Springer-Verlag, Berlin, 1986.