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Stability of Fréchet functional equation in non-Archimedean normed spaces*

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Abstract. We will establish stability of Fréchet functional equation

$$\Delta^n_{x_1,\dots,x_n} f(y) = 0$$

in non-Archimedean normed spaces for some unbounded control function. Among some applications of our results, we will give a counterexample to show that the nature of stability in non-Archimedean normed spaces is different from one in classical normed spaces.
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1. Introduction

In 1821, in his famous book [5] A. L. Cauchy proved that a continuous mapping $f : \mathbb{R} \to \mathbb{R}$ is additive if and only if there is some $c \in \mathbb{R}$ such that $f(x) = cx$ for each $x \in \mathbb{R}$. Since then, the additive functional equation $f(x+y) = f(x)+f(y)$ is known by his name.

Let X and Y be linear spaces. For a function $f : X \to Y$ and $x \in X$, let

$$\Delta_x f(y) = f(x + y) - f(y) \quad (y \in X).$$

Inductively, we define

$$\Delta^n_{x_1,\dots,x_n} f(y) = \Delta^{n-1}_{x_1,\dots,x_{n-1}}(\Delta_{x_n} f(y)) \quad (y, x_1, \dots, x_n \in X).$$

If $x_1 = \cdots = x_n = x$, we write $\Delta^n_x f(y) = \Delta^n_{x_1,\dots,x_n} f(y)$, where $x, y \in X$.

It is known that a function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ satisfies the Cauchy equation if and only if $\Delta^2_x f(y) = 0$ for each $x, y \in \mathbb{R}$ (see e. g. [2]).

In 1909, M. Fréchet [7] had showed that a continuous mapping $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree n if and only if $\Delta^{n+1}_{x_1,\dots,x_{n+1}} f(0) = 0$ for each $x_1, \dots, x_{n+1} \in \mathbb{R}$ (a simpler proof of this fact can be found in Lemma 2 of [2]).

A function $f : X \to Y$ is called a *polynomial* of degree n if it is a solution of the Fréchet functional equation of degree $n + 1$,

$$\Delta^{n+1}_{x_1,\dots,x_{n+1}} f(0) = 0. \tag{1}$$

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The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [21] posed the first stability problem. In 1941, Hyers [10] gave the first significant partial solution to his problem. Th. M. Rassias [19] improved Hyers' theorem by weakening the condition for the Cauchy difference controlled by $||x||^p + ||y||^p$, $p \in [0, 1)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called the Hyers–Ulam–Rassias stability.

In [3], L. M. Arriola and W. A. Beyer initiated the study of the stability of functional equations in non-Archimedean spaces [20]. In fact they established stability of Cauchy functional equations over p-adic fields. In [15], [16] and [18] the stability of Cauchy, quadratic and quartic functional equations in non-Archimedean normed spaces were investigated.

The stability of Fréchet functional equation was initiated by D. H. Hyers in [11]. In 1999 this result was generalized by Borelli et al. [4]. Other versions of this problem have been recently considered by some authors (see, e. g., [1, 6, 8, 12, 14, 17, 22, 23] and the references therein).

In this paper, we adopt some ideas from [4], [11] and [15] to establish stability of Fréchet functional equation of degree $m-1$, $m > 2$, in non-Archimedean normed linear spaces. More precisely, we will show that if $f : X \to Y$ satisfies

$$||\Delta^m_{x_1,\ldots,x_m} f(0)||_Y \leq \varphi_m(||x_1||_X, \ldots, ||x_m||_X) \quad (x_1, \ldots, x_m \in X),$$

(where X and Y are two non-Archimedean normed vector spaces over the same non-Archimedean vector field $\mathbb{K}$) for a suitable control function $\varphi_m : \mathbb{R}^m_+ \to \mathbb{R}_+$, there exists a unique polynomial $p_{m-1} : X \to Y$ of degree at most $m-1$ such that

$$||f(x) - p_{m-1}(x)||_Y \leq |k|^{-pm}\varphi_m(||x||_X, \ldots, ||x||_X) \quad (x \in X),$$

where k is the smallest positive integer $k \in \mathbb{K}$ with $|k| < 1$ and $0 \leq p < 1$. In section 3, among some applications of our results, we will give an example to show that Hyers' theorem in [11] cannot be applied in non-Archimedean normed spaces. Therefore, Fréchet stability phenomenon in non-Archimedean normed spaces is of different nature from the one in classical normed spaces.

2. Results

Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that for any $a, b \in \mathbb{K}$, $|a+b| \leq \max\{|a|, |b|\}$, $|ab| = |a||b|$, and $|a| = 0$ if and only if $a = 0$. The last inequality is called the strong triangle inequality or ultrametric inequality. It is important to note that all valued field $\mathbb{K}$ has zero characteristic. In particular, this implies that, if $(\mathbb{K}, |\cdot|)$ is a non-Archimedean field with a non trivial absolute value $|\cdot|$, then $\mathbb{Q} \hookrightarrow \mathbb{K}$ and we will assume in all what follows that $\mathbb{Q} \subseteq \mathbb{K}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $||\ .\ || : X \to \mathbb{R}_+$ is a non-Archimedean norm (valuation) if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric inequality); namely,

$$||x + y|| \leq \max\{||x||, ||y||\} \quad (x, y \in X).$$

Then $(X, ||\ .\ ||)$ is called a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent. It is important, for our objectives, to note that any non-Archimedean vector space X over a non-Archimedean valued field $\mathbb{K}$ is also $\mathbb{Q}$-vector space, since $\mathbb{Q} \subseteq \mathbb{K}$.

Hereafter, unless otherwise is explicitly stated, we will assume that X and Y are non-Archimedean normed spaces over a non-Archimedean field $\mathbb{K}$ with a valuation $|\cdot|$ and Y is complete. Furthermore, we suppose that $k \in \mathbb{K}$ is the smallest positive integer with $|k| < 1$ and, for each $m \geq 2$, we assume that $\varphi_m : \mathbb{R}_+^m \to \mathbb{R}_+$ is a non-decreasing mapping with respect to each variable on $\mathbb{R}_+^m$ such that for some $0 \leq p < 1$,

$$\varphi_m(|k|^{-1}t_1, \dots, |k|^{-1}t_m) \leq |k|^{-p}\varphi_m(t_1, \dots, t_m) \quad (t_1, \dots, t_m) \in \mathbb{R}_+^m. \tag{2}$$

For example, $\varphi_m(t_1, \dots, t_m) = \max\{t_1^p, \dots, t_m^p\},\ t_1, \dots, t_m \in \mathbb{R}_+$, satisfies the above conditions. We first prove the main result of this paper in the following special case. Although its proof is similar to that of [15, Theorem 2.1], but for the sake of completeness and self-containment, we give here a direct proof.

Theorem 1. *Let $f : X \to Y$ satisfy the inequality*

$$||\Delta_{x_1,x_2} f(0)||_Y \leq \varphi_2(||x_1||_X, ||x_2||_X) \quad (x_1, x_2 \in X), \tag{3}$$

Then there exists a unique additive mapping $\mathcal{M}_1 : X \to Y$ such that

$$||f(x) - f(0) - \mathcal{M}_1(x)||_Y \leq |k|^{-p}\varphi_2(||x||_X, ||x||_X) \quad (x \in X). \tag{4}$$

The function $\mathcal{M}_1$ is given by the formula

$$\mathcal{M}_1(x) = \lim_{n\to\infty} k^n \Delta_{k^{-n}x} f(0) \quad (x \in X).$$

Proof. By (3), we have

$$||f(x_1 + x_2) - f(x_1) - f(x_2) + f(0)||_Y \leq \varphi_2(||x_1||_X, ||x_2||_X) \quad (x_1, x_2 \in X). \tag{5}$$

Let $g = f - f(0)$. Then by (5) we have

$$||g(x_1 + x_2) - g(x_1) - g(x_2)||_Y \leq \varphi_2(||x_1||_X, ||x_2||_X) \quad (x_1, x_2 \in X). \tag{6}$$

We will show that for each $x \in X$ and $2 \leq j \leq k$,

$$||g(jx) - jg(x)||_Y \leq \varphi_2(||x||_X, ||x||_X), \quad (x \in X). \tag{7}$$

Put $x_1 = x_2 = x$ into (6) to obtain

$$||g(2x) - 2g(x)||_Y \leq \varphi_2(||x||_X, ||x||_X), \quad (x \in X).$$

This proves (7) for $j = 2$. Let (7) hold for some $2 < j < k$. Replacing x_1 by x and y by jx in (6), we see that

$$||g((j + 1)x) - g(x) - g(jx)||_Y \leq \varphi_2(||x||_X, ||jx||_X) = \varphi_2(||x||_X, ||x||_X), \tag{8}$$

for each $x \in X$. Since

$$g((j+1)x) - (j+1)g(x) = g((j+1)x) - g(x) - g(jx) + g(jx) - jg(x)$$

for each $x \in X$, it follows from (8) and our induction hypothesis that

$$\begin{aligned} ||g((j+1)x) - (j+1)g(x)||_Y &\leq \max\{||g((j+1)x) - g(x) - g(jx)||_Y, \\ &\qquad ||g(jx) - jg(x)||_Y\} \\ &\leq \varphi_2(||x||_X, ||x||_X) \quad (x \in X). \end{aligned}$$

This proves (7). In particular,

$$||g(kx) - kg(x)||_Y \leq \varphi_2(||x||_X, ||x||_X) \quad (x \in X). \tag{9}$$

It follows that for each $n \in \mathbb{N}$ and $x \in X$,

$$\begin{aligned} ||k^{(n-1)}g(k^{-(n-1)}x) - k^n g(k^{-n}x)||_Y &\leq |k|^{(n-1)}\varphi_2(||k^{-n}x||_X, ||k^{-n}x||_X) \\ &\leq |k|^{n-1-pn}\varphi_2(||x||_X, ||x||_X). \end{aligned} \tag{10}$$

Since the right-hand side of the above inequality tends to zero as n tends to infinity, it follows from the altrametric inequality and (10) that $\{k^n g(k^{-n}x)\}$ is a Cauchy sequence in Y. Thanks to completeness of Y, $\mathcal{M}_1(x) = \lim_{n\to\infty} k^n \Delta_{k^{-n}x} f(0) = \lim_{n\to\infty} k^n g(k^{-n}x)$ for each $x \in X$ exists. Since for each $n \geq 1$ and $x \in X$,

$$\begin{aligned} ||g(x) - k^n g(k^{-n}x)||_X &= ||\sum_{i=1}^{n} k^{i-1} g(k^{-(i-1)}x) - k^i g(k^{-i}x)||_Y \\ &\leq \max\{||k^{i-1}g(k^{-(i-1)}x) - k^i g(k^{-i}x)||_Y : \ 1 \leq i \leq n\} \\ &\leq |k|^{-p}\varphi(||x||_X, ||x||_X), \end{aligned}$$

the inequality (4) holds. The additivity of $\mathcal{M}_1$ follows from the following inequality.

$$\begin{aligned} &||\mathcal{M}_1(x+y) - \mathcal{M}_1(x) - \mathcal{M}_1(y)||_Y \\ &\quad = \lim_{n\to\infty} ||k^n g(k^{-n}(x+y)) - k^n g(k^{-n}x) - k^n g(k^{-n}y)||_Y \\ &\quad \leq \lim_{n\to\infty} |k|^{n(1-p)}\varphi(||x||_X, ||y||_X) = 0 \quad (x, y \in X). \end{aligned}$$

Let $\mathcal{M}'_1$ be another additive map such that

$$||f(x) - f(0) - \mathcal{M}'_1(x)||_Y \leq |k|^{-p}\varphi_2(||x||_X, ||x||_X) \quad (x \in X).$$

Then by the altrametric inequality

$$||\mathcal{M}_1(x) - \mathcal{M}'(x)|| \leq |k|^{-p}\varphi_2(||x||_X, ||x||_X) \quad (x \in X).$$

Therefore for each $n \in \mathbb{N}$ and $x \in X$, we have

$$\begin{aligned} ||\mathcal{M}_1(x) - \mathcal{M}'(x)|| &= ||k^n \mathcal{M}_1(k^{-n}x) - k^n \mathcal{M}'(k^{-n}x)|| \\ &\leq |k|^n \varphi_2(||k^{-n}x||_X, ||k^{-n}x||_X) \\ &\leq |k|^{n(1-p)}\varphi_2(||x||_X, ||x||_X). \end{aligned}$$

Since the right-hand side of the above inequality tends to zero as n tends to infinity $\mathcal{M}_1 = \mathcal{M}'_1$.

□

In order to extend Theorem 1, we need to the following definition.

Definition 1. *Let X and Y be two arbitrary $\mathbb{Q}$-linear spaces. A function $T : X^n \to Y$ is called n-additive if it is additive with respect to each variable. It follows from the definition that if $T : X^n \to Y$ is n-additive and $f : X \to Y$ is defined by $f(x) = T(x, \dots, x)$, then for each $r \in \mathbb{Q}$ and $x \in X$, $f(rx) = r^n f(x)$.*

A function $\mathcal{M} : X \to Y$ is said to be a monomial of degree n *if $\mathcal{M}(rx) = r^n\mathcal{M}(x)$ for all $x \in X$ and $r \in \mathbb{Q}$.*

We call a function $p : X \to Y$ a transformation of degree n *if $p(x) = \mathcal{M}_0(x) + \cdots + \mathcal{M}_n(x)$, where $\mathcal{M}_i$ is a monomial of degree i for $0 \le i \le n$ and $\mathcal{M}_n$ is not identically zero.*

S. Mazur and W. Orlicz proved the following.

Theorem 2 (see [13])**.** *Let $\mathcal{M} : X \to Y$, where X and Y are $\mathbb{Q}$-linear spaces. If $\mathcal{M}$ is a monomial of degree m, then there is a unique symmetric m-additive mapping $T : X^m \to Y$ such that*

$$\mathcal{M}(x) = T(x, \dots, x) \quad (x \in X).$$

The mapping T is defined by the formula

$$T(x_1, \dots, x_m) = \frac{1}{m!}\Delta^m_{x_1,\dots,x_m}\mathcal{M}(x) \quad (x, x_1, \dots, x_m \in X).$$

In particular, if $\mathcal{M}$ is a monomial of degree at most m, then $\Delta^{m+1}_{x_1,\dots,x_{m+1}}\mathcal{M}(x) = 0$ for each $x, x_1, \dots, x_{m+1} \in X$.

It follows immediately from Theorem 2 that for any transformation $p : X \to Y$ of degree at most m,

$$\Delta^{m+1}_{x_1,\dots,x_{m+1}} p\ (x) = 0 \quad (x, x_1, \dots, x_{m+1} \in X).$$

The authors in [13] have shown that the converse of this statement is also true. So that we have the following.

Corollary 1. *Let X and Y be $\mathbb{Q}$-linear spaces. Then for a mapping $p : X \to Y$, the following is equivalent.*

(1) p is a transformation of degree (at most) m;

(2) p is a polynomial of degree (at most) m.

Lemma 1. *Let $f : X \to Y$ satisfy the inequality*

$$||f(x+y)-f(x)-f(y)+f(0)-\mathcal{Q}(x,y)-\mathcal{M}(x,y)||_Y \le \varphi_2(||x||_X, ||y||_X) \quad (x, y \in X), \tag{11}$$

where $\mathcal{Q}(x, y)$ is a polynomial of degree at most $m-2$ with respect to x and $\mathcal{M}(x, y)$ is a monomial of degree $m-1$ with respect to x, $(m > 1)$. Then

$$\mathcal{M}_m(x) = \frac{1}{m}\mathcal{M}(x, x) \quad (x \in X)$$

defines a monomial of degree m. Moreover, we have

$$\mathcal{M}_m(x) = \frac{1}{m!}\lim_{n\to\infty} k^{nm}\Delta^m_{k^{-n}x} f(0) \quad (x \in X),$$

and

$$\mathcal{M}(x,y) = \frac{1}{(m-1)!}\lim_{n\to\infty} k^{(m-1)n}\Delta^{m-1}_{k^{-n}x} g(0,y) \quad (x,y \in X)$$

where $g(x,y) = f(x+y) - f(x)$ for each $x, y \in X$.

Proof. By Theorem 2, there exits a function $T_1 : X^m \to Y$ which is additive and symmetric with respect to the first $m-1$ variables such that

$$\mathcal{M}(x,y) = T_1(x,\dots,x,y) \quad (x,y \in X) \tag{12}$$

and

$$T_1(x_1,\dots,x_{m-1},y) = \frac{1}{(m-1)!}\Delta_{x_1,\dots,x_{m-1}}\mathcal{M}(x,y) \quad (x,x_1,\dots,x_{m-1},y \in X).$$

Put $T(x_1,\dots,x_{m-1},y) = (m-1)!T_1(x_1,\dots,x_{m-1},y)$, $(x_1,\dots,x_{m-1},y \in X)$. Then

$$\Delta_{x_1,\dots,x_{m-1}}\mathcal{M}(x,y) = T(x_1,\dots,x_{m-1},y) \quad (x,x_1,\dots,x_{m-1},y \in X).$$

Let $\mathcal{P}_{m-1}$ denote the set of all permutations on $\{1,\dots,m-1\}$. Thanks to (11), it follows that for each $x_1,\dots,x_{m-1},y \in X$,

$$\begin{aligned} &||\Delta^m_{x_1,\dots,x_{m-1},y} f(0) - \Delta^{m-1}_{x_1,\dots,x_{m-1}}\mathcal{Q}(0,y) - T(x_1,\dots,x_{m-1},y)||_Y \\ &\quad \le \max\{\varphi_2(||\sum_{i=1}^{j} x_{\sigma(i)}||_X, ||y||_X) : 1 \le j \le m-1, \sigma \in \mathcal{P}_{m-1}\}. \end{aligned} \tag{13}$$

Since $\mathcal{Q}$ is a polynomial of degree (at most) $m-2$, by Corollary 1, $\Delta^{m-1}_{x_1,\dots,x_{m-1}}\mathcal{Q}(0,y) = 0$ for each $x_1,\dots,x_{m-1},y \in X$. Moreover, by the ultrametric inequality for each $x_1,\dots,x_{m-1} \in X$, we have

$$||\sum_{i=1}^{j} x_{\sigma(i)}||_X \le \max\{||x_i||_X : 1 \le i \le m-1\} \quad (1 \le j \le m-1). \tag{14}$$

Since φ_2 is non-decreasing, it follows from (13) and (14) that

$$\begin{aligned} &||\Delta_{x_1,\dots,x_{m-1},y} f(x) - T(x_1,\dots,x_{m-1},y)||_Y \\ &\quad \le \max\{\varphi_2(||x_i||_X, ||y||_X) : 1 \le i \le m-1\}. \end{aligned} \tag{15}$$

Since m-th difference in the above inequality is symmetric in all its increments, by interchanging x_1 with y in (15), we obtain

$$\begin{aligned} &||\Delta_{x_1,x_2,\dots,x_{m-1},y} f(0) - T(y,x_2,\dots,x_{m-1},x_1)||_Y \\ &\quad \le \max\{\varphi_2(||x_i||_X, ||x_1||_X), \varphi_2(||y||_X, ||x_1||_X) : 2 \le i \le m-1\} \end{aligned} \tag{16}$$

for each $x_1, \dots, x_{m-1}, y \in X$. It follows from (15) and (16) that

$$\begin{aligned} &||T(x_1, \dots, x_{m-1}, y) - T(y, x_2, \dots, x_{m-1}, x_1)||_Y \qquad (17)\\ &\quad \le \max\{\varphi_2(||x_i||_X, ||y||_X), \varphi_2(||x_j||_X, ||x_1||_X), \varphi_2(||y||_X, ||x_1||_X) :\\ &\qquad 1 \le i \le m-1, 2 \le j \le m-1\} \quad (x_1, \dots, x_{m-1}, y \in X). \end{aligned}$$

Since T is $(m-1)$-additive, by replacing x_1 by $k^{-n}x_1$ in (17), we get to the following inequality

$$\begin{aligned} &||T(x_1, \dots, x_{m-1}, y) - k^n T(y, x_2, \dots, x_{m-1}, k^{-n}x_1)||_Y \qquad (18)\\ &\quad \le |k|^n \max_{2\le i\le m-1}\{\varphi_2(|k|^{-n}||x_1||_X, ||y||_X), \varphi_2(||x_i||_X, ||y||_X),\\ &\qquad \varphi_2(||x_i||_X, |k|^{-n}||x_1||_X), \varphi_2(||y||_X, |k|^{-n}||x_1||_X)\}\\ &\quad \le |k|^n \max_{2\le i\le m-1}\{\varphi_2(|k|^{-n}||x_1||_X, |k|^{-n}||y||_X), \varphi_2(||x_i||_X, ||y||_X),\\ &\qquad \varphi_2(|k|^{-n}||x_i||_X, |k|^{-n}||x_1||_X), \varphi_2(|k|^{-n}||y||_X, |k|^{-n}||x_1||_X)\}\\ &\quad \le |k|^{n(1-p)} \max_{1\le i\le m-1,\ 2\le j\le m-1}\{\varphi_2(||x_i||_X, ||y||_X), \varphi_2(||x_j||_X, ||x_1||_X),\\ &\qquad \varphi_2(||y||_X, ||x_1||_X)\} \quad (x_1, \dots, x_{m-1}, y \in X). \end{aligned}$$

Since $0 \le p < 1$ and $|k| < 1$, the right-hand side of (18) tends to zero as $n \to \infty$. Therefore

$$T(x_1, \dots, x_{m-1}, y) = \lim_{n\to\infty} k^n T(y, x_2, \dots, x_{m-1}, k^{-n}x_1) \quad (x_1, \dots, x_{m-1}, y \in X).$$

By the additivity of T with respect to first variable, we obtain

$$\begin{aligned} T(x_1, \dots, x_{m-1}, y_1 + y_2) &= \lim_{n\to\infty} k^n T(y_1 + y_2, x_2, \dots, x_{m-1}, k^{-n}x_1)\\ &= \lim_{n\to\infty} k^n T(y_1, x_2, \dots, x_{m-1}, k^{-n}x_1)\\ &\quad + \lim_{n\to\infty} k^n T(y_2, x_2, \dots, x_{m-1}, k^{-n}x_1)\\ &= T(x_1, \dots, x_{m-1}, y_1) + T(x_1, \dots, x_{m-1}, y_2) \end{aligned}$$

for each $x_1, \dots, x_{m-1}, y_1, y_2 \in X$. This means that T is additive with respect to each variable. Replace x_i by $k^{-n}x_i$ for each $1 \le i \le m-1$ in (15) and multiply both sides of this inequality by $k^{n(m-1)}$ to obtain

$$\begin{aligned} &||\ k^{n(m-1)}\Delta_{k^{-n}x_1,\dots,k^{-n}x_{m-1}} g(0, y) - T(x_1, \dots, x_{m-1}, y)||_Y \qquad (19)\\ &\quad \le |k|^{n(m-1)} \max_{1\le i\le m-1} \varphi_2(|||k|^{-n}x_i||_X, ||y||_X)\\ &\quad \le |k|^{n(m-1-p)} \max_{1\le i\le m-1} \varphi_2(||x_i||_X, ||y||_X) \end{aligned}$$

for each $x_1, \dots, x_{m-1}, y \in X$. Since the right-hand side of the above inequality tends to zero as $n \to \infty$, we have

$$T(x_1, \dots, x_{m-1}, y) = \lim_{n\to\infty} k^{n(m-1)}\Delta_{k^{-n}x_1,\dots,k^{-n}x_{m-1}} g(0, y) \quad (x_1, \dots, x_{m-1}, y \in X).$$

In particular,

$$\mathcal{M}(x,y) = \frac{1}{(m-1)!}T(x,\ldots,x,y) = \lim_{n\to\infty} k^{n(m-1)}\Delta^{m-1}_{k^{-n}x}g(0,y) \quad (x,y\in X).$$

Let $\mathcal{M}_m(x) = \frac{1}{m}\mathcal{M}(x,x)$. The additivity of T with respect to all of its variables implies that $\mathcal{M}_m$ is a monomial of degree m. Since for each $x\in X$ and $n\in\mathbb{N}$, we have $g(0,k^{-n}x) = \Delta_{k^{-n}x}f(0)$, by putting $x_1 = \ldots, x_{m-1} = y = k^{-n}x$ in (19) we see that

$$\mathcal{M}_m(x) = \frac{1}{m!}T(x,\ldots,x) = \frac{1}{m!}\lim_{n\to\infty} k^{nm}\Delta^m_{k^{-n}x}f(0) \quad (x\in X). \tag{20}$$

This completes our proof. □

Lemma 2. *Let $f,\mathcal{Q}$ and $\mathcal{M}$ satisfy the conditions of Lemma 1 for some $m>1$ and $f'(x) = f(x) - \mathcal{M}_m(x)$ for each $x\in X$. Then there are $\mathcal{Q}',\mathcal{M}' : X\times X\to Y$, where $\mathcal{Q}'(x,y)$ is a polynomial of degree at most $m-3$ in x and $\mathcal{M}'(x,y)$ is a monomial of degree $m-2$ in x such that for each $x,y\in X$,*

$$||f'(x+y) - f'(x) - f'(y) + f'(0) - \mathcal{Q}'(x,y) - \mathcal{M}'(x,y)||_Y \le \varphi_2(||x||_X, ||y||_X). \tag{21}$$

Proof. We have

$$\begin{aligned} &f(x+y) - f(x) - f(y) + f(0) - \mathcal{Q}(x,y) - \mathcal{M}(x,y) \\ &\quad = f'(x+y) - f'(x) - f'(y) + f'(0) + \mathcal{M}_m(x+y) \\ &\quad\quad -\mathcal{M}_m(x) - \mathcal{M}_m(y) - \mathcal{Q}(x,y) - \mathcal{M}(x,y) \end{aligned} \tag{22}$$

Thanks to (20), we have

$$\begin{aligned} &\mathcal{M}_m(x+y) - \mathcal{M}_m(x) - \mathcal{M}_m(y) \\ &\quad = \frac{1}{m!}\Big(T(x+y,\ldots,x+y) - T(x,\ldots,x) - T(y,\ldots,y)\Big) \end{aligned}$$

for each $x,y\in X$. Since T is m-additive and symmetric, for each $x,y\in X$,

$$\begin{aligned} T(x+y,\ldots,x+y) &= \sum_{i=0}^{m}\binom{m}{i}T(\underbrace{x,\ldots,x}_{i-terms},\ \underbrace{y,\ldots,y}_{(m-i)-terms}\ ) \\ &= T(x,\ldots,x) + mT(x,\ldots,x,y) + T(y,\ldots,y) \\ &\quad + \sum_{i=1}^{m-2}\binom{m}{i}T(\underbrace{x,\ldots,x}_{i-terms},\ \underbrace{y,\ldots,y}_{(m-i)-terms}\ ). \end{aligned}$$

Therefore for each $x,y\in X$, we have

$$\begin{aligned} &\mathcal{M}_m(x+y) - \mathcal{M}_m(x) - \mathcal{M}_m(y) \\ &\quad = \frac{1}{m!}\Big(mT(x,\ldots,x,y) + \sum_{i=1}^{m-2}\binom{m}{i}T(\underbrace{x,\ldots,x}_{i-terms},\ \underbrace{y,\ldots,y}_{(m-i)-terms}\ )\Big) \quad (23) \\ &\quad = \mathcal{M}(x,y) + \frac{1}{m!}\sum_{i=1}^{m-2}\binom{m}{i}T(\underbrace{x,\ldots,x}_{i-terms},\ \underbrace{y,\ldots,y}_{(m-i)-terms}\ ), \end{aligned}$$

for each $x, y \in X$. Since T is m-additive, for each $1 \le i \le m-2$,

$$T(\underbrace{x, \dots, x}_{i-terms}, \underbrace{y, \dots, y}_{(m-i)-terms})$$

defines a monomial of degree i in x. Therefore, the last term in (23) is a polynomial of degree at most $m-2$, which vanishes at $x = 0$. Let

$$h(x,y) = \frac{1}{m!} \sum_{i=1}^{m-2} \binom{m}{i} T(\underbrace{x, \dots, x}_{i-terms}, \underbrace{y, \dots, y}_{(m-i)-terms}) \quad (x, y \in X).$$

Then for each $x, y \in X$, we have

$$\begin{aligned} \mathcal{M}_m(x+y) - \mathcal{M}_m(x) - \mathcal{M}_m(y) - \mathcal{M}(x,y) - \mathcal{Q}(x,y) &= h(x,y) - \mathcal{Q}(x,y) \\ &= \mathcal{Q}'(x,y) + \mathcal{M}'(x,y), \quad (24) \end{aligned}$$

where $\mathcal{Q}'(x,y)$ is a polynomial of degree $m-3$ in x and $\mathcal{M}'(x,y)$ is a monomial of degree $m-2$ in x. Therefore, the Lemma follows from (11), (22) and (24). □

Now, we are ready to state the main result of this paper.

Theorem 3. *Let $f : X \to Y$ for some $m > 1$ satisfy*

$$||\Delta_{x_1,\dots,x_m} f(0)||_Y \le \varphi_m(||x_1||_X, \dots, ||x_m||_X) \quad (x_1, \dots, x_m \in X). \tag{25}$$

Then there exists a unique polynomial p_{m-1} of degree at most $m-1$ such that

$$||f(x) - p_{m-1}(x)||_Y \le |k|^{-(m-1)p} \varphi_m(||x||_X, \dots, ||x||_X) \quad (x \in X). \tag{26}$$

The polynomial p_{m-1} is given by the formula

$$p_{m-1}(x) = f(0) + \mathcal{M}_1(x) + \cdots + \mathcal{M}_{m-1}(x) \quad (x \in X),$$

where each $\mathcal{M}_i$ is either a monomial of degree i or identically zero ($1 \le i \le m-1$). Finally, for each $x \in X$,

$$\mathcal{M}_{m-1}(x) = \frac{1}{(m-1)!} \lim_{n\to\infty} k^{n(m-1)} \Delta^{m-1}_{k^{-n}x} f(0)$$

and for each $1 \le i < m-1$,

$$\mathcal{M}_i(x) = \frac{1}{i!} \lim_{n\to\infty} k^i \left\{ \Delta^i_{k^{-n}x} f(0) - \sum_{j=i+1}^{m-1} \Delta^j_{k^{-n}x} \mathcal{M}_j(0) \right\} \quad (x \in X). \tag{27}$$

Proof. We first prove uniqueness assertion of the theorem. Let p_{m-1} and p'_{m-1} be two polynomials such that for each $x \in X$

$$\begin{aligned} ||f(x) - f(0) - p_{m-1}(x)|| &\le \varphi_m(||x||_X, \dots, ||x||_X), \\ p_{m-1}(x) &= f(0) + \mathcal{M}_1(x) + \cdots + \mathcal{M}_{m-1}(x) \end{aligned}$$

and

$$\|f(x)-f(0)-p'_{m-1}(x)\|_Y \leq \varphi_m(\|x\|_X,\dots,\|x\|_X),$$
$$p'_{m-1}(x)=f(0)+\mathcal{M}'_1(x)+\cdots+\mathcal{M}'_{m-1}(x),$$

where $\mathcal{M}_i$ and $\mathcal{M}'_i$ are either a monomial of degree i or identically zero ($1 \leq i \leq m-1$). We have

$$p_{m-1}(x)-p'_{m-1}(x)=\mathcal{M}_1(x)-\mathcal{M}'_1(x)+\cdots+\mathcal{M}_{m-1}(x)-\mathcal{M}'_{m-1}(x) \quad (x\in X). \tag{28}$$

Let $p_{m-1} \neq p'_{m-1}$ and i be the greatest index for which $\mathcal{M}_i \neq \mathcal{M}'_i$, $1 \leq i \leq m-1$. By the ultrametric inequality for each $x \in X$, we have

$$\begin{aligned}|k|^{-(m-2)p}\ \varphi_{m-1}(\|x\|_X,\dots,\|x\|_X) \geq \|p_m(x)-p'_m(x)\|_Y &= \|\sum_{j=1}^{i}\mathcal{M}_j(x)-\mathcal{M}'_j(x)\|_Y\\ &\geq \|\mathcal{M}_i(x)-\mathcal{M}'_i(x)\|_Y\\ &\quad -\max_{1\leq j\leq i-1}\|\mathcal{M}_j(x)-\mathcal{M}'_j(x)\|_Y \quad (x\in X).\end{aligned} \tag{29}$$

By replacing x by $k^{-n}x$ in (29), we obtain

$$\begin{aligned}&|k|^{-np-(m-2)p}\varphi_{m-1}(\|x\|_X,\dots,\|x\|_X)\\ &\quad\geq |k|^{-(m-2)p}\varphi_{m-1}(|k|^{-n}\|x\|_X,\dots,|k|^{-n}\|x\|_X)\\ &\quad\geq |k|^{-ni}\|\mathcal{M}_i(x)-\mathcal{M}'_i(x)\|_Y\\ &\qquad -\max_{1\leq j\leq i-1}|k|^{-nj}\|\mathcal{M}_j(x)-\mathcal{M}'_j(x)\|_Y\ (x\in X).\end{aligned} \tag{30}$$

It follows from (30) that for each $x \in X$,

$$\begin{aligned}\|\mathcal{M}_i(x)-\mathcal{M}'_i(x)\|_Y \leq{}& |k|^{n(i-p)-(m-2)p}|\varphi_{m-1}(\|x\|_X,\dots,\|x\|_X)\\ &+\max_{1\leq j\leq i-1}|k|^{n(i-j)}\|\mathcal{M}_j(x)-\mathcal{M}'_j(x)\|_Y.\end{aligned}$$

Since $|k| < 1$ and $i > \max\{p, j; 1 \leq j \leq i-1\}$, the right-hand side of the above inequality tends to zero as $n \to \infty$. It follows that $\mathcal{M}_i(x) = \mathcal{M}'_i(x)$ for each $x \in X$. This contradiction proves the uniqueness assertion of the theorem.

Next, we will prove the existence of p_{m-1} by induction on m. For $m = 2$, the result follows from Theorem 1. Let the theorem hold for some $m \geq 2$ and $f : X \to Y$ satisfy the inequality

$$\|\Delta^{m+1}_{x_1,\dots,x_{m+1}}f(0)\|_Y \leq \varphi_{m+1}(\|x_1\|_X,\dots,\|x_{m+1}\|_X) \quad (x\in X). \tag{31}$$

Fix some $y \in X$, and let

$$\varphi_m(x_1,\dots,x_m)=\varphi_{m+1}(\|x_1\|_X,\dots,\|x_m\|_X,\|y\|_X) \text{ for each } x_1,\dots,x_m\in X.$$

Then we have

$$\|\Delta^m_{x_1,\dots,x_m}(\Delta_y f)(0)\|_Y \leq \varphi_m(\|x_1\|_X,\dots,\|x_m\|_X) \quad (x_1,\dots,x_m\in X). \tag{32}$$

By our hypothesis, there exists a polynomial $p_{m-1}(x,y)$ of degree $m-1$ in x on X such that

$$||\Delta_y f(x) - p_{m-1}(x,y)||_Y \le |k|^{-p(m-1)}\varphi_{m+1}(||x||_X, \dots, ||x||_X, ||y||_X) \quad (x \in X). \tag{33}$$

Moreover,

$$p_{m-1}(x,y) = \Delta_y f(0) + \mathcal{Q}(x,y) + \mathcal{M}(x,y) \quad (x \in X), \tag{34}$$

where $\mathcal{Q}(x,y)$ is a polynomial of degree $m-2$ in x, $\mathcal{Q}(0,y)=0$ and $\mathcal{M}(x,y)$ is a monomial of degree $m-1$ in x. Define

$$\varphi_2(||x||_X, ||y||_X) = |k|^{-p(m-1)}\varphi_{m+1}(||x||_X, \dots, ||x||_X, ||y||_X) \quad (x, y \in X).$$

By substituting (34) in (33), for each $x, y \in X$, we obtain

$$||f(x+y) - f(x) - f(y) + f(0) - \mathcal{Q}(x,y) - \mathcal{M}(x,y)||_Y \le \varphi_2(||x||_X, ||y||_X). \tag{35}$$

The inequality (35) shows that the conditions of Lemma 1 hold. Therefore

$$\mathcal{M}_m(x) = \frac{1}{m!}\lim_{n\to\infty} k^{nm}\Delta_{k^{-n}x^n} f(0) \quad (x \in X)$$

is either zero or a monomial of degree m. Thanks to Lemma 2, $f_1(x) = f(x) - \mathcal{M}_m(x)$ defines a mapping from X to Y which satisfies the conditions of Lemma 1 for $m-1$. Therefore

$$\begin{aligned}\mathcal{M}_{m-1}(x) &= \frac{1}{(m-1)!}\lim_{n\to\infty} k^{(m-1)n}\Delta^{m-1}_{k^{-n}x} f_1(0)\\ &= \frac{1}{(m-1)!}\lim_{n\to\infty} k^{(m-1)n}\Big(\Delta^{m-1}_{k^{-n}x} f(0) - \Delta^{m-1}_{k^{-n}x}\mathcal{M}_m(0)\Big) \quad (x \in X)\end{aligned}$$

defines a mapping from X to Y which is either identically zero or a monomial of degree $m-1$. By continuing this manner, we get to $f_{m-2} : X \to Y$, which is defined by

$$f_{m-2}(x) = f(x) - \mathcal{M}_3(x) - \cdots - \mathcal{M}_m(x) \quad (x \in X),$$

where $\mathcal{M}_i$ is given by (27) and

$$\begin{aligned}&||f_{m-2}(x+y) - f_{m-2}(x) - f_{m-2}(y) + f_{m-2}(0) - \mathcal{T}(x,y)||_Y\\ &\quad \le \varphi_2(||x||_X, ||y||_X) \quad (x, y \in X),\end{aligned}$$

where either $\mathcal{T}(x,y) = 0$ for each $x, y \in X$ or $\mathcal{T}(x,y)$ defines a monomial of degree one in x. Apply Lemma 1 once more and put $\mathcal{M}_2(x) = \frac{1}{2}\mathcal{T}(x,x)$, $x \in X$, then

$$\mathcal{M}_2(x) = \frac{1}{2!}\lim_{n\to\infty} k^{2n}\Delta^2_{k^{-n}x} f_{m-2}(0) \quad (x \in X).$$

Define $f_{m-1}(x) = f_{m-2}(x) - \mathcal{M}_2(x) = f(x) - \sum_{i=2}^m \mathcal{M}_i(x)$ for each $x \in X$. By applying Lemma 2 once again, we see that

$$\begin{aligned}&||f_{m-1}(x+y) - f_{m-1}(x) - f_{m-1}(y) + f_{m-1}(0)||_Y\\ &\quad \le \varphi_2(||x||_X, ||y||_X) \quad (x, y \in X).\end{aligned} \tag{36}$$

Thanks to Theorem 1, there exists an additive (monomial of degree one) $\mathcal{M}_1 : X \to Y$ such that for each $x \in X$

$$\begin{aligned} ||f_{m-1}(x) - f_{m-1}(0) - \mathcal{M}_1(x)||_Y &\le |k|^{-p}\varphi_2(||x||_X, ||x||_X) \\ &= |k|^{-mp}\varphi_{m+1}(||x||_X, \dots, ||x||_X). \end{aligned}$$

Since $f_{m-1}(0) = f(0)$, we have

$$||f(x) - p_m(x)||_Y \le |k|^{-mp}\varphi_{m+1}(||x||_X, \dots, ||x||_X) \quad (x \in X),$$

where $p_m(x) = f(0) + \mathcal{M}_1(x) + \dots + \mathcal{M}_m(x)$ for each $x \in X$. This proves our theorem with m replaced by $m+1$. Thus by induction on m, the existence assertion of our theorem has been proved. This completes the proof of the theorem. □

3. Applications

In [11], D. H. Hyers proved the following.

Theorem 4. *Let X be a vector space over the rational numbers, S be a convex cone in X and B be Banach space. If β is fixed positive number and if $f : S \to B$ satisfies the condition*

$$||\Delta_h^2 f(x)||_Y \le \beta \quad (x, h \in S), \tag{37}$$

then there exists an additive mapping $T : S \to B$ such that $||f(x) - f(0) - T(x)||_Y \le \beta$ for all $x \in S$. The function T is given by the formula $T(x) = \lim_{n\to\infty} n^{-1} f(nx)$.

The following example shows that the theorem of Hyers is not true in non-Archimedean normed spaces.

Example 1. *Let $p > 2$ be a prime number. For any nonzero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field [9]. Define $f : \mathbb{Q}_p \to \mathbb{Q}_p$ by $f(x) = x + p$ for each $x \in \mathbb{Q}_p$. Then $\Delta_h^2 f(x) = 0$. Therefore* (37) *holds. However, $\lim_{n\to\infty} n^{-1} f(nx)$ is not Cauchy. In fact for the subsequence $\{p^n\}$ of $\{n\}$, we have $p^{-n} f(p^n x) = x + p^{-n+1}$. Therefore*

$$\begin{aligned} |p^{-n} f(p^n x) - p^{-(n+1)} f(p^{n+1} x)|_p &= |x + p^{-(n-1)} - x - p^{-n}|_p \\ &= |p^{-n}|_p |p-1|_p = p^n. \end{aligned}$$

Since the right-hand side of the above equation tends to infinity as $n \to \infty$, the subsequence $\{p^{-n} f(p^n x)\}$ is not Cauchy. Hence $\lim_{n\to\infty} n^{-1} f(nx)$ in $(\mathbb{Q}_p, |\cdot|_p)$ does not exist.

Here we give some applications of Theorem 3.

Corollary 2. *Let $f : X \to Y$ for some $0 \le p < 1$ and $\varepsilon > 0$ satisfy the inequality*

$$||\Delta^m_{x_1,\dots,x_m} f(0)||_Y \le \varepsilon \max\{||x_1||_X^p, \dots, ||x_m||_X^p\} \quad (x_1, \dots, x_m \in X).$$

Then there exists a unique polynomial p_{m-1} of degree $m-1$ such that

$$||f(x) - p_{m-1}(x)||_Y \le \varepsilon |k|^{-(m-1)p} ||x||_X^p \quad (x \in X).$$

Proof. Take

$$\varphi_m(||x_1||_X, \dots, ||x_m||_X) = \varepsilon \max\{||x_1||_X^p, \dots, ||x_m||_X^p\} \quad (x_1, \dots, x_m \in X),$$

in Theorem 3. □

The following result can be considered as a generalization of the main result in [19].

Corollary 3. *Let $f : X \to Y$ for some $0 \le p < 1$ and $\varepsilon > 0$ satisfy the inequality*

$$||\Delta^m_{x_1,\dots,x_m} f(0)||_Y \le \varepsilon \sum_{i=1}^{m} ||x_i||_X^p \quad (x_1, \dots, x_m \in X).$$

Then there exists a unique polynomial p_{m-1} of degree $m-1$ such that

$$||f(x) - p_{m-1}(x)||_Y \le m\varepsilon |k|^{-(m-1)p} ||x||_X^p \quad (x \in X).$$

Proof. Apply Theorem 3 for

$$\varphi_m(||x_1||_X, \dots, ||x_m||_X) = \varepsilon \sum_{i=1}^{m} ||x_i||_X^p \quad (x_1, \dots, x_m \in X).$$

□

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