MATHEMATICAL COMMUNICATIONS
Math. Commun. **17**(2012), 525–535

Robust solutions to uncertain weighted least squares problems*

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Received November 3, 2009; accepted March 25, 2012

Abstract. Robust optimization is a rapidly developing methodology for handling optimization problems affected by non-stochastic uncertain-but-bounded data perturbations. In this paper, we consider the weighted least squares problems where the coefficient matrices and vector belong to different uncertain bounded sets. We introduce the robust counterparts of these problems and reformulate them as the tractable convex optimization problems. Two kinds of approaches for solving the robust counterpart of weighted least squares problems with ellipsoid uncertainty sets are also given.
AMS subject classifications: 90C34, 90C05, 90C20

Key words: robust optimization, weighted least squares problems, robust counterpart, convex programming

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1. Introduction

Many real-world optimization problems involve input data that is noisy or uncertain, due to measurement or modelling errors, or simply the unavailability of the information at the time of decision. So addressing data uncertainty in mathematical programming models has long been recognized as a central problem in optimization.

In recent years, a body of literature is developing under the name of robust optimization which is based on a description of uncertainty sets, as opposed to probability distribution. The uncertain parameters are only known to belong to known sets, and one associates with the uncertain problem its *robust counterpart* where the constraints are enforced for every possible value of the parameters within their prescribed sets; under such constraints, the worst-case value of the cost function is then minimized to obtain a *robust solution* of the problem. Mulvey et al. [23] presented an approach that integrates goal programming formulations with scenario-based description of the problem data. Soyster, in the early 1970s, [25] proposed a linear optimization model to construct a solution that is feasible for all input

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*This work was supported by the Key program of NSFC (Grant No. 70831005), the National Natural Science Foundation of China (11171237, 11126346), and the "Project 211 (Phase III)" Research Fund of the Southwestern University of Finance and Economics (211QN10009).
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data such that each uncertain input data can take any value from an interval. This approach, however, tends to find solutions that are over-conservative. Ben-Tal and Nemirovski [2, 3, 4, 5, 6, 7], Ben-Tal et al. [8], El-Ghaoui and Lebret [15] and El-Ghaoui et al. [16] addressed the over-conservatism of robust solutions by allowing the uncertainty sets for the data to be ellipsoids, and proposed some efficient algorithms to solve convex optimization problems under data uncertainty. Bertsimas and Sim [9, 10, 11] proposed a different approach to control the level of conservatism in the solution that has the advantage that leads to a linear optimization model. Iyengar [20] and Iyengar and Erdogan [21] studied the problem with chance (probabilistic) constraints which are ambiguous in the sense that the underlying distribution of the random parameters is uncertain. They used a robust sampled problem to get the good approximations to the ambiguous chance constraints. Recently, Yan et al. [28] treated the split feasibility problem with the uncertain linear operator and reformulated it as a tractable convex optimization problem. Very recently, Zhao et al. [29] considered the uncertain extended weighted Steiner problem and reformulated it as a semidefinite program under the ellipsoidal uncertainty.

There exist many references on the least squares problem. To avoid trying to list all at the expense of omitting some we adopt the excellent book by Bjorck [12] as our desktop reference. The problem of uncertainty in (A, b) is addressed by using several remedies, such as total least squares and variants thereof, Tikhonov regularization, iterated regularization, L-curve analysis and so on. The interested reader is directed to Section 7 of Chapter 2 and Section 6 of Chapter 4 in [12]. An important reference on total least squares and its applications in engineering is by Van Huffel and Vandevelle [26]. Ample information on regularization methods can be found in the book by Hansen [19]. For recent related articles on least squares problems under uncertainty, the reader is directed to [13, 14, 27] as well. Another important line of research dealing with uncertainty in linear systems of equations summarized by Kreinovich et al. [22] is the subject of interval computations, with an emphasis on complexity issues.

It has been well recognized that vector optimization has its roots in economic modeling and general equilibrium theory. Recently, Bao and Mordukhovich [1] and Habte and Mordukhovich [18] considered the general nonconvex models of welfare economics involving both private and public goods with finite-dimensional and infinite-dimensional spaces of commodities. Based on advanced tools of variational analysis and generalized differentiation, they established appropriate approximate and exact versions of the extended second welfare theorem for Pareto, weak Pareto, and strong Pareto optimal allocations in both marginal price and decentralized price forms.

Motivated by the works mentioned above, in this paper, we use robust methodology to solve the weighted least squares problem (for short, WLSP), in which the coefficient matrices A_i and b_i belong to uncertainty sets. The rest of paper is organized as follows. In Section 2, we introduce the robust counterpart of WLSP (for short, RWLSP) with uncertainty sets. In Section 3, we illustrate the general uncertainty sets and show that RWLSP with general uncertainty sets is equivalent to a convex programming problem. In Section 4, we use two kinds of approaches to solve RWLSP with ellipsoid uncertainty sets.

2. Preliminaries

Throughout this paper, we need the following notations. For a vector x, $\|x\|$ denotes the 2-norm. For a matrix $A \in R^{m\times n}, \|A\|_2$ denotes the spectral norm which is induced by 2-norm in vector space. $\|A\|_F$ denotes the Frobenious norm and

$$vec(A) = (a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn})$$

denotes the matrix-vector. For given $x \in R^n$ and $z \in R^m$, the notion $x \otimes y$ refers to the Kronecker product with x and y, i.e.,

$$x \otimes z = (x_1z_1, \cdots, x_1z_m, x_2z_1, \cdots, x_2z_m, \cdots, x_nz_1, \cdots, x_nz_m).$$

In this section, we give an optimization reformulation of Weighted Least Square problem (WLSP),

$$\min_{x\in R^n} \sum_{i=1}^{N} \omega_i\|A_ix - b_i\|, \tag{1}$$

where $A_i \in R^{m\times n}, b_i \in R^m, i = 1, 2, \cdots, N$ and $\omega_i, i = 1, \cdots, N$ are fixed positive weights. When the data (A_i, b_i) is uncertain and is only known to belong to some uncertainty set $\mathcal{U}_i$, we speak about the following uncertain weighted least square problem (UWLSP),

$$\min_{x\in R^n} \{\sum_{i=1}^{N} \omega_i\|A_ix - b_i\|, \quad (A_i, b_i) \in \mathcal{U}_i, \quad i = 1, \cdots, N\}. \tag{2}$$

The robust counterpart of (2) is defined to be the following optimization problem (RWLSP)

$$\min_{x\in R^n} \{\max_{(A_i,b_i)\in\mathcal{U}_i} \sum_{i=1}^{N} \omega_i\|A_ix - b_i\|, \quad (A_i, b_i) \in \mathcal{U}_i, \quad i = 1, \cdots, N\}, \tag{3}$$

An optimal solution of (3) is called a *robust optimal solution* of (2). The importance of these solutions is motivated and illustrated in [3, 4, 5, 6, 7, 8], of course, a crucial issue regarding the usefulness and applicability of the robust optimization methodology is the extent of the computational effort needed to solve problems such as (RWLSP). The goal of this paper is to reformulate the robust counterparts of these problems as the tractable convex optimization problems.

3. General uncertainty

In this section, we consider the general uncertainty in matrices (A_i, b_i), i.e., (A_i, b_i) belong to the following sets

$$\mathcal{U}_i^1 = \{(A_i, b_i) = (A_{i0}, b_{i0}) + (\Delta A_i, \Delta b_i) | \|(\Delta A_i, \Delta b_i)\|_F \le \rho\},$$

where $i = 1, \cdots, N, (A_{i0}, b_{i0})$ are the nominal values of WLSP, $(\Delta A_i, \Delta b_i)$ are unknown-but-bounded matrices and ρ is a given positive constant.

For x fixed, we define the worst-case residual of RWLSP with $\mathcal{U}_i^1$ as

$$r_i^1(A_i, b_i, \rho, x) = \max_{\|(\Delta A_i, \Delta b_i)\|_F \le \rho} \sum_{i=1}^{N} \omega_i(\|(A_{i0} + \Delta A_i)x - (b_{i0} + \Delta b_i)\|),$$

where $i = 1, \cdots, N$.

For every $\rho > 0$, it is easy to see that $r_i^1(A_i, b_i, \rho, x) = \rho r_i^1(A_i/\rho, b_i/\rho, 1, x)$. Thus, we take $\rho = 1$ in what follows unless otherwise stated.

From (3), we know that RWLSP with $\mathcal{U}_i^1$ is equivalent to the following reformulation

$$\min_{x \in R^n} \max_{\|(\Delta A_i, \Delta b_i)\|_F \le \rho} \sum_{i=1}^{N} \omega_i(\|(A_{i0} + \Delta A_i)x - (b_{i0} + \Delta b_i)\|). \tag{4}$$

Theorem 1. *When $\rho = 1$, RWLSP (4) is equivalent to the following convex programming problem:*

$$\min_{x \in R^n} \sum_{i=1}^{N} \omega_i(\|A_{i0}x - b_{i0}\| + \sqrt{\|x\|^2 + 1}). \tag{5}$$

Proof. It follows from the definition of $r_1(A_i, b_i)$ that

$$\begin{aligned}
r_i^1(A_i, b_i, 1, x) &= \max_{\|(\Delta A_i, \Delta b_i)\|_F \le 1} \sum_{i=1}^{N} \omega_i \|(A_{i0} + \Delta A_i)x - (b_{i0} + \Delta b_i)\| \\
&= \max_{\|(\Delta A_i, \Delta b_i)\|_F \le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\| \le 1} z^T[(A_{i0}x - b_{i0}) + (\Delta A_i x - \Delta b_i)] \\
&= \max_{\|(\Delta A_i, \Delta b_i)\|_F \le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\| \le 1} [z^T(A_{i0}x - b_{i0}) + z^T(\Delta A_i x - \Delta b_i)] \\
&\le \max_{\|(\Delta A_i, \Delta b_i)\|_F \le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\| \le 1} z^T(A_{i0}x - b_{i0}) \\
&\quad + \max_{\|(\Delta A_i, \Delta b_i)\|_F \le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\| \le 1} z^T(\Delta A_i x - \Delta b_i) \\
&= \sum_{i=1}^{N} \omega_i \|A_{i0}x - b_{i0}\| + \sum_{i=1}^{N} \omega_i \sqrt{\|x\|^2 + 1}.
\end{aligned}$$

Moreover, choose $\Delta = (\Delta A_i, \Delta b_i)$ as

$$\Delta = \frac{z_0}{\sqrt{\|x\|^2 + 1}}(x^T, 1),$$

where

$$z_0 = \begin{cases} \frac{A_{i0}x - b_{i0}}{\|A_{i0}x - b_{i0}\|}, & \text{if } A_{i0}x \ne b_{i0}, \\ q, & \text{otherwise,} \end{cases}$$

with $\|q\| = 1$. Then we can conclude that $\|\Delta\|_F = \|\Delta\| = 1, \|z\| = 1$ and

$$\sum_{i=1}^{N} \omega_i z_0^T (A_{i0}x - b_{i0}) + \sum_{i=1}^{N} \omega_i z_0^T (\Delta A_i x - \Delta b_i)$$
$$= \sum_{i=1}^{N} \omega_i \|A_{i0}x - b_{i0}\| + \sum_{i=1}^{N} \omega_i \sqrt{\|x\|^2 + 1}.$$

This shows the equivalence between (4) and (5). This completes the proof. □

Remark 1. *When $N = 1$, Theorem 1 is reduced to Theorem 3.1 of El-Ghaoui and Lebret [15].*

4. Ellipsoid uncertainty

In this section, we will consider RWLSP with the ellipsoid uncertainty sets as follows:

$$\mathcal{U}_i^2 = \{(A_i, b_i) = (A_{i0}, b_{i0}) + \sum_{l=1}^{L} (A_{il}, b_{il}) u_l \| \|u\| \le \rho\}, \tag{6}$$

where $i = 1, \cdots, N, (A_{i0}, b_{i0})$ are the nominal values of the WLSP, (A_{il}, b_{il}) are the given directions of perturbation, u_l are the uncertain variables with $\|u\| \le \rho$. When $\rho > 0$, let $v = u/\rho$, then $\|v\| \le 1$, so we take $\rho = 1$ in what follows unless otherwise stated.

Next, we will use two kinds of new approaches to solve RWLSP with ellipsoid uncertainty sets.

4.1. Optimizing the worst-case residual

For x fixed, we define the worst-case residual of RWLSP with $\mathcal{U}_i^2$ as

$$r_i^2(A_i, b_i, \rho, x) = \max_{\|u\| \le \rho} \sum_{i=1}^{N} \omega_i \|(A_{i0}x + \sum_{l=1}^{L} A_{il} u_l)x - (b_{i0} + \sum_{l=1}^{L} b_{il} u_l)\|,$$

where $i = 1, \cdots, N$. Then RWLSP with $\mathcal{U}_i^2$ is equivalent to the following reformulation

$$\min_{x \in R^n} \max_{\|u\| \le \rho} \sum_{i=1}^{N} \omega_i \|(A_{i0}x + \sum_{l=1}^{L} A_{il} u_l)x - (b_{i0} + \sum_{l=1}^{L} b_{il} u_l)\| \tag{7}$$

Theorem 2. *When $\rho = 1$, RWLSP (7) is equivalent to the following convex programming problem*

$$\min_{x \in R^n} \sum_{i=1}^{N} \omega_i (\|A_{i0}x - b_{i0}\| + \|A_i[x]\|_F) \tag{8}$$

where $A_i[x] = (A_{i1}x - b_{i1}, A_{i2}x - b_{i2}, \cdots, A_{iL}x - b_{iL}) \in R^{m \times L}, \quad i = 1, \cdots, N.$

Proof. We first prove that

$$\max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1}[z^T(A_{i0}x-b_{i0})+(u\otimes z)^T vec(A_i[x])] = \sum_{i=1}^N \omega_i\|A_{i0}x-b_{i0}\| + \sum_{i=1}^N \omega_i \|A_i[x]\|_F.$$

In fact, we have

$$\begin{aligned}&\max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1}[z^T(A_{i0}x-b_{i0})+(u\otimes z)^T vec(A_i[x])]\\ &\le \max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1} z^T(A_{i0}x-b_{i0}) + \max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1}[(u\otimes z)^T vec(A_i[x])]\\ &= \sum_{i=1}^N \omega_i\|A_{i0}x-b_{i0}\| + \sum_{i=1}^N \omega_i \|A_i[x]\|_F.\end{aligned}$$

Moreover, if we take

$$z_0 = \begin{cases} \frac{A_{i0}x-b_{i0}}{\|A_{i0}x-b_{i0}\|}, & \text{if } A_{i0}x \ne b_{i0},\\ q, & \text{otherwise,}\end{cases}$$

with $\|q\|=1$, and

$$u_0 = \begin{cases} \frac{(A_i[x])^T}{\|A_i[x]\|_F} z_0, & \text{if } \|A_i[x]\|_F \ne 0,\\ 0, & \text{otherwise,}\end{cases}$$

then we conclude that $\|z_0\|=1$, $\|u_0\|\le 1$ and

$$\sum_{i=1}^N \omega_i z_0^T(A_{i0}x-b_{i0}) + \sum_{i=1}^N \omega_i (u_0\otimes z_0)^T vec(A_i[x]) = \sum_{i=1}^N \omega_i\|A_{i0}x-b_{i0}\| + \sum_{i=1}^N \omega_i\|A_i[x]\|_F.$$

Hence the desired equation holds.

Next, from the definition of $r_2(A_i, b_i, 1, x)$, we have

$$\begin{aligned} r_i^2(A_i,b_i,1,x) &= \max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \|(A_{i0}+\sum_{l=1}^L A_{il}u_l)x - (b_{i0}+\sum_{l=1}^L b_{il}u_l)\|\\ &= \max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1} z^T((A_{i0}x-b_{i0})+\sum_{l=1}^L (A_{il}x-b_{il})u_l)\\ &= \max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1} (z^T(A_{i0}x-b_{i0}) + z^T(\sum_{l=1}^L (A_{il}x-b_{il})u_l))\\ &= \max_{\|u\|\le 1}\sum_{i=1}^N \omega_i \max_{\|z\|\le 1} (z^T(A_{i0}x-b_{i0}) + \sum_{l=1}^L (z^T(A_{il}x-b_{il})u_l))\end{aligned}$$

$$= \max_{\|u\|\le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\|\le 1} \Bigg(z^T(A_{i0}x - b_{i0})$$

$$+ z^T(A_{i1}x - b_{i1}, A_{i2}x - b_{i2}, \cdots, A_{iL}x - b_{iL}) \begin{pmatrix} u_1 \\ \vdots \\ u_L \end{pmatrix} \Bigg)$$

$$= \max_{\|u\|\le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\|\le 1} \left(z^T(A_{i0}x - b_{i0}) + z^T A_i[x] \begin{pmatrix} u_1 \\ \vdots \\ u_L \end{pmatrix} \right)$$

$$= \max_{\|u\|\le 1} \sum_{i=1}^{N} \omega_i \max_{\|z\|\le 1} (z^T(A_{i0}x - b_{i0}) + (u \otimes z)^T vec(A_i[x]))$$

$$= \sum_{i=1}^{N} \omega_i \|A_{i0}x - b_{i0}\| + \sum_{i=1}^{N} \omega_i \|A_i[x]\|_F.$$

This shows the equivalence between (7) and (8). This completes the proof. □

4.2. Inner approximation

It is clear that the RWLSP (3) with uncertainty sets $\mathcal{U}_i^2$ is equivalent to the following optimization problem:

$$\min_{\tau,x} \left\{ \tau : \sum_{i=1}^{N} \omega_i \|A_i x - b_i\| \le \tau, \quad (A_i, b_i) \in \mathcal{U}_i^2, \quad i = 1, 2, \cdots, N. \right\}. \tag{9}$$

By introducing the variables y_i with $i = 1, \cdots, N$, it is easy to see that problem (9) rewritten equivalently as the following problem:

$$\begin{aligned} \min_{x,y,\tau} \quad & \tau \\ s.t. \quad & \mathbf{1}^T y \le \tau, \ y_i \ge 0, \\ & \lambda_i = y_i/\omega_i, \\ & \|A_i x - b_i\| \le \lambda_i, \\ & \forall (A_i, b_i) \in \mathcal{U}_i^2, \ i = 1, \cdots, N, \end{aligned} \tag{10}$$

where $\mathbf{1}^T = (1, \cdots, 1)^T$ and $y = (y_1, \cdots y_N)$.

For a perturbation set as given in (6), the verification of (10) is an NP-hard problem (see [6]). Therefore, we shall build an inner approximation of the set in problem (10). To this end, we will use the ideas of semidefinite relaxation (Theorem 3 below). We need the following useful lemmas.

Lemma 1 (see [3])**.** *Let P and Q be two symmetric matrices such that there exists z_0 satisfying $z_0^T P z_0 > 0$. Then the implication*

$$z^T P z \ge 0 \Rightarrow z^T Q z \ge 0$$

holds true if and only if there exists $\lambda \ge 0$ such that $Q \ge \lambda P$.

Lemma 2 (see [6]). *Let*

$$A = \begin{pmatrix} B & C^T \\ C & D \end{pmatrix}$$

be a symmetric matrix with $k \times k$ block B and $l \times l$ block D. Assume that B is positive definite. Then A is positive (semi)definite if and only if the matrix $D - CB^{-1}C^T$ is positive (semi)definite (this matrix is called the Schur complement of B in A).

Theorem 3. *When $\rho = 1$, WRMLSP (10) corresponding to the uncertainty sets $\mathcal{U}_i^2$ is equivalent to the following semidefinite programming:*

$$\min \quad \tau$$

w.r.t. $x \in R^n$, $\lambda_1, \cdots, \lambda_N \in R$, $\mu_1, \cdots, \mu_N \in R$
subject to $\mathbf{1^T} y \leq \tau$, $\quad y_i \geq 0$, $\quad \lambda_i = y_i/\omega_i$ and

$$\begin{pmatrix} \lambda_i - \mu_i & 0 & 0 & \dots & 0 & (A_{i0}x - b_{i0})^T \\ 0 & \mu_i & 0 & \cdots & 0 & (A_{i1}x - b_{i1})^T \\ 0 & 0 & \mu_i & \cdots & 0 & (A_{i2}x - b_{i2})^T \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \mu_i & (A_{iL}x - b_{iL})^T \\ A_{i0}x - b_{i0} & A_{i1}x - b_{i1} & A_{i2}x - b_{i2} & \dots & A_{iL}x - b_{iL} & \lambda_i I \end{pmatrix} \geq 0 \quad (11)$$

for $i = 1, \cdots, N$.

Proof. We consider every uncertain constraint of (10)

$$\|A_i x - b_i\| \leq \lambda_i, \qquad \forall (A_i, b_i) \in \mathcal{U}_i^2.$$

It follows that in order to understand what is, analytically, WRMLSP, it suffices to understand what is the robust version of a single constraint of WRMLSP; to simplify notation, we drop the index i, so that the robust constraint in question becomes

$$\|Ax - b\| \leq \lambda, \qquad \forall (A, b) \in \mathcal{U}^2. \quad (12)$$

$$\mathcal{U}^2 = \Big\{(A, b) = (A_0, b_0) + \sum_{l=1}^{L} (A_l, b_l) u_l \mid \|u\| \leq 1\Big\}.$$

Let us set

$$a[x] = A_0 x - b_0, \quad A[x] = (A_1 x - b_1, A_2 x - b_2, \cdots, A_L x - b_L).$$

In the above notation the robust constraint (12) becomes the following constraints

$$\lambda \geq 0, \quad (13)$$

and

$$\|a[x] + A[x]u\|^2 \leq \lambda^2, \ \forall u : \|u\| \leq 1. \quad (14)$$

The constraint (14) is equivalent to the constraint

$$\forall\big((t,u),\ u^T u \le t^2\big)\ :\ \|a[x]t + A[x]u\|^2 \le \lambda^2 t^2.$$

In other words, a pair (x, λ) satisfies (13) and (14) if and only if λ is nonnegative, and nonnegativity of quadratic form $(t^2 - u^T u)$ of variables t and u implies the nonnegativity of quadratic form

$$\lambda^2 t^2 - \|a[x]t + A[x]u\|^2$$

of the same variables. By Lemma 1, the indicated property is equivalent to the existence of nonnegative ν such that the quadratic form

$$\Psi(t,u) = \lambda^2 t^2 - \|a[x]t + A[x]u\|^2 - \nu(t^2 - u^T u)$$

is positive semidefinite. We claim that ν can be represented as $\mu\lambda$ with some nonnegative μ and $\mu = 0$ in the case of $\lambda = 0$. Indeed, our claim is evident if $\lambda > 0$. In the case of $\lambda = 0$, the form $\Psi(t,u)$ clearly can be positive semidefinite only if $\nu = 0$, and we indeed have $\nu = \mu\lambda$ with $\mu = 0$.

We have demonstrated that a pair (x, λ) satisfies (13) and (14) if and only if there exists a μ such that the triple (x, λ, μ) possesses the following property (π):

(π) : $\lambda, \mu \ge 0$; when $\lambda = 0$, then the quadratic form

$$\Psi(t,u) = \lambda(\lambda - \mu)t^2 + \lambda\mu u^T u - (t, u^T)R^T(x)R(x)\begin{pmatrix} t \\ u \end{pmatrix}$$

of t, u is positive semidefinite, where

$$R(x) = (a[x], A[x]).$$

Now let us prove that the property (π) of (x, λ, μ) is equivalent to positive semidefiniteness of the matrix $S = S(x, \lambda, \mu)$ in left hand side of (11). Indeed, if $\lambda > 0$, positive semidefiniteness of Ψ is equivalent to positive semidefiniteness of quadratic form

$$(\lambda - \mu)t^2 + \mu u^T u - (t, u^T)R^T(x)(\lambda I)^{-1}R(x)\begin{pmatrix} t \\ u \end{pmatrix},$$

which, by Lemma 2, is exactly the same as positive semidefiniteness of $S(x, \lambda, \mu)$. Of course, the matrix in the left-hand side of (11) can be positive semidefinite only when $\lambda \equiv \lambda_i, \mu \equiv \mu_i$ are nonnegative. Thus, for triple (x, λ, μ) with $\lambda > 0$ the property (π) indeed is equivalent to the positive semidefiniteness of $S(x, \lambda, \mu)$. Now we consider the case of $\lambda = 0$, and let $(x, 0, \mu)$ satisfy (π). Due to (π), $\mu = 0$ and Ψ is positive semidefinite, which for $\lambda = 0$ is possible if and only if $R(x) = 0$; of course, in the case of $\mu = 0$, $S(x, 0, 0)$ is positive semidefinite. Vice versa, if $\lambda = 0$ and $S(x, \lambda, \mu)$ is positive semidefinite, then, of course, $R(x) = 0$ and $\mu = 0$ and triple (x, λ, μ) possesses the property (π).

The summary of our equivalences is that (x, λ) with $\lambda = \lambda_i$ satisfies (10) if and only if there exists $\mu = \mu_i$ such that the triple (x, λ, μ) satisfies (11). This is exactly the assertion of the theorem. □

5. Conclusion

This paper suggests a robust optimization approach to formulate and solve the weighted least squares problems where the coefficient matrices A_i and b_i belong to two different uncertain-but-bounded sets. We introduce the robust counterpart of the weighted least squares problems with uncertainty sets and illustrate the general uncertainty sets. We also show that the robust counterpart of the weighted least squares problems with general uncertainty sets is equivalent to a convex programming problem under some suitable conditions and use two kinds of approaches to solve the robust counterpart of the weighted least squares problems with ellipsoid uncertainty sets.

As pointed out by the referee, "In the context of robust optimization, it might help to use some ideas from parametric programming such as input optimization, regions of stability and structural optima (see, for example, [30]). It may lead to robust optimization over regions of stability". In the future work, these regions need to be established. Furthermore, we should construct some effective algorithms for solving the weighted least squares problems.

Acknowledgement

The authors are grateful to the editor and referees for their valuable comments and suggestions.

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